Location Domination Number of Sum of Graphs

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Abstract

Locating dominating set is the subset $S$ of the vertex set $V(G)$ which dominate and uniquely identify all vertices of the set $V(G) - S$. In this paper we formulated the apt method for finding the location domination number of sum of graphs $G_1 + G_2$ based on the nature of the graphs $G_1$ and $G_2$.

Keywords: 1-vertex non locating dominating set; Dominating set; Locating domination set; Locating set; Sum of graphs

1. Introduction

Dominating set [6] was defined by Oystein Ore in the year 1962 and it is stated as a subset $S$ of the vertex set $V(G)$ such that every vertex of the $G$ is either an element of $S$ or adjacent to the elements of $S$. Cardinality of minimal dominating set is the domination number.

A set $S = \{v_1, v_2, ..., v_n\}$ of vertices in a connected graph $G$ is a locating set if for every pair of distinct vertices $u, w \in V(G)$, 
\[(d(u, v_1), ..., d(u, v_n)) \neq (d(w, v_1), ..., d(w, v_n))\]
where $d(x, y)$ is the distance between $x$ and $y$. The locating number $\beta'(G)$ is the minimum cardinality of a locating set of $G$ [4].

Locating-dominating set (LD-set) was introduced by Slater [11,12] and it is defined as the subset $S$ of the vertex set $V(G)$ which can detect and uniquely identify the position of every vertices of the set $V(G) - S$. Locating dominating set of $G$ is called referencing-dominating set or an RD-set if $G$ has no locating dominating set of smaller cardinality. Cardinality of RD-set is called the location-dominating number of $G$ and it is symbolised by $RD(G)$.

For simple graph $G = (V(G), E(G))$, open neighbourhood $N_o(v)$ of a vertex $v \in V(G)$ is $N_o(v) = \{u \in V(G) \mid u \notin E(G)\}$. And $N(v) = \{u \in V(G) \mid u \notin E(G)\} \cup \{v\}$. is the closed neighbourhood of $v \in V(G)$.

Colbourn et al. [2] proved that Locating-dominating set is NP-complete. Rajasekar et al.[8,9] have found the location domination number of graph connected by a bridge and graphs obtained by fusion of single vertex. Locating dominating set was studied by various authors in [3,4,6,10,11].

In [1] Sergio R. Canoy, Jr. et al. have found location domination number of sum of graphs by using the definition of locating set. But locating set has its own downfall, as it is defined only for connected graphs. To overcome this we define 1-vertex non locating dominating graph, with this we have conveyed a mode for finding location domination number of sum of any graphs.

2. Location Domination Number of Sum of Graphs

Theorem 2.1

Let $G_1$ and $G_2$ be any two graphs, then location domination number of graph $G_1 + G_2$ satisfies the condition
$RD(G_1) + RD(G_2) - 2 \leq RD(G_1 + G_2) \leq RD(G_1) + RD(G_2) + 1$.

Proof

Let us first prove that $RD(G_1 + G_2) \leq RD(G_1) + RD(G_2) + 1$. For that it is enough to show that the graph $G_1 + G_2$ has a LD-set with cardinality $RD(G_1) + RD(G_2) + 1$.

Let $S_1$ and $S_2$ be the RD-set of $G_1$ and $G_2$. Now for the set $S = S_1 \cup S_2$ we have that
$S(u) = \begin{cases} S_1(u) \cup S_1(u), & u \notin V(G_1) - S_1 \\ S_2 \cup S_2(u), & u \notin V(G_2) - S_2 \end{cases}$

If $S$ is an LD-set then there is nothing to prove. If not, then there exist at least one pair of vertices $v_1, v_2 \in V(G_1 + G_2) - S$ such that
$S(v_1) = S(v_2)$

(1)

If $v_1, v_2 \in V(G_1) - S_1$, then from Equation 1 one may conclude that $S_1(v_1) \cup S_2 = S_1(v_2)$

That is $S_1(v_1) = S_1(v_2)$, this implies that $S_1$ is not an RD-set of $G_1$. Hence it contradict the assumption that $S_1$ is a RD-set of $G_1$. Thus both $v_1$ and $v_2$ cannot belongs to $V(G_1) - S_1$ simultaneously. For same reason, both $v_1$ and $v_2$ cannot be simultane-
ously in \(V(G_1) - S_2\). Hence if \(v_i \in V(G_1) - S_1\) then \(v_i\) must be in the set \(V(G_2) - S_1\). Therefore we have \(S_i(v_i) \cup S_i = S_i \cup S_i(v_i)\).

