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***Research paper***

# Advanced theory of vibration of uniform beams

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**Abstract**

In classical theory the equation of a dynamic Euler – Lagrange beam is solved by using the composition of the displacements into the sum of harmonic vibrations to obtain the ordinary differential equation. The solution of this equation with prescribed set of boundary conditions is a typical Sturm – Liouville problem with the infinite, discrete eigenvalues and modes of vibration. The purpose of this paper is to reveal that an elastic beam is a limited continuum with limited domain of physically existing, continuous eigenvalues and modes of vibration. In contrast to the classical theory, the advanced theory of free vibration of beams without damping in present investigation is based on the analysis with transversal and angular stiffness, initiated by external transient excitations and inherited by beams in compliance with energy conservation law. The output of this investigation demonstrates the fundamental distinction between the dynamic characteristics of uniformed beams established by classical theory with infinite, discrete eigenvalues and derived characteristics of beams with continuous eigenvalues and modes of vibration in limited domains. The theoretical investigation shows that only few, natural, discrete eigenvalues and normal modes of vibration physically exist in limited domains.
Nomenclature: $k^4 = \omega^2 a^{-2}$, $a^2 = E I g (A \gamma)^{-1}$, E − modulus of elasticity, g − gravitational acceleration, A − area of the beam's cross - section, γ − specific gravity of the beam's material, ω = 2πf, f − frequency of vibration per second, angular frequency $p = a (kl)^2/l^2$, I – length of beam. I − moment of inertia of area.

***Keywords***: *Eigenvalues; Modes; Resonance; Stiffness; Vibration.*

## 1. Introduction

The numerical approach for analysis of vibration of uniform beams without damping was presented by S. P. Timoshenko in 1928. The results for uniform beams are available in numerous books, monographs and handbooks (see, for example [1 - 4]) and widely used in engineering applications. In spite of the conventional completeness of this theory the experimental investigations show the significant disparity compared to these theoretical results. Especially insuperable difficulties arise to observe the modes with high frequencies, and this is not just due to the effect of damping. The purpose of this investigation is to present the advanced theory of beams vibration with fitness to physical reality. For preliminary elucidation consider the single degree – of – freedom system without damping excited by half – wave sine impulse with frequency $\omega$. The differential equation of motion is

$$m\, d^2x/dt^2 + kx = F(t)$$

The Fourier expansion of this impulse is the following [5]

$$F(t) = F_0 [(\pi)^{-1} + 0.5 \sin \omega t - 2(\pi)^{-1} \sum (n^2 - 1)^{-1} \cos n\omega t ],$$

Where n = 2, 4, 6…

The obvious solutions of this equation show that free, periodic vibrations with dominant frequency $\omega$ occur after $t > \pi/\omega$ and resonance occurs if $p = (k/m)^{0.5} = \omega$. Since the free vibrations, excited by impulse, can be with any frequency, we can conclude that the model of free vibrations with any frequency without force but with suitable "dynamic stiffness" is valid theoretical equivalent for analysis. The free vibration with natural frequency p can be excited only by initial, static displacement $x(0)$ = const. The concept of "dynamic stiffness" is used analyzing the free vibrations of beams. The tabulated Krylov's functions
$R_1(kx)$, $R_2(kx)$, $R_3(kx)$ and $R_4(kx)$ are used for computations [6].

$$R_1(kx) = 0.5(\cosh kx + \cos kx),\quad R_2(kx) = 0.5(\sinh kx + \sin kx),$$

$$R_3(kx) = 0.5(\cosh kx - \cos kx),\quad R_4(kx) = 0.5(\sinh kx - \sin kx).$$

Note: In classical theory of free vibrations of beams the Euler – Lagrange equation is solved by separation of variables. With specified boundary conditions only first two discrete modes can be excited by the initial, static displacements: one for symmetrical mode and one for asymmetric mode. So, they are natural. All higher modes are resonant and can be excited only by dynamic forces or moments.

## 2. Method and results

The governing equation for the modes of the uniform beams is [1]:

$$d^4X/dx^4 = k^4X \qquad (1)$$

The solution of Eq. (1) can be presented in the following form:

$$X = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx) +$$

$$C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx), \qquad (2)$$

Where $C_1$, $C_2$, $C_3$ and $C_4$ are constants.

