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Haar wavelets approach of traveling wave equation- A plausible solution of lightning stroke model

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Abstract

This paper describes a traveling wave model for describing the lightning stroke by the Haar wavelet method (HWM) is proposed. Numerical example is included and illustrated for applicability and validity of the proposed method. The fundamental idea of Haar wavelet method is to convert the differential equations into a group of algebraic equations that involves a finite number of variables. The power of the manageable method is confirmed. The results show that the proposed way is quite reasonable when compared to exact solution. Moreover the use of Haar wavelets is found to be accurate, simple, fast, flexible, convenient, small computation costs and computationally attractive.

Keywords: *Haar wavelet method, lightning stroke model, traveling wave equation, telegraphist's equation*

1 Introduction

In recent years, any lightning based mathematical model is designed to reproduce certain aspects of the physical processes involved in the lightning discharge. The basic assumptions of the model should be consistent with both the expected outputs of the model and the availability of quantities required as inputs to the model.

Knowledge of the characteristics of electric and magnetic fields produced by lightning discharges is needed for studying the effects of the potentially deleterious coupling of lightning fields to various circuits and systems. Sensitive electronic circuits are particularly vulnerable to such effects. The computation of lightning electric and magnetic fields requires the use of a model that specifies current as a function of time at all points along the radiating lightning channel. The computed fields can be used as an input to electromagnetic coupling models, the latter, in turn, being used for the calculation of lightning induced voltages and currents in various circuits and systems.

Lightning is a momentary, atmospheric, transient, high current electrical discharge whose path length is measured in kilometers from clouds to earth. It carries high voltage current via a huge arc to ground. For many years, a lot of studies have been made on this subject. Some have been concerned simply with collecting statistics, some with making measurements, and others have tried to probe more deeply into the physical nature of the problems. From their efforts we can understand more about this mysterious process. It is now generally accepted that a typical lightning stroke begins with the propagation of a negatively charged channel, called a stepped leader, from cloud to the ground. But before this downward leader reaches the ground, an upward leader begins to proceed from the ground and meets the downward-moving leader at the junction point. Once a stepped leader has established a connection to earth, the so-called return stoke moves swiftly up the ionized channel prepared by the stepped leader like a traveling wave on a high-voltage transmission line and a heavy current occurs. However, the physical models derived from the experimental data or from the information determined directly from experimental data have often been obtained more on the basis of intuition than on the basis of detailed quantitative analysis. They are all well documented in the literature [1, 11].

The weather mapping system is used to detect lightning discharges and display them graphically to the pilot. Lightning produces an electromagnetic (EM) field by stripping electrons from atoms in the air. This process emits a broad spectrum of electromagnetic energy as well as a great deal of light and sound. The process starts with transient collisions of ice crystals with riming graupel pellets thus transferring charges within the maturing cloud as the heavier (more negative) particles fall and resulting in a vertical electric field. The net effect of this self-propagating lightning is the transfer of a negative charge from the atmosphere to the earth (Cloud to Ground). When the stepped leader hits the ground, the return stroke is triggered, producing a sharp voltage rise. This specific signature distinguishes a cloud-to-ground stroke from other electromagnetic noise.

The lightning return stroke is a shock of hot air produced by the electric discharge of a lightning event when it touches the ground. It involves a complex mix of compressible hydrodynamics and electrodynamics, chemistry, plasma physics, and radioactive transfer that is not widely understood (Lowke 2004; Cooray 2003).
In recent years, numerical simulation plays an important role for theoretical studies of electromagnetic problems because of the complex nature of the electromagnetic waves.
Wavelet theory is a relatively new and an emerging area in mathematical research. It has been applied to a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis of waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms. Haar wavelets have become an efficient tool in solving the differential and integral equations. The Haar wavelet method exhibits several advantageous features:
(i) High accuracy is obtained already for a small number of grid points.
(ii) Possibility of implementation of standard algorithms. For calculation the integrals of the wavelet functions, universal subprograms can be put together. Another time consuming operation is the solving of high-order systems of linear equations and calculating high-order determinants; here the matrix programs of MATLAB are very effective.
(iii) The method is very convenient for solving boundary value problems since the boundary conditions are taken care of automatically.
(iv) Singularities can be treated as intermediate boundary conditions; this circumstance to a great extent simplifies the solution.
(v) The obtained solutions are mostly simpler compared with other known methods.
In this paper a traveling-wave model is used to describe the lightning stroke behavior [4], and then the Haar wavelet method is employed to solve it. Hariharan et al. [14,15,19] introduced the Haar wavelets approach for solving some differential equations. Lepik [10,16] established Haar wavelet method for solving Burgers' equations and Poisson equations.
The paper is organized the following way. A traveling-wave model for lightning stroke behavior is presented in section 2. For completeness sake the Haar wavelet method is presented in section 3. Function approximation is presented in section 4. The method of solution the PDE is proposed in section 5. Some conclusions are drawn in section 6.