This is possible only if
\[
S_i(v_i) = S_i \text{ and } S_i(u) = S_2
\]  

(2)

Suppose, there exist another pair of vertices \(u, v \in S_i\) such that \(S_i(u) = S_i(v)\). Then by above argument as in case of \(v_i, v_j\), one may come to conclusion that both \(u, v\) does not belong to \(V(G_1) - S_1\) or \(V(G_2) - S_2\) simultaneously.

Hence let us assume that \(u \in V(G_1) - S_1\) and \(u_2 \in V(G_2) - S_2\). Thus \(S_i(u) = S_2 \cup S_i(u_2)\), this implies that
\[
S_i(u) = S_2 \text{ and } S_i(u_2) = S_2
\]  

(3)

From Equation 2 and 3 we have that \(S_i(v_i) = S_i(u) = S_i\) and \(S_i(v_j) = S_i(u_j) = S_j\). That is there exist \(u, v \in V(G_1) - S_1\) such that \(S_i(u) = S_i(v)\), but this contradict the assumption that \(S_i\) is an RD-set of \(G_1\). Similarly it also contradict the assumption that \(S_2\) is a RD-set of \(G_2\).

Hence with respect to the set \(S = S_1 \cup S_2\) there may exist only one pair of vertex \(v_i, v_j \in V(G_1 + G_2) - S_1\) such that \(S_i(v) = S_i(v_j)\).

Thus except \(v_i, v_j\), all vertices are uniquely located and dominated by the set \(S\). Hence \(S \cup \{v_i\}\) and \(S \cup \{v_j\}\) must be the LD-set of the graph \(G_1 + G_2\). Therefore
\[
RD(G_1 + G_2) \leq |S| \oplus \{v\}
\]

(4)

Now let us prove that the lower bound of \(RD(G_1 + G_2)\) is \(RD(G_1) + RD(G_2) - 2\). Suppose there exist an RD-set \(S\) of \(G_1 + G_2\) such that
\[
|S| \leq RD(G_1) + RD(G_2) - 3
\]  

(5)

Clearly \(S = S \cap \{V(G_1) \cup V(G_2)\} = \{S \cap V(G_1)\} \cup \{S \cap V(G_2)\}\)

and \(|S| = |S \cap V(G_1)| + |S \cap V(G_2)| - |S \cap V(G_1) \cap S \cap V(G_2)|.\)

As \(V(G_1)\) and \(V(G_2)\) don’t have any vertex in common we have that
\[
|S| = |S \cap V(G_1)| + |S \cap V(G_2)|
\]  

(5)

From Equation 4 and 5 we have
\[
|S| = |S \cap V(G_1)| + |S \cap V(G_2)| \leq RD(G_1) + RD(G_2) - 3
\]  

This is possible only if either \(|S \cap V(G_1)| \leq RD(G_1) - 2\) or \(|S \cap V(G_2)| \leq RD(G_2) - 2\). Without loss of generality let us assume that \(|S \cap V(G_1)| \leq RD(G_1) - 2\). In the graph \(G\), \(S \cap V(G_1)\) dominate all the vertices of \(V(G_1) - S\) and it can atmost uniquely locate only one vertex of \(V(G_1) - S\), say \(u\), provided \(S \cap V(G_1)\) uniquely locate remaining all vertices of \(V(G_1) - \{u\}\). Thus \(S \cap V(G_1)\) is a LD-set of \(V(G_1) - \{u\}\), so \((S \cap V(G_1)) \cup |u|\) is a LD-set of \(V(G_1)\) with cardinality less than or equal to \(RD(G_1) - 1\). This contradicts the definition of RD-set. Hence \(|S \cap V(G_1)| \geq RD(G_1) - 1\). Similarly \(|S \cap V(G_2)| \geq RD(G_2) - 1\). Therefore
\[
RD(G_1) + RD(G_2) - 2 \leq |S \cap V(G_1)| + |S \cap V(G_2)|
\]

