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Research paper

Construction of Inverse Unit Regular Monoids from a Semilattice and a Group

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Abstract

This paper is a continuation of a previous paper [6] in which the structure of certain unit regular semigroups called *R*-strongly unit regular monoids has been studied. A monoid *S* is said to be unit regular if for each element $s \in S$ there exists an element *u* in the group of units *G* of *S* such that *s* = *sus*. Hence $s = suu^{-1}$ where *su* is an idempotent and u^{-1} is a unit. A unit regular monoid *S* is said to be a unit regular inverse monoid if the set of idempotents of *S* form a semilattice. Since inverse monoids are *R* unipotent, every element of a unit regular inverse monoid can be written as *s* = *eu* where the idempotent part *e* is unique and *u* is a unit. Here we give a detailed study of inverse unit regular monoids and the results are mainly based on [10]. The relations between the semilattice of idempotents and the group of units in unit regular inverse monoids are better identified in this case.

Keywords: *Inverse monoids, Unit regular monoids, Semi lattice, group.*

1. Introduction

Throughout this paper let *E* (= *E*(*S*)) denote the semilattice of idempotents and *G* (= *G*(*S*)) denote the group of units of *S*.

Proposition 1.1 ([4]). *Let S be a regular monoid. Then S is unit regular if and only if for* each $s \in S$ *there is an idempotent* $x \in E$ and $g \in G$ *such that s = xg.*

Let *L*, *R* be two of Green's relations and let L_e, R_e be the *L*-class containing *e* and *R*-class containing *e* respectively. Then we have the following proposition.

Proposition 1.2 ([2]). Let *S* be an inverse monoid with *G* = *G*(*S*) and *E* = *E*(*S*). Then the following conditions are equivalent on *S*. (i) *S* is unit regular (ii) $L_e = Ge$ for every $e \in E$. (iii) $R_e = eG$ for every $e \in E$.

Definition 1.3 ([4]). Let *G* be a group and *E* a non empty set. Then *G* is said to act on *E* if there is a function from $G \times E$ to *E* usually denoted by $(u,e) \to u*e$ such that $1*e = e$ for every $e \in E$ and for all $u_1, u_2 \in G$ and $e \in E$, $(u_1u_2)*e = u_1*(u_2*e)$

Definition 1.4. ([5]). Let *S* be a regular semigroup and the sandwich set of $e, f \in E(S)$ be denoted by $S(e,f)$. Then

$S(e,f) = \{h \in E(S) : he = fh = h \text{ and } ehf = ef\}$.

It is well known that the set of idempotents *E*(*S*) of an inverse semigroup *S* is a semilattice. Further the relations $L|E(S)$ and $R|E(S)$ are trivial.

2. Inverse Unit Regular Monoids

In this section we study about the construction of some unit regular inverse monoids. Throughout this section let *x*, *y*, *z*, k, w denote the elements of *E* and g, *h* elements of *G*.

Theorem 2.1. Let *E* be a semilattice (that is a commutative band) with a maximum element 1 and *G* be a group acting on *E*. That is for each $g \in G$, the map $x \to g*x$ is an isomorphism of *E*. For each $x \in E$ suppose there exist a collection of subgroups of *G* say G(*x*) satisfying the following conditions.

(i) G (1) = {1}

(ii) $gG(x)g^{-1} = G(g*x)$, $g \in G, x \in E$

(iii) For any *k*, *x* in *E*, $G(x) \subseteq G(kx)$ = G(*xk*)

(iv) If $g \in G(x)$, then $g*x = x$ and $xy = x(g*y)$ for $x, y \in E$.

Then on $E \times G$ define a relation ~ as follows. For (*x*, *g*), (*y*, *h*) $\in E \times G$, $(x,g) \sim (y,h)$ if *x* = *y* and $gh^{-1} \in G(x)$. Let $T = (E \times G)/\sim$ and define a product on *T* as given below. For [*x*, *g*] , [*y*, *h*] $\in T$, [*x*, *g*] [*y* , *h*] = [*x*(*g***y*), *gh*] where the equivalence class of (*x*, *g*) of $E \times G$ under ~ is denoted by [*x*, *g*]

.Then T is a unit regular inverse monoid with semilattice of idempotents isomorphic to E and group of units isomorphic to G.

Proof:

In order to prove the theorem we shall prove the following lemmas.

Lemma 2. 2. Let $F = E \times G$. Let $\psi : F \times F \rightarrow F/\sim$ be defined as $((x, g), (y, h))\psi = [x(g \ast y), gh]$. Then $((x, g), (y, h))\psi = ((x, g'), (y, h'))\psi$ whenever $(x, g) \sim (x, g')$ and $(y, h) \sim (y, h')$.