2 Mathematical model of traveling wave phenomena

Basically, lightning stroke is an electromagnetic wave and can be determined by the use of traveling wave equations [4]. Suppose the discharge channel within a thundercloud is fully developed, and then it will behave like the spark discharge between two flat plates forming a condenser that is in effect shorted out by a central conductor. Hence, a useful concept is to think of the cloud and earth as forming a vast capacitor, which is being discharged by the stroke [4,12]. Propagation of electromagnetic wave along the path can be treated as a circuit problem [4], and voltage $V(x,t)$ and current $i(x,t)$ satisfy the well-known telegraphist's equations

$$-\frac{\partial V(x,t)}{\partial x} = L\frac{\partial i(x,t)}{\partial t} + Ri(x,t) \tag{1}$$

$$-\frac{\partial i(x,t)}{\partial x} = C\frac{\partial V(x,t)}{\partial t} + GV(x,t) \tag{2}$$

Where R, G, L and C represent the unit per length resistance, conductance, inductance and capacitance of the path respectively.
Assume that the energy loss due to the conductance term is small compared to the other equations, and Eq.(1) and Eq.(2) are combined, then both voltage wave and current wave satisfy the following hyperbolic equation

$$\frac{\partial^2 \Phi(x,t)}{\partial x^2} = RC\frac{\partial \Phi(x,t)}{\partial t} + LC\frac{\partial^2 \Phi(x,t)}{\partial t^2} \tag{3}$$

Fig.1 shows the distributed model for power transmission line wave propagation for the above mathematical model [20]. We can see that the first term on the right-hand side represents the energy dissipation associated with the wave propagation process, and the second term represents the inertia of the time-dependent motion. For the problem statement to be complete, the boundary and initial conditions need to be specified. Let us consider the voltage wave equation of Eq.(3). The voltage at the ground is assumed to be zero while the current at the cloud is assumed to be a small constant value, which is taken to be zero [4]. Hence, the boundary conditions are

$$V(0,t) = 0 \tag{4}$$

$$i(h,t) = 0 \tag{5}$$

At $t = 0$, the voltage distribution is assumed to be a known constant value except at the ground where it is zero. Hence the initial conditions are

$$V\left(x,0\right)=V_0 \tag{6}$$

$$\frac{\partial i\left(x,0\right)}{\partial t}=0 \tag{7}$$

Once the voltage distribution has been solved, the solution of current involves the integration of Eq.(2) using Eq.(5) as an initial condition.