(4)

\[
= |S| = RD(G_1) + RD(G_2)
\]

Theorem 2.2

Let \(G\) be any graph with RD-set \(S\). If for every RD-set of \(G\), there exist a vertex \(v \in V(G) - S\) such that \(S(v) = S\), then \(RD(G + K_1) = RD(G) + 1\) otherwise \(RD(G + K_1) = RD(G)\).

Proof

Let \(V(K_1) = \{v_i\}\). By Theorem 2.1, we have
\[
RD(G) + RD(K_1) \leq 2 \leq RD(G + K_1)
\]

Suppose that \(G + K_1\) has an RD-set \(S\) with cardinality \(RD(G) - 1\), then \(v_i\) must belongs to \(S_1\). Otherwise \(G\) would have a RD-set with cardinality less than \(RD(G)\). But in the graph \(G + K_1\), \(v_i\) can uniquely locate only one vertex of \(V(G + K_1) - S_1\) (say \(u\)). Therefore the set \(\{S \cup \{u\}\} \cap \{v_i\}\) must be the RD-set of \(G + K_1\). So we have constructed a RD-set which doesn’t include the vertex \(v_i\) with cardinality \(RD(G) - 1\). But it must also be the RD-set of \(G\). This contradict the definition of RD-set. So \(RD(G + K_1) \geq RD(G)\).

Case 1: Let \(S(v) \neq S\) for all \(v \in V(G) - S\).

By definition of \(G + K_1\), \(v_i\) is adjacent to all vertices of \(G\). Hence \(v_i\) is dominated by \(S\) and \(S(v_i) = S \cup \{u\}\) for all \(u \in V(G) - S\), so \(S\) is a LD-set of \(G + K_1\). Since the set with the cardinality less than \(RD(G)\) would not be a LD-set, \(S\) must be the RD-set of \(G + K_1\). Hence \(RD(G + K_1) = RD(G)\).

Case 2: Let for all RD-set \(S\) of graph \(G\), \(S(v) = S\) for some \(v \in V(G) - S\).

Assume that \(G + K_1\) has an RD-set \(S\) such that \(|S| = RD(G)\). If \(v_i \notin S\) then by our assumption there must exist a vertex \(v \in V(G) - S\) such that \(S(v) = S\). As \(v_i\) is adjacent to all vertices of \(G\), \(S(v_i) = S = S(v)\), so \(v_i\) must belongs \(S_1\).

Now consider the set \(S_2 = S_1 - \{v_i\}\), its’ cardinality is \(RD(G) - 1\). So it would not be the LD-set of \(G\). As \(S_1\) is the RD-set of \(G + K_1\) and the vertex \(v_i\) can atmost uniquely locate only one vertex \(w\) of \(V(G) - S_1\), we must have that \(S_2 = S_1 - \{v_i\}\) as an RD-set of \(G - \{w\}\) and \(N_1(w) \cap S_2 = \Phi\).

Now consider the set \(S = S_2 \cup \{w\}\), which is clearly the RD-set of \(G\). But by our assumption there exist some vertex \(v \in V(G) - S_2\) such that \(S_2(v) = S_1\). That is \(S_2(v) = S_2 \cup \{w\} = S\).