### 2.1. Beam with hinged ends (BHH)

For beams with hinged ends the conditions at x = 0 are the following:

$$X(0) = 0 \text{ and } d^2X/dx^2 = 0 \quad (3)$$

To satisfy these conditions $C_1 = C_2 = 0$ and solution (2) becomes

$$X = C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx) = 2[C_3R_2(kx) - C_4R_4(kx)] \quad (4)$$

and successive derivatives with respect to x

$$DX/dx = C_3k(\cos kx + \cosh kx) + C_4k(\cos kx - \cosh kx) = 2k[C_3R_1(kx) - C_4R_3(kx)]; \quad (5)$$

$$d^2X/dx^2 = C_3k^2(-\sin kx + \sinh kx) - C_4k^2(\sin kx + \sinh kx) = 2k^2[C_3R_4(kx - C_4R_2(kx)]; \quad (6)$$

$$d^3X/dx^3 = C_3k^3(-\cos kx + \cosh kx) - C_4k^3(\cos kx + \cosh kx) =$$

$$2k^3[C_3R_3(kx) - C_4R_1(kx)] \quad (7)$$

Since all modes of BHH are symmetrical or asymmetric, it is clear that each half of that beam vibrates with the frequencies inherent in the whole of the beam. It is expected that modeling of the half of BHH for determination of the frequency equation will provide comprehensive information about BHH's dynamic characteristics. The springs with "dynamic stiffness" $N_1$ and $N_2$ represent the transversal and angular effects of the reciprocity between left and right halves of BHH, Fig. 1.

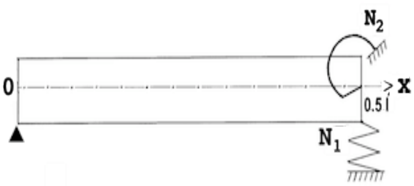


**Fig. 1:** BHH with Two Reciprocal Springs in the Middle of the Beam.

### 2.1. The boundary conditions in the middle of model under consideration for symmetrical modes, excited by transversal transient forces, are the following (4, 5, 7):

$$N_1X = EId^3X(s)dx^3 \text{ or } X(s) = 2EIN_1^{-1}k^3[C_3R_3(s) - C_4R_1(s)] \quad (8)$$

$$dX(s)/dx = 2k[C_3R_1(s) - C_4R_3(s)] = \beta = 0 \quad (9)$$

Here, s = 0.5 kl.
The solutions (4) and (9) with end condition (8) give two following equations:

$$C_3[R_2(s) + \alpha R_3(s)] - C_4[R_4(s) + \alpha R_1(s)] = 0$$

$$C_3R_1(s) - C_4R_3(s) = 0$$

Here $\alpha = EIk^3N_1^{-1}$, $N_1$ − transversal dynamic stiffness and $\beta = EIkN_2^{-1} = 0$.
The constants $C_3$ and $C_4$ are different from zero only if the determinant of these two equations equals zero. Thus, the frequency equation is

$$[R_1(s)R_4(s) - R_2(s)R_3(s)]_* [R_3(s)^2 - R_1(s)^2]^{-1} = \alpha$$

For s = 0.5kl = 3.927, $R_1(s)R_4(s) - R_2(s)R_3(s) = 0$, α = 0 and X(s) = dX(s)/dx = 0.
For s = 1.5708, $R_3(s)^2 - R_1(s)^2 = 0$, α = ∞.

Thus, the limited domain of continuous eigenvalues for the symmetrical modes of BHH's with α = 0 and β = 0 is [0, 7.854] and first natural, resonant eigenvalue kl = 3.14516.

### 2.2. The boundary conditions at the middle of model under consideration for asymmetric modes, excited by transient angular moments, are the following [solutions (4), (5), (6)]:

$$N_2dX(s)/dx = EId^2X(s)/dx^2, \; dX/dx = 2k\beta[C_3R_4(s) - C_4R_2(s)] \quad (10)$$

and $X(s) = 2[C_3R_2(s) - C_4R_4(s)] = 0$

Here, s = 0.5kl, $\beta = EIkN_2^{-1}$, α = 0 and $N_2$ − angular frequency stiffness.
The solutions (4) and (5) with boundary condition (10) give 2 following equations:

$$C_3[R_1(s) - \beta R_4(s)] - C_4[R_3(s) - \beta R_2(s)] = 0$$

$$C_3R_2(s) - C_4R_4(s) = 0$$

The constants $C_3$ and $C_4$ are different from zero, only if the determinant of these two equations equals zero. Thus, the frequency equation is

$$[R_2(s)R_3(s) - R_1(s)R_4(s)]_* [R_2(s)^2 - R_4(s)^2]^{-1} = \beta$$

For s = 0.5kl = 3.927, $R_2(s)R_3(s) - R_1(s)R_4(s) = 0$, β = 0 and X(s) = dX(s)/dx = 0.