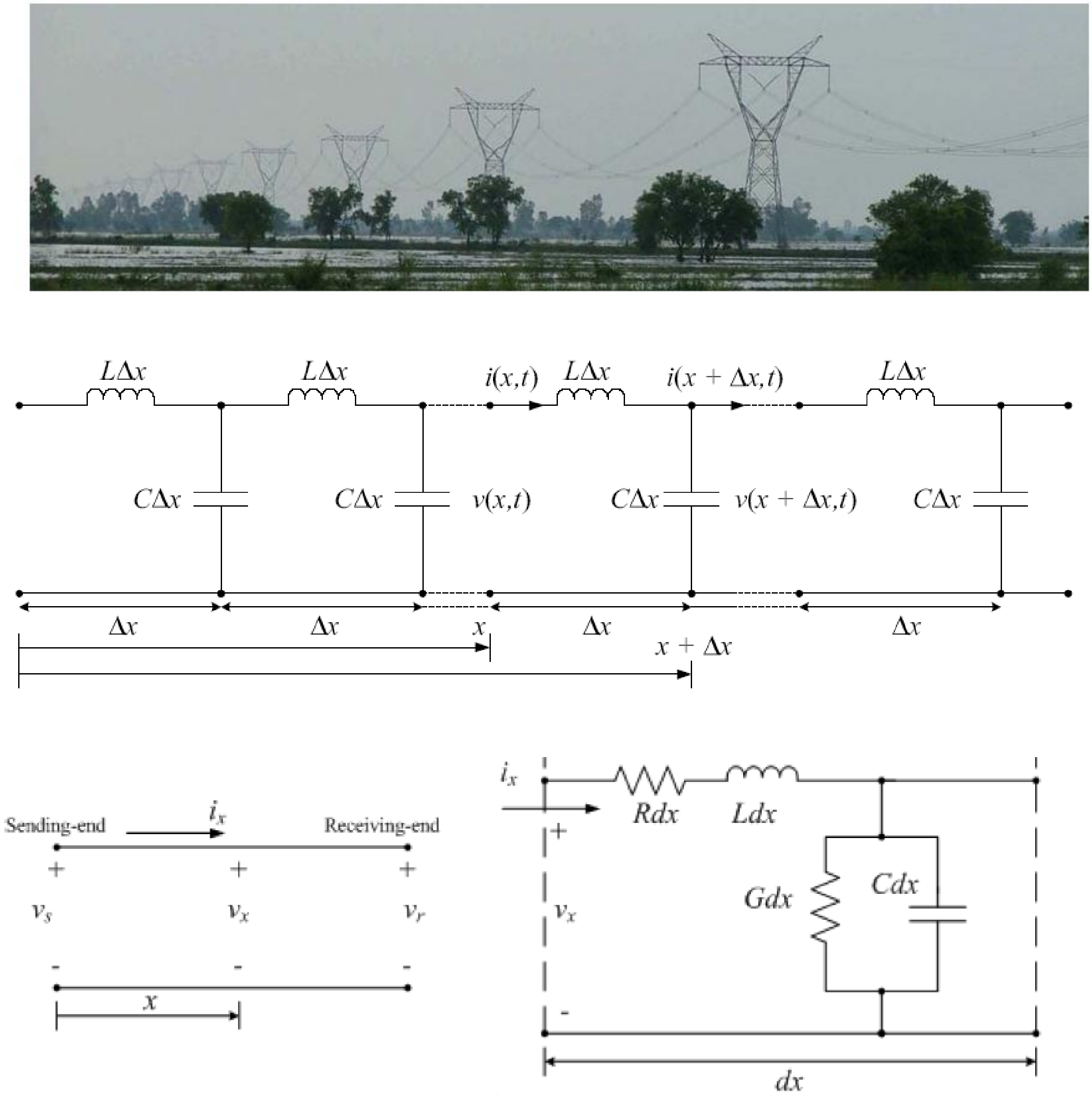

Fig.1: Distributed line model for power transmission line wave propagation

3 Haar wavelets preliminaries

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar [5]. The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform [6,10,17,18]. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.
Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.
The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [3], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [7], who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang [8] proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix. The orthogonal set of Haar function $h_i(t)$ is shown in Fig.2. This is a group of square waves with magnitudes of ±1 in certain intervals and zeros elsewhere.
For applications of the Haar transform in logic design, efficient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed.
The Haar wavelet family for $t \in [0,1]$ is defined as follows.

$$h_i(t) = \begin{cases} 1, & for \ \ t \in \left[\dfrac{k}{m}, \dfrac{k+0.5}{m} \right) \\ -1, & for \ \ t \in \left[\dfrac{k+0.5}{m}, \dfrac{k+1}{m} \right) \\ 0, & elsewhere \end{cases} \tag{8}$$

Integer $m = 2^j \ (j = 0,1,2...J)$ indicates the level of the wavelet; $k = 0,1,2,.....m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$; in the case of minimal values. $m = 1, k = 0$ we have $i = 2$, the maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1 \ in \ [0,1]$. Let us define the collocation points $t_l = (l - 0.5) / 2M, \ \ (l = 1,2....2M)$ and discretise the Haar function $h_i(t)$; in this way we get the coefficient matrix $H(i,l) = (h_i(t_l))$, which has the dimension $2M \times 2M$.
The operational matrix of integration P, which is a 2M square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t)\,dt \tag{9}$$

The elements of the matrices H and P can be evaluated according to Eq.(8) and Eq.(9).

