International Journal of Engineering & Technology, 7 (4.36) (2018) 810-812

SPC

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

A Generalization of the bounds of copulas on quantum logic

Ahmed Al-Adilee[1*], Sami Kadhem Kareem Althabhawi[2]

[1]*Department of Computer Sciences, Faculty of Education, Iraq.*
[2]*Department of Computer Sciences, Faculty of Education, Iraq.*

Abstract

Our paper is concerned with a generalization of the boundaries of copula that are well-known by Frechet-Hoeffding conditions [4], and study copula boundaries as functions defined on quantum logic ., While there are many properties that have been presented according to our proposed generalization throughout these modified functions and show that some important propositions that have different forms to those of classical copulas. Indeed, it has been shown two main results relevant to the lower and upper boundaries of what we have defined and named by quantum logic copulas, see [1].

Keywords: *Frechet, boundary, copulas.*

1. Introduction

Frechet-Hoeffding conditions are very important in statistical inference because they represent the boundaries of each copula. These types of copula functions are well-known as minimum copula ($M(u,v) = \min(u,v)\,, u,v \in [0,1]$), and maximum copula ($\mathrm{W}(u,v) = \max(u+v-1,0), u,v \in [0,1]$). It has been shown that $M(u,v)$, and $W(u,v)$ are the upper and lower limits of each copula function, respectively.

From a historical point of view, these copulas were firstly appeared in the study of Hoeffding in 1941, see[4]. They have been explicitly studied such functions without referring to them as copulas. Subsequently, they have been shown in the study of Sklar in 1959, where they have officially been referred to them as copulas.

Furthermore, we can notice that these copulas as bounds to each copula are very useful in the study of dependence structure, for examples, in finance, risk management, and many other real life applications. Their importance comes from the fact that these types of copula represent the restrictions that can easily restrict any other copula.

Our study can be divided to two main parts. First part will address some preliminaries, and basic concepts. While our second part will be devoted to demonstrate our thoughts of the types of Frechet-Hoeffding conditions as boundaries of copulas on quantum logic.

2. Preliminaries

The bivariate copula that has been presented in [4,5] is the main concept that has the major role in statistical inference so that we are interested in the study of its properties. These properties are essential to examine whether a function is copula or not.

As explained in [4] each copula should has a grounded property, for each $\mathrm{u,v} \in [0,1], \mathrm{C(u,1) = u, C(1,v) = v}$, and eventually, a function should has the 2-increasing property.

On other hand, the boundaries of any copula function are indeed the minimum and maximum copulas, respectively, and satisfy the following property, see [5].

$$\mathrm{W(u,v) \le C(u,v) \le M(u,v)} \qquad (1)$$

Obviously, $\mathrm{W, M}$ are also copulas because they satisfy the properties of being copula, see [4,5]. Indeed, it is also important to know that these copulas are well-known as a lower and upper bound of any copula.

We emphasize once again that the inequality in equation (1) is very important in the description of dependence structure and known as Frechet-Hoeffiding bounds because they can examine what is defined in [5] by the lower tail and the upper tail. Moreover, and in order to set a reasonable generalization for these extremes of copula functions and defined them rightly on quantum logic, we need to recall two basic definitions, see [3].

Definition1:- Any system $\mathcal{L} = (\mathrm{L}, 0, \mathrm{I}, \vee, \wedge, \perp)$ is said to be an orthomodular lattice and partial ordering $\le$, if $\perp : \mathrm{L} \to \mathrm{L}$, and satisfy the following conditions:

i) $(u^{\perp})^{\perp} = u$;
ii) $u \le v \Rightarrow v^{\perp} \le u^{\perp}$;
iii) $u \vee u^{\perp} = I$;
iv) $u \le v \Rightarrow v = u \vee (u^{\perp} \wedge v)$.

where, 0 is the smallest element, and I is the greatest element of the system. Also, property number (iv) is well-known as an orthomodular law, see [3].

Definition2:- Let $\mathcal{L}$be an orthomodularσ-lattice. A state on L is a map $\mathrm{m} : \mathrm{L} \to [0,1]$with the following properties

1. $\mathrm{m(I) = 1}$;
2. $m(\vee_{i=1}^{n} a_i) = \sum_{i=1}^{n} m(a_i)$, *for any* $a_1, \dots, a_n \in L$ such that$a_i \perp a_j$, whenever$i \neq j$, see[3].

It is clear that $m(O) = 0$, because the state m corresponds to probability measure. Also it is additive, when $''i''$ goes to infinity.

In [1], we have presented several types of a generalization of copula function on quantum logic without referring to the extremes of copulas. We have denoted to such type of copulas by QL-copulas, and defined them as follow

Definition3:- Let $\mathcal{L}$ be a quantum logic. A QL-copula is a function $Q : L^2 \to [0,1]$ that fulfills the following conditions:

Q1. For each $a \in L, Q(O,a) = Q(a,O) = 0$;
Q2.$Q(I,.), Q(.,I)$ are states on L;
Q3.For each $a,b,c,d \in L,\ a \le b$and $c \le d$, then

$$Q(b,d) + Q(a,c) \ge Q(a,d) + Q(b,c).$$

***Remark1*:-**Since the notion of 2-increasing copula in [5] is important and necessary to build our constructions as bounds of QL-copulas, then according to this notion,we can easily see that elements in L^2 are also non-negatives.