By combining the fact that \(N(w) \cap S_2 = \Phi\), \(v \in N(w)\) and \(S_2(v) \neq S_2\) for all \(y\) other than \(v\), we have that the set \(S_2 \cup \{v\}\) is an RD-set of \(G\). But it has no vertex which is dominated by
all the vertices of $S_1\cup\{v_1\}$. This contradiction shows that $G + K_r$

 cannot have any $LD$-set with cardinality $RD(G)$.

And so $S_1\cup\{v_1\}$ is the $RD$-set of $G + K_r$ with cardinality $RD(G) + K_r$.

**Remark 2.1**

If $G$ is disconnected then $S(v) \neq S$ for any $RD$-set of $G$.

Hence $RD(G + K_r) = RD(G)$.

**Remark 2.2**

Consider the graph $K_n$ where $n \geq 2$. Clearly $K_n$ is a disconnected graph and by Remark 2.1, $RD(K_n + K_r) = RD(K_n) = n$.

**Definition 2.1**

For the given graph $G$, a set $S\subseteq V(G)$ is said to be 1-vertex non locating dominating set if it satisfies the following conditions:

(i) With respect to some vertex $v \in V(G)$, $S_m$ must be the $RD$-set of $G - \{v\}$

(ii) $N_G(v) \cap S_m = \emptyset$

(iii) $|S_m| \leq RD(G)$

(iv) Any set with cardinality less than $|S_m|$ will not satisfies (i), (ii) and (iii)

(v) If $|S_m| = RD(G)$ with $S_m(u) = S_m$ for some vertex $u \in V(G) - S_m$, then for every $RD$-set $S$ of the graph $G$

there exist some vertex $w \in V(G) - S$ such that $S(w) = S_m$.

Graphs for which the 1-vertex non location domination set can be found are said to be 1-vertex non locating dominating graph.

**Remark 2.3**

Every graph need not be a 1-vertex non location dominating graph.

**Remark 2.4**

1-vertex non location domination set for the given graph can be obtained from the $RD$-set $S$ of the graph $G$ by removing a vertex from $S$ or interchanging a vertex from $S$ with vertex from $V(G) - S$. Hence cardinality of $S_m$ can be either $RD(G)$ or $RD(G) - 1$.

**Note:** Graph $G_1, G_2$ which are mentioned in Theorem 2.3, 2.4, 2.5, 2.6 and Remark 2.5 can be any graph except the graph $K_1$.

**Theorem 2.3**

The set $S$ is a $RD$-set of $G_1 + G_2$, then the following one of the conditions must be true in regarding to the set $S \cap V(G_1)$.

(i) If $G_1$ has 1-vertex non location domination set $S_m$ with cardinality $RD(G_1) - 1$ then $S_m \subseteq S \cap V(G_1)$.

(ii) If $G_1$ has 1-vertex non location domination set $S_m$ with cardinality $RD(G)$ then $S_m \subseteq S \cap V(G_1)$ or $S \cap V(G_1)$ may or may not contains $S_1$ the $RD$-set of $G_1$.

(iii) If $G_1$ is not a 1-vertex non locating dominating graph then $S_1 \subseteq S \cap V(G_1)$, where $S_1$ is the $RD$-set of $G_1$.

Similar condition are also true for the set $S \cap V(G_2)$.

**Proof**

In the graph $G_1 + G_2$, $S \cap V(G_1)$ can almost uniquely locate only one vertex of $G_1$ provided $S \cap V(G_1)$ locates and dominate remaining all vertices of $S \cap (V(G_1))$. That is, if $G_1$ has a 1-vertex non location domination set and $S$ is a $RD$-set of $G_1 + G_2$ then $S \cap V(G_1)$ contains 1-vertex non location domination set (6)

**Case 1:** Assume $G_1$ has 1-vertex non location domination set $S_m$ with cardinality $RD(G_1) - 1$. By Equation (6), $S_m \subseteq S \cap V(G_1)$.