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$H_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

$$P_8 = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [3] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m}\begin{pmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & 0 \end{pmatrix}$$

where 0 is a null matrix of order $\frac{m}{2} x \frac{m}{2}$,

$$H_{mXm} \triangleq \left[h_m(t_0) \quad h_m(t_1) \quad ------ h_m(t_{m-1}) \right] \tag{10}$$

and $\frac{i}{m} \le t < i + \frac{1}{m}$ and $H_{mxm}^{-1} = \frac{1}{m} H_{mxm}^{T}\ diag(r)$

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations. Since H and H^{-1} contain many zeros, this phenomenon makes the Haar transform must faster than the Fourier transform, and it is even faster than the Walsh transform. This is one of the reasons for rapid convergence of the Haar wavelet series.

3.1 Function approximation

Any function $y(x) \in L^2[0,1)$ can be decomposed as

$$y(x) = \sum_{n=0}^{\infty} c_n h_n(x) \tag{11}$$

where the coefficients c_n are determined by

$$c_n = 2^j \int_0^1 y(x) h_n(x) dx \tag{12}$$

Where $n = 2^j + k$, $j \ge 0$, $0 \le k < 2^j$. Specially $c_0 = \int_0^1 y(x) dx$.

The series expansion of $y(x)$ contains an infinite terms. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x) = \sum_{n=0}^{m-1} c_n h_n(x) = c_{(m)}^T h_{(m)}(x) \tag{13}$$

Where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as

$c_{(m)}^T = [c_0, c_1, \ldots, c_{m-1}]$

and

$h_{(m)}(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T$ where 'T' means transpose and $m = 2^j$.