Proof : We prove that $((x, g), (y, h))\psi = ((x, g'), (y, h'))\psi$ whenever $(x, g) \sim (x, g')$ and $(y, h) \sim (y, h')$. We will show this in two steps. That is we will show that

(i) $((x, g), (y, h))\psi = ((x, g'), (y, h))\psi$ if $(x, g) \sim (x, g')$ and

(ii) $((x, g), (y, h))\psi = ((x, g), (y, h'))\psi$ if $(y, h) \sim (y, h')$.

Consider the first case. Let $(x, g) \sim (x, g')$. Hence $gg'^{-1} \in G(x)$. Let $k \in S(x, g * y)$. Let $gg'^{-1} = h$. Then $h \in G(x)$ and $g = hg'$. Hence $k \in S(x, (hg') * y)$. Hence $h^{-1} * k \in S(h^{-1} * x, g' * y)$.

Since $h \in G(x), h^{-1} \in G(x)$. Hence $h^{-1} * x = x$, by property (iv). That is $h^{-1} * k \in S(x, g' * y)$. Let $h^{-1} * k = k'$ .If $h \in G(x)$, then $h^{-1} \in G(x)$ and $xk = x(h^{-1} * k)$, by property (iv). That is $xk = xk'$. Choose $k = x(g * y)$ and $k' = x(g' * y)$. Then $x\, x(g * y) = x\, x(g' * y)$. Hence $x(g * y) = x(g' * y)$ .Also $[x(g*y), gh] = [x(g'*y), g'h]$ only if $(gh)(g'h)^{-1} = gg'^{-1} \in G(x(g * y))$. By property (iii) $G(x) \subseteq G((g * y)x) = G(x(g*y))$. Hence $((x, g), (y, h))\psi = ((x, g'), (y, h))\psi$ whenever $(x, g) \sim (x, g')$. Next let $(y, h) \sim (y, h')$. Then we will prove that $((x,g), (y,h))\psi = ((x, g), (y, h'))\psi$. Since $(y, h) \sim (y, h')$ we get that $hh'^{-1} \in G(y)$. We have to prove $[x(g*y), gh] = [x(g*y), gh']$. That is we have to show that $(gh)(gh')^{-1} = ghh'^{-1}g^{-1} \in G(x(g * y))$ .By property **(ii)** since $hh'^{-1} \in G(y)$ we get $ghh'^{-1}g^{-1} \in G(g \ast y)$. By property (iii), $ghh'^{-1}g^{-1} \in G(x(g \ast y))$. Now coming to the general case,

$((x, g),(y, h))\psi = ((x, g'), (y, h))\psi$, by step 1

$= ((x, g'),(y, h'))\psi$, by step 2

Since the mapping ψ is well defined, the product in T namely $[x, g][y, h] = [x(g \ast y), gh]$ .In the following sections let T denote $(E \times G)/\sim$.

Lemma 2.3. *T* **with** *the product defined by* $[x, g][y, h] = [x(g \ast y), gh]$ *is a monoid.*

Proof: To prove T is a semigroup it is enough to show that the associative property holds.

Now

$[x, g]([y, h][z, j]) = [x, g]([y(h*z), hj])$

$= [x(g*(y(h*z)), ghj]$

$= [x(g*y)(g*(h*z)), ghj]$

$= [x(g*y)((gh)*z), ghj]$, since G is a group acting on E.

Also $([x, g][y, h])[z, j] = [x(g*y), gh][z, j]$

$= [x(g*y)((gh)*z), ghj]$

Now we prove that $[1, 1]$ is the identity element of T. Now, $[x, h][1, 1] = [x(h*1), h] = [x, h]$, since the map $x \rightarrow h * x$ is an isomorphism of E. Also $[1, 1][x, h] = [(1*x), h] = [x, h]$ by property (iv).Therefore $[1, 1][x, h] = [x, h]$. Hence T is a monoid.

Lemma 2.4. *The set of idempotents of T is given by* $E(T) = \{[x,1] : x \in E\}$.

Proof: First we trace out the idempotents of T namely $E(T)$). We will show that $E(T) = \{[x,1]: x \in E\}$. Now $[x, 1][x, 1] = [x(1*x), 1] = [x^2, 1] = [x, 1]$. So $[x, 1][x, 1] = [x, 1]$. Also if $[x, g] \in E(T)$, then $[x, g][x, g] = [x, g]$ implies $[x(g*x), g^2] = [x, g]$. Thus $x(g*x) = x$ and $g \in G(x)$. Hence $[x, g] = [x, 1]$.

Now to identify $G(T)$ it is necessary to have the following result.

Lemma 2.5. *The group of units of T is given by* $G(T) = \{[1,h] : h \in G\}$.