**Case 2:** By Equation (6), $S_m \subseteq S \cap V(G_1)$. As $|S_m| = RD(G_1)$, instead of 1-vertex non location domination set $S_m$, $S \cap V(G_1)$ may contains a locating dominating set $S_1$ of the graph $G_1$ also, provided $S_m$ satisfies some conditions. The following are various situations:

i) $S_m(u) = S_m$ for all $u \in V(G_1) - S_m$ and $S_1(v) = S_1$ for some $v \in V(G_1) - S_1$ then $S_m \subseteq S \cap V(G_1)$ and $S_1 \subseteq S \cap V(G_1)$.

ii) $S_m(u) = S_m$ for all $u \in V(G_1) - S_m$ and $S_1(v) = S_1$ for all $v \in V(G_1) - S_1$ then $S_m \subseteq S \cap V(G_1)$ or $S_1 \subseteq S \cap V(G_1)$.

iii) $S_m(u) = S_m$ for some $u \in V(G_1) - S_m$ and $S_1(v) = S_1$ for some $v \in V(G_1) - S_1$ then $S_m \subseteq S \cap V(G_1)$ or $S_1 \subseteq S \cap V(G_1)$.

iv) $S_m(u) = S_m$ for some $u \in V(G_1) - S_m$ and $S_1(v) = S_1$ for all $v \in V(G_1) - S_1$ then $S_m$ is not a 1-vertex non locating dominating set. This case will not occur, as $G_1$ is a 1-vertex non locating dominating graph.

Hence based on nature of $S_m$ and $S_1$, $S_m \subseteq S \cap V(G_1)$ for all cases and $S_1 \subseteq S \cap V(G_1)$ or $S_1 \subseteq S \cap V(G_1)$.

**Case 3:** Assume $G_1$ is not a 1-vertex non locating dominating graph.

As $G_1$ does not have a 1-vertex non location domination set and the set $S \cap V(G_1)$ doesn’t play any role in locating the vertices of $V(G_1) - S$, the set $S \cap V(G_1)$ must locate and dominate all the vertices of $G_1$. That is $S \cap V(G_1)$ is a $LD$-set of $G_1$.

Similarly, with regarding to the graph $G_2$ the set $S \cap V(G_2)$ will satisfy one of the conditions.

**Theorem 2.4**

Let $G_1$ and $G_2$ be any two 1-vertex non locating dominating graph with 1-vertex non location domination set $S_m$ and $S_n$, respectively. If for all 1-vertex non location domination set $S_m$ and $S_n$ of the graph $G_1$ and $G_2$ there exist some vertex $v_i \in V(G_1) - S_m$ and $v_j \in V(G_2) - S_n$ such that $S_m(v_i) = S_n$ and $S_n(v_j) = S_m$ then $RD(G_1 + G_2) = |S_m| + |S_n| + 1$ otherwise $RD(G_1 + G_2) = |S_m| + |S_n|$. 

**Proof**

As both $G_1$ and $G_2$ is 1-vertex non locating dominating graphs, by Theorem 2.3 the $RD$-set $S$ of the graph $G_1 + G_2$ is as follows, $S_m \subseteq S \cap (V(G_1))$ and $S_n \subseteq S \cap (V(G_2))$. That is
\[ S_1 \cup S_{a_1} \subseteq S \]  

(7)

Case 1: Assume that for all 1-vertex non location domination set \( S_{a_1} \), \( S_{a_1} \) of graph \( G_1 \), \( G_2 \) respectively, there exist some vertex 
\( v_1 \in V(G_1) - S_1 \) and \( v_2 \in V(G_1) - S_1 \) such that \( S_1(v_1) = S_{a_1} \) and 
\( S_2(v_1) = S_{a_1} \).

Now consider the set \( S = S_1 \cup S_{a_1} \), then we have 
\[ S(v_1) = S_1(v_1) \cup S_{a_1} = S_1 \cup S_{a_1} \] 

and 
\[ S(v_2) = S_1 \cup S_{a_1} = S_1 \cup S_{a_1} \].