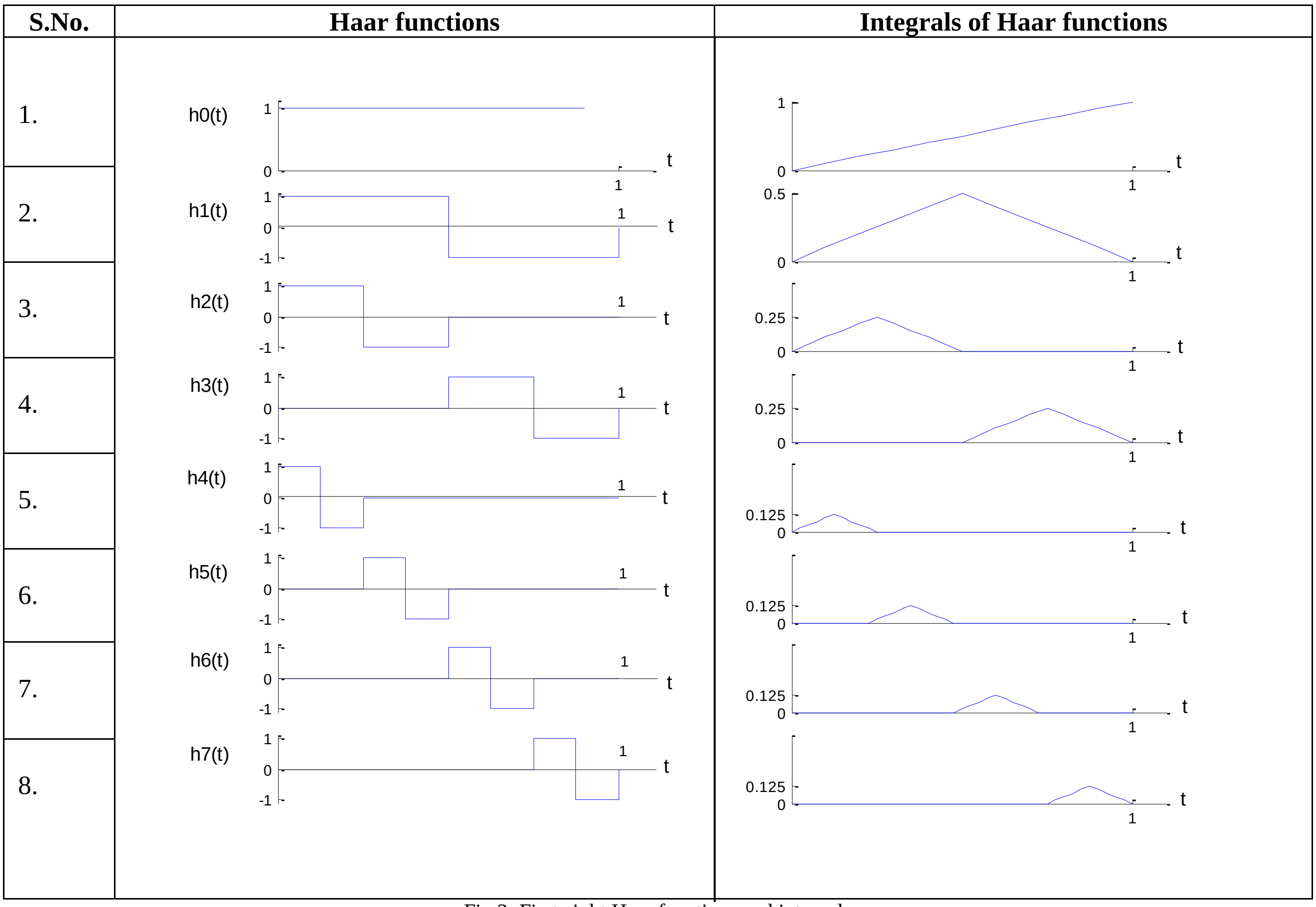

Fig.2: First eight Haar functions and integrals

3.2 Method of solution

The telegraphist's equations Eq.(1) and Eq.(2) satisfy the hyperbolic equations

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (RC + LG)\frac{\partial v}{\partial t} + RGv \tag{14}$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + LG)\frac{\partial i}{\partial t} + RGi \tag{15}$$

In the Haar domain, we assume that $\partial^2 v/\partial t^2$ can be expanded as a Haar series as

$$\frac{\partial^2 v}{\partial t^2} = a^t(x)H(t) \tag{16}$$

Where $a(x)$ is an m -vector function of x , x denotes the space distance. The x in $a(x)$ will be dropped to simplify the notation

$$\frac{\partial v}{\partial t} = \int^t \frac{\partial^2 v}{\partial t^2}dt + \frac{\partial v(x,0)}{\partial t} = a^t PH(t) \tag{17}$$

$$v(x,t) = \int^t \frac{\partial v}{\partial t} + v(x,0) = a^t P^2 H(t) \tag{18}$$

Where $\partial v(x,0)/\partial t$ and $v(x,0)$ have been set to zero for the initially relaxed system. Equations (16),(17) and (18) can be applied to Eq.(14)

$$\ddot{a}^t - a^t M^2 = 0 \tag{19}$$

$$\ddot{a} \triangleq d^2 a/dx^2 \tag{20}$$

$$M \triangleq \left[LCI + \left(RC + LG \right) P + RGP^2 \right]^{\frac{1}{2}} P^{-1} \tag{21}$$

Solving Eq.(19), we have

$$a^t = a_1^t \exp\left(-Mx\right) + a_2^t \exp\left(Mx\right) \tag{22}$$

where a_1 and a_2 are two constant vectors to satisfy the boundary conditions. The voltage $v\left(x,t\right)$ must be finite when $x \to \infty$. It is necessary that $a_2 = 0$,

$$v\left(x,t\right) = a_1^t \exp\left(-Mx\right) P^2 H\left(t\right) \tag{23}$$

When a unit step voltage is applied at $x = 0$, it means $v\left(0,t\right) = 1$

$$v\left(0,t\right) = a_1^t P^2 H\left(t\right) = [1,1,...,1]$$

$$a_1^t P^2 = [1,0,...,0] \triangleq a_0^t \tag{24}$$

$$a^t = [1,0,...,1] P^{-2} = a_0^t P^{-2} \tag{25}$$

Applying the Haar transform to eqn. (1) at the driving point, namely $x = 0$

$$di\left(0,t\right)/dt = b^t H\left(t\right) \tag{26}$$

$$i\left(0,t\right) = \int^t \frac{di\left(0,t\right)}{dt} dt + i\left(0,0\right) = b^t P H\left(t\right) \tag{27}$$