For s = 3.1415, $R_2(s)^2 - R_4(s)^2 = 0$, β = ∞.

The limited domain of continuous eigenvalues for the asymmetrical modes of BHH's with α = 0 and β = 0 is [0, 7.854]. Second natural, resonant eigenvalue kl = 6.283.

## 3. Beam with clamped - free ends (BCF)

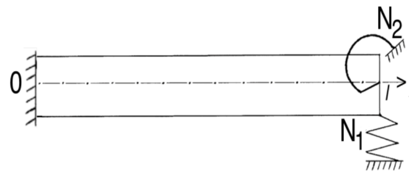


**Fig. 2:** BCF with Two Reciprocal Springs at the Free End.

For beams with clamped – free ends the conditions at clamped end (x = 0) are the following:

X (0) = 0 and dX/dx = 0. To satisfy these conditions $C_1 = C_3 = 0$ and Eq. (2) is:

$$X = C_2(\cos s - \cosh s) + C_4(\sin s - \sinh s) = -2[C_2R_3(s) + C_4R_4(s)] \quad (11)$$

The successive derivatives with respect to x:

$$dX/dx = -C_2k(\sin s + \sinh s) + C_4k(\cos s - \cosh s) = -2k[C_2R_2(s) + C_4R_3(s)]; \quad (12)$$

$$d^2X/dx^2 = -C_2k^2(\cos s + \cosh s) - C_4k^2(\sin s + \sinh s) = -2k^2[C_2R_1(s) + C_4R_2(s)]; \quad (13)$$

$$d^3X/dx^3 = -C_2k(\sinh s - \sin s) - C_4k^3(\cos s + \cosh s) = -2k^3[C_2R_4(s) + C_4R_1(s)]. \quad (14)$$

**3.1. The boundary conditions at x = l of the model under consideration for the modes, excited by transversal transient forces, are the following (7):**

$N_1X = EId^3X(s)/dx^3$ or $X(s) = -2EIN_1^{-1}k^3[C_2R_4(s) + C_4R_1(s)] = -2\alpha[C_2R_4(s) + C_4R_1(s)]$ (15)

and $d^2X(s)/dx^2 = -2k^2[C_2R_1(s) + C_4R_2(s)] = 0$. Here, $s = kl$ and $\alpha = EIk^3N_1^{-1}$.

The solution (11) with first condition (15) and $d^2X(s)/dx^2 = 0$ gives 2 equations:

$C_2[R_3(s) - \alpha R_4(s)] + C_4[R_4(s) - \alpha R_1(s)] = 0$

$C_2R_1(s) + C_4R_2(s) = 0$

The constants $C_2$ and $C_4$ are different from zero, only if the determinant of these two equations equals zero. The frequency equation is

$[R_2(s)R_3(s) - R_1(s)R_4(s)]*[R_2(s)R_4(s) - R_1(s)^2)]^{-1} = \alpha$.

For $s = kl = 3.927$, $R_1(s)R_4(s) - R_2(s)R_3(s) = 0$, $\alpha = 0$ and $X(s) = dX(s)/dx = 0$.

For $s = kl = 1.875$, $R_2(s)R_4(s) - R_1(s)^2 = 0$, $\alpha = \infty$.

The limited domain of continuous eigenvalues for the modes of BCF, excited by transient, transversal forces for $\alpha = 0$ and $d^2X/dx^2 = 0$ is [0, 3.927]. Natural, resonant eigenvalue is kl = 1.875.