Therefore \( S(v_1) = S(v_2) \) and so \( S_1 \cup S_{a_1} \) cannot be the LD-set of \( G_1 + G_2 \). Clearly \( S_1 \cup S_1 \cup \{v_1\} \) and \( S_1 \cup S_{a_1} \cup \{v_2\} \) must be the LD-set of \( G_1 + G_2 \) with minimal cardinality. Hence 
\[ RD(G_1 + G_2) = |S_1| + |S_{a_1}| + 1. \]

Case 2: Assume that \( G_1 \) and \( G_2 \) has 1-vertex non location domination set which can be any form other than in Case 1. Let us consider the set \( S = S_1 \cup S_{a_1} \), where \( S_1, S_{a_1} \) are 1-vertex location domination sets of \( G_1 \) and \( G_2 \). For any vertex \( v \),
\[ S(v) = \begin{cases} S_1(v) \cup S_{a_1}, & \text{if } v \in V(G_1) - S_1 \\ S_1(v) \cup S_{a_1}, & \text{if } v \in V(G_2) - S_2 \\ S_1(v) \cup S_{a_1}, & \text{if } v \in V(G_2) - S_2 \\ \end{cases} \]

Clearly \( S(u) = S(w) \) is possible only if \( S(u) = S(w) = S_1 \cup S_{a_1} \). But by our assumption both \( G_1 \) and \( G_2 \) cannot simultaneously have a vertex \( u \in (G_1) - S_1 \), \( w \in (G_2) - S_2 \) such that \( S_1(u) = S_{a_1} \) and \( S_1(w) = S_{a_1} \). Hence \( S(u) = S(w) \) for all \( u, w \in V(G_1 + G_2) - S \). Thus \( S \) is the RD-set of \( G_1 + G_2 \) with cardinality \( |S_1| + |S_{a_1}| \).

**Theorem 2.5**

Let \( G_1 \) and \( G_2 \) be two graphs which do not have any 1-vertex non location domination set. If for every RD-set \( S_1 \) and \( S_2 \) of graphs \( G_1 \) and \( G_2 \) respectively, there exist some vertex 
\( v_1 \in V(G_1) - S_1 \) and \( v_2 \in V(G_1) - S_1 \) such that \( S_1(v_1) = S_{a_1} \) and 
\( S_2(v_1) = S_{a_1} \) then 
\[ RD(G_1 + G_2) = RD(G_1) + RD(G_2) + 1. \]

**Proof**

Let \( S \) be the RD-set of \( G_1 + G_2 \). By applying Theorem 2.3 to the hypothesis of the theorem we get 
\[ S_1 \subseteq S \cup V(G_1) \] 

and 
\[ S_2 \subseteq S \cup V(G_2) \]. Thus \( S_1 \cup S_2 \subseteq S \) and hence 
\[ |S_1| + |S_2| \leq |S| \]

Proof to the rest of the theorem is similar to the Theorem 2.4.

**Remark 2.5**

If at least one of the graph \( G_1 \) and \( G_2 \) is disconnected, then based on the nature of the graph, \( RD(G_1 + G_2) \) can take value from either one of the following 
(i) \( |S_1| + |S_{a_1}| \)
(ii) \( |S_2| + |S_{a_2}| \)
(iii) \( |S_1| + |S_{a_2}| \)
(iv) \( |S_2| + |S_{a_1}| \)

where \( S_1 \) and \( S_{a_1} \) denote the 1-vertex non location domination set of \( G_1 \) and \( G_2 \) respectively. And \( S_2, S_{a_2} \) denotes the RD-set of \( G_1 \) and \( G_2 \) respectively.

**Remark 2.6**

Consider the graph \( R_m, R_n \), where \( m,n \geq 2 \). Clearly \( R_m \) and \( R_n \) are disconnected, 1-vertex non locating dominating graphs. Therefore by Remark 2.5 we get 
\[ RD(R_m + R_n) = RD(R_m) + RD(R_n) - 2 = m + n - 2. \]

**3. Conclusion**

In this paper we have define a graph namely 1-vertex non locating dominating graph. With the help of this definition we have found the location domination number of sum of any graphs.

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