Proof: Now $[1, h][1, h^{-1}] = [h*1, 1] = [1, 1]$ .That is, $[1, h][1, h^{-1}] = [1, 1]$. Similarly $[1, h^{-1}][1, h] = [1,1]$.Therefore the elements of the form $[1, h] \in G(T)$. Also if $[x, h] \in G(T)$, then there exists $[y, h'] \in T$ such that $[x, h][y, h'] = [1, 1]$ and $[y, h'][x, h] = [1, 1]$. Hence $[x(h*y), hh'] = [1, 1]$ and $[y(h'*x), h'h] = [1, 1]$ .So $x(h*y) = y(h'*x) = 1$. Since $x(h*y) \leq x$, we get $1 \leq x$ .Also $x \leq 1$. Hence $x = 1$. So $h*y = 1$. Similarly $y = (h'*x) = 1$. Hence hh', $h'h \in G(1) = \{1\}$,by Property **(i).** Therefore $hh' = h'h = 1$. So, $h' = h^{-1}$. That is $[x, h] = [1, h]$ and $[y, h'] = [1, h^{-1}]$. Consequently $G(T) = \{[1, h]: h \in G\}$, with $[1, h]^{-1} = [1, h^{-1}]$.

Lemma 2.6. *T is a unit regular monoid.*

Proof: We prove the unit regularity of T by showing that every element $[x, h]$ of T is a product $[x, 1][1, h]$ where $[x, 1] \in E(T)$ and $[1, h] \in G(T)$. Now, $[x, 1][1, h] = [x, h]$. So T is a unit regular monoid.

Remark: It can be seen that for $[x, h] \in T$ we can write $[x, h] = [1, h][h^{-1} \ast x, 1]$.

Lemma 2. 7. $E(T)$ is a semilattice.

Proof: If x and y are elements in E , xy is an idempotent since E is a semilattice. Also $[x,1][y,1] = [x(1*y), 1] = [xy, 1]$, since G is a group acting on E, $1 * y = y$. Also $[y, 1][x, 1] = [y(1*x), 1] = [yx, 1]$. Since E is a semilattice, $xy = yx$. Hence $[x,1][y,1] = [y, 1][x, 1]$.

Lemma 2. 8. *G (T) is isomorphic to G (as groups) and E (T) is isomorphic to E as monoids*

Proof: Now let $\varphi_1 : G \rightarrow G(T)$ be defined as $g\varphi_1 = [1, g]$. Then $(g_1g_2)\varphi_1 = [1, g_1g_2]$. Also $(g_1\varphi_1)(g_2\varphi_1) = [1,g_1][1,g_2] = [1(g_1 \ast 1), g_1g_2] = [1, g_1g_2]$, since $x \rightarrow g * x$ is an isomorphism and $g_1 \ast 1 = 1$. Therefore φ_1 is a homomorphism. φ_1 is evidently onto. φ_1 is one one since $g_1\varphi_1 = g_2\varphi_1$ implies that $[1, g_1] = [1, g_2]$. Hence $g_1g_2^{-1} \in G(1) = \{1\}$, by property (i) .So $g_1 = g_2$. Therefore φ_1 is an isomorphism of groups.

Let $\varphi_2 : E \rightarrow E(T)$ be defined as $x\varphi_2 = [x, 1]$. Then φ_2 is evidently one one and onto. We will prove that φ_2 is a isomorphism. $xy\varphi_2 = [xy, 1]$.Also $x\varphi_2\, y\varphi_2 = [x, 1][y, 1] = [x(1*y), 1] = [xy, 1]$ So φ_2 is an isomorphism of semilattices. Also $1\varphi_2 = [1, 1]$.

From the above lemmas we have the **Theorem 2.1**.

Corollary 2.9. Let E be a semilattice (that is a commutative band) with a maximum element 1 and G be a group acting on E. Suppose that for each $x \in E$ their corresponds a subgroup $G(x)$ of G satisfying the following:

(i) $G(1) = \{1\}$

(ii) For any k, x in E, $G(x) \subseteq G(kx) = G(xk)$

(iii) $gG(x)g^{-1} = G(g * x)$, $g \in G, x \in E$

Then on $E \times G$ define a relation ~ as follows. For $(x, g), (y, h) \in E \times G$, $(x, g) \sim (y, h)$ if $x = y$ and $gh^{-1} \in G(x)$. Let $T = (E \times G)/\sim$ and define a product on T as given below.

For $[x, g]$, $[y, h] \in T$, $[x, g]$ $[y , h]$ = $[xy, gh]$ where the equivalence class of (x, g) of $E \times G$ under ~ is denoted by $[x, g]$ .Then T is a unit regular inverse monoid with semilattice of idempotents isomorphic to E and E- centralizing group of units isomorphic to G. T is in particular a Clifford semigroup.

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