**3.2. The boundary conditions at x = l of model under consideration for modes, excited by transient angular moments, are the following:**

$N_2dX/dx = EId^2X(s)/dx^2$, or $dX(s)/dx = -2k\beta[C_2R_1(s) + C_4R_2(s)]$, $\beta = EIkN_2^{-1}$.

and $d^3X(s)/dx^3 = -2k^3[C_2R_4(s) + C_4R_1(s)] = 0$.

Eq. (13) with first condition and $X(s) = 0$ gives two equations:

$C_2[R_2(s)\ \beta R_1(s)] + C_4[R_3(s) - \beta R_2(s)] = 0$ and $C_2R_4(s) + C_4R_1(s) = 0$

The constants $C_2$ and $C_4$ are different from zero, only if the determinant of these two equations equals zero. The frequency equation is:

$[R_1(s)R_2(s) - R_3(s)R_4(s)]*[R_1(s)^2 - R_2(s)R_4(s)]^{-1} = \beta$.

For $s = kl = 4.73$, $R_1(s)R_2(s) - R_3(s)R_4(s) = 0$, $\beta = 0$ and $X(s) = dX(s)/dx = 0$.

For $s = kl = 3.927$, $R_1(s)^2 - R_2(s)R_4(s) = 0$, $\beta = \infty$.

The limited domain of continuous eigenvalues for the modes of BCF, excited by angular transient moments, for $\beta = 0$ and $d^3X(s)/dx^3 = 0$ is [0, 4.73] with natural, resonant eigenvalue kl = 3.927.

## 4. Beams with clamped ends (BCC) and clamped - hinged ends (BCH)

**4.1. The solutions (11), (12) and (14) for BCF are used for analysis of eigenvalues of symmetrical modes of BCC, Fig. 2. The boundary conditions at x = 0. 5l are**

$N_1X = EId^3(s))X/dx^3$, or $X = -2EIN_1^{-1}k^3[C_2R_4(s) + C_4R_1(s)] = -2\alpha[C_2R_4(s) + C_4R_1(s)]$

and $dX(s)/dx = -2k[C_2R_2(s) + C_4R_3(s)] = \beta = 0$.

The frequency equation is $[R_3(s)^2 - R_2(s)R_4(s)]*[R_3(s)R_4(s) - R_1(s)R_2(s)]^{-1} = \alpha$.

For $s = 0.5kl = 4.73$, $R_3(s)^2 - R_2(s)R_4(s) = 0$, $\alpha = 0$ and $X(s) = dX(s)/dx = 0$.

For $s = 2.365$, $R_3(s)R_4(s) - R_1(s)R_2(s) = 0$, $\alpha = \infty$.

The limited domain of continuous eigenvalues for the symmetrical modes of BCC with $\alpha = 0$ and $dX(s)/dx = 0$ is [0, 9.46] with natural, resonant eigenvalue kl = 4, 73.

**4.2. The solutions (11), (12) and (13) for BCF are used for analysis of eigenvalues of asymmetric modes of BCC. The boundary conditions at x = 0. 5l are**

$N_2dX(s)/dx = EId^2X(s)/dx^2$ or $dX/dx = -2k\beta[C_2R_1(s) + C_4R_2(s)]$, $\beta = EIkN_2^{-1}$.

and $X(s) = -2[C_2R_3(s) + C_4R_4(s)] = 0$.

The frequency equation is: $[R_2(s)R_4(s)^2 - R_3(s)^2]*[R_1(s)R_4(s) - R_2(s)R_3(s)]^{-1} = \beta$.

For $s = 0.5kl = 4.73$, $R_2(s)R_4(s)^2 - R_3(s)^2 = 0$, $\beta = 0$ and $X(s) = dX(s)/dx = 0$.

For $s = 3.927$, $R_1(s)R_4(s) - R_2(s)R_3(s) = 0$, $\beta = \infty$.

The limited domain of continuous eigenvalues for asymmetric modes of BCC with $\alpha = 0$ and $\beta = 0$ is [0, 9.46] with natural, resonant eigenvalue kl = 7.853.