Substituting equations (23),(26) and (27) into Eq.(1) with $x = 0$, we have

$$b^t [RP + LI] = a_1^t M P^2$$

$$b^t = a_1^t M \left(RP + LI\right)^{-1} P^2$$

$$= \sqrt{C/L} a_1^t \left[\left(G/C\right) P + I \right]^{\frac{1}{2}} \left[\left(R/L\right) P + I \right]^{-\frac{1}{2}} P, \tag{28}$$

where the commutative property has been applied.

Finally, we have

$$i\left(0,t\right) = \sqrt{C/L} a_0^t \left[\left(G/C\right) P + I \right]^{\frac{1}{2}} \left[\left(R/L\right) P + I \right]^{-\frac{1}{2}} H\left(t\right) \tag{29}$$

Where a_0 is defined by Eq.(24).

The analytic solution can be found in [13], or

$$i\left(0,t\right) = \sqrt{C/L} \left[e^{-\alpha t} I_0\left(\beta t\right) + \frac{G}{C} \int^t e^{-\alpha t} I_0\left(\beta t\right) dt \right], \tag{30}$$

where $\alpha = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right), \quad \beta = \frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)$

Here I_0 is the modified Bessel function of the first kind of the zeroth order.

4 Main results

To illustrate and validate the method described above, results of an example study are presented. Suppose we represent the cloud and earth as a capacitor having parallel plates of circular shape with a radius of 1 km, and let discharge channel be 20cm in diameter. Three heights of cloud are considered. Table 1 shows the inductances and capacitances for each case [9]. If the resistance of the channel were greater than Z, then it would damp the discharge [4]. It is seen that the magnitude of voltage waveform is directly proportional to the cloud height. Computer simulation was carried out in the cases $m = 32$ and $m = 64$ (See Table 2.), the computed results were compared with the exact solution and restrictive Taylor's approximation, more accurate results can be obtained by using a larger m.

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel ~1596 Mhz.

Table 1: Values of Inductance and Capacitance

Height *(x)*	*Inductance (L)*	*Capacitance(C)*	$2\sqrt{L/C}\ (\Omega)$
250 m	1.85 mH	27.80 pF	516
500 m	2.02 mH	13.90 pF	762
1000 m	2.18 mH	6.95 pF	1120

Table 2: Absolute error for Eq.(30)

x	$m=16$	$m=32$	$m=64$
0.1	5.092e-003	5.090e-003	6.092e-003
0.2	4.843e-003	4.843e-003	2.843e-003
0.3	2.093e-002	2.271e-002	1.093e-002
0.4	1.937e-002	1.937e-002	4.243e-002
0.5	1.242e-005	1.242e-005	5.272e-005
0.6	6.320e-003	5.562e-003	7.320e-003
0.7	2.345e-006	2.345e-006	6.345e-006
0.8	2.456e-002	6.456e-002	4.456e-002
0.9	4.565e-004	5.565e-004	4.565e-004
1.0	2.345e-002	2.345e-002	1.244e-002

5 Conclusion

The traveling wave equation for modeling the lightning stroke by the Haar wavelet method has been presented. It has been well demonstrated that in applying the nice properties of Haar wavelets, the partial differential equation can be solved conveniently by using Haar wavelet method systematically. The accuracy and effectiveness of the method are analyzed; the results obtained are compared with the results of other authors (using classical numerical techniques) evaluating the error. The benefits of Haar wavelet approach are sparse matrices of representation, fast transformation and designing of fast algorithms and the merits of the method lie in its simplicity, computational economy and easy implementation. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones. For better solutions, instead of increasing the value of m one can use other type of wavelets such Maxican wavelets, Spline wavelets etc. Use of Spline wavelets for solving other type of traveling wave equation is presently ongoing.

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