**4.3. The dynamic characteristics of BCH are not considered since the location of transient forces and moments is uncertain, however, due to the resemblance with BCC, it is reasonable to assume that for almost symmetrical modes [0, 7.854] with first natural eigenvalue kl = 3.927 and for almost asymmetric modes [0, 7.854] with first natural, resonant eigenvalue kl = 7.069**

## 5. Beams with free - hinged ends (BFH) and with free ends (BFF)

The discrete, (classical) dynamic characteristics of BFH and BFF were recently revised and presented in [7]. The investigation was initiated and fulfilled owing to inconsistency of the lowermost eigenvalues and number of nodal points, obtained by means of classical theory, with Rayleigh theorem [8], theorems for nodal points [9,10], Sturm-Liouville theory [11] and physical reality. The limited domains for BFH and BFF with continuous eigenvalues and modes of vibration, excited by transient forces or moments, are defined below.

**5.1. BFH with free left end and transversal N 1 and angular N 2 "dynamic stiffness" on the right end is shown conditionally in horizontal position in Fig. 3.**

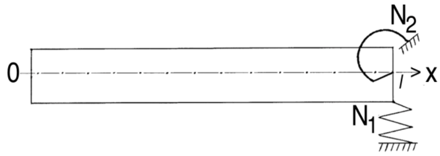


**Fig. 3:** Two Reciprocal Springs at the Right End of A Beam.

On the right end $\alpha = EIk^3N_1^{-1} = 0$ and $\beta = EIkN_2^{-1}$. With $\beta = \infty$ BFH is motionless or sways as rigid pendulum around the hinge. The revised dynamic characteristics of BFH reveals the missing, discrete eigenvalue $kl = 0.5\pi$ within range $0 < \beta < \infty$ [7]. The natural mode of vibration with $kl = 0.5\pi$ can be excited by initial, static displacement applied to free end. With $\beta = 0$ BFH vibrates as BCF and can turn around the hinge. So, the transient moment, applied around the hinge, can excite vibration in limited domain with continuous eigenvalues coinciding with the limited domain of BCF, i.e. [0, 4.73].

### 5.2. The lowermost discrete modes of vibration of BFF are shown in Fig. 4

These three missing, lowermost modes and eigenvalues were revealed in [7]. Two first natural modes can be excited by static angular (a) or transversal (b) displacements.

On the middle of BFF $\alpha = EIk^3N_1^{-1} = 0$ and $\beta = EIkN_2^{-1}$.

**a** **b** **c**

**Fig. 4:** a – Natural Asymmetric Mode with $kl = \pi$, b – Natural, Symmetrical Mode with $Kl = 3.75$, c – Resonant, Asymmetric Mode with $kl = 3.75$.

The limited domain of eigenvalues for BFF can be determent based on the resemblance of the geometrical and dynamic characteristics of each half of BFF with BFH. As corollary, the limited domains of continuous eigenvalues created by transient forces or moments for BFF symmetrical modes is [0, 7.854] and for asymmetric modes [0, 9.46].

## 6. Conclusion

The determined limited domains of continuous eigenvalues for all uniform beams with various boundary conditions are given in Table 1.

**Table 1:** Limited Domains of Eigenvalues

| | BFH | BFF | BHH | BCF | BCH | BCC |
|---|---|---|---|---|---|---|
| 1. | - | [0, 7.854]* | [0, 7.854]* | [0, 3.927] | [0, 7.854]* | [0, 9.46]* |
| 2. | [0, 4.73]** | [0, 9.46] ** | [0, 7.854]** | [0, 4.73] | [0, 7.854]** | [0, 9.46]** |

* Symmetrical modes, ** asymmetric modes.

The essential peculiarity of analysis of beams dynamic characteristics in present investigation is taking into account the physical properties in boundary conditions, created by transient forces or moments, instead of the commonly used kinematic. This method is more informative and is useful for determination the limited domains of continuous eigenvalues, which are valid for the free vibrations as well as for forced vibrations. In contrast to the classical theory the solutions revealed by the advanced theory are not the solutions of the Sturm – Liouville problem, although they include few, natural, discrete eigenvalues and normal modes of classical theory. The obtained results reveal that the magnitudes of continuous eigenvalues in domains, corresponding to stiffness $N_1 = \infty$ and $N_2 = \infty$, indicate the limit of excitation's frequencies having impact on vibration of beams. The stiffness $N_1 = 0$ and $N_2 = 0$ indicate resonant eigenvalues. The modes of vibrations can be depicted with any eigenvalue within limited domains. Many of the problems faced today by applied mathematicians and engineers involve basic principals of vibration of beams. The obtained results are recommended for engineering applications and for educational purposes.

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