3. Bounds of QL-copula on quantum logic

The maximum copula W, and minimum copula M that we have presented in the previous part can be generalized on quantum logic. Therefore according to the definition of QL-copula, we need to prove that W, M are also copulas on quantum logic. This can be introduced by the following propositions

Proposition1:-Let$Q$be a QL-copula.Let$\overline{W}(a,b) = \max(m(a \vee b) - m(I), m(O))$,is a function defined on $\mathcal{L}$. If $\overline{W}$is QL-copula, then it is ***the lower bound QL-copula***.

Proof

First of all, we should mention that the expression of $\overline{W}$is equivalent to the classical one and since it has been shown in [4] that W is a lower bound of copula then the prove needs only to show that $\overline{W}$holds the properties of being QL-copula Q.

1. In advance, it is clear that the first property is satisfied since $\overline{W}(a,O) = \max(m(a \vee O) - m(I), m(O)) = \max(m(a) - m(I), m(O)) = m(O) = 0 = \overline{W}(O,a)$.
2. Our second part that we need to prove is to show that $\overline{W}(I,.), \overline{W}(.,I)$are states. Then,
   i) $\max(m(I \vee I) - m(I), m(O)) = \max(m(I) + m(I) - m(I), m(O)) = \max(m(I) + m(I) - m(I), 0) = m(I) = 1$(***definition2***);
   ii) Let $a \perp b$, then

$$\overline{W}(I, a \vee b) = \max\left(m(I \vee (a \vee b)) - m(I), m(O)\right) = \max(m(I) + m(a \vee b) - m(I), m(O)) = \max(m(I) + m(a) + m(b) - m(I), m(O)) = \max(m(I) + m(a) + m(b) - m(I), m(O)) = m(a) + m(b)\ \ (\textit{definition2});$$

On other hand,

$$\overline{W}(I,a) = \max(m(I \vee a) - m(I), m(O)) = \max(m(a), m(O)) = m(a)$$

Hence, and by the same way, $\overline{W}(I,b) = m(b)$.

Then by adding $\overline{W}(I,a)$ to $\overline{W}(I,b)$, we obtain that $\overline{W}(I,a) + \overline{W}(I,b) = m(a) + m(a)$;

Hence , $\overline{W}(I, a \vee b) = \overline{W}(I,a) + \overline{W}(I,b)$

Therefore, $\overline{W}(I,.)$ is state

Similarly, we can easily see that $\overline{W}(.,I)$ is also state.

1. Finally, we have to prove that $\overline{W}$ has the 2-increasing property

   Let $a_1 \leq a_2, b_1 \leq b_2, a_1, a_2, b_1, b_2 \in L$. Then we should prove that

$$\overline{W}(a_1,b_1) + \overline{W}(a_2,b_2) - \overline{W}(a_1,b_2) - \overline{W}(a_2,b_1) \geq 0$$

So, we know that

$$\overline{W}(a_1,b_1) = \max(m(a_1 \vee b_1) - m(I), m(O))$$
$$\overline{W}(a_2,b_2) = \max(m(a_2 \vee b_2) - m(I), m(O))$$
$$\overline{W}(a_{m(I)}, b_2) = \max(m(a_1 \vee b_2) - m(I), m(O))$$
$$\overline{W}(a_2,b_1) = \max(m(a_2 \vee b_1) - m(I), m(O))$$

Then

$$\begin{aligned}\overline{W}(a_1,b_1) + \overline{W}(a_2,b_2) - \overline{W}(a_1,b_2) - \overline{W}(a_2,b_1) &= \max(m(a_1 \vee b_1) - m(I), m(O)) \\ &- \max(m(a_2 \vee b_2) - m(I), m(O)) \\ &- \max(m(a_1 \vee b_2) - m(I), m(O)) \\ &- \max(m(a_2 \vee b_1) - m(I), m(O)) \\ = \max(m(a_1) + m(b_1) - m(I), m(O)) &- \max(m(a_2) + m(b_2) - m(I), m(O)) \\ &- \max(m(a_1) + m(b_2) - m(I), m(O)) \\ &- \max(m(a_2) + m(b_1) - m(I), m(O))\end{aligned}$$

According to ***remark1***, we obtain that

$$\max(m(a_1) + m(b_1) - m(I), m(O)) = m(a_1) + m(b_1) - m(I),$$
$$\max(m(a_2) + m(b_2) - m(I), m(O)) = m(a_2) + m(b_2) - m(I),$$
$$\max(m(a_1) + m(b_2) - m(I), m(O)) = m(a_1) + m(b_2) - m(I),$$

and $\max(m(a_2) + m(b_1) - m(I), m(O)) = m(a_2) + m(b_1) - m(I)$

Hence

$$m(a_1) + m(b_1) - m(I) + m(a_2) + m(b_2) - m(I) - (m(a_1) + m(b_2) - m(I)) - (m(a_2) + m(b_1) - m(I)) \geq 0.$$

Therefore, $\overline{W}(a,b)$ is a QL-copula and it is the lower bound of any other copulas.

Proposition2:-Let Q be aQL-copula. Let $\overline{M}(a,b) = \min(m(a), m(b))$, is a function defined on $\mathcal{L}$. If $\overline{M}$ is QL-copula then it is ***the upper bound QL-copula***.

The prove of***proposition2*** is clear and not difficult, so we have left it as an exercise to the readers.

According to equation (1), one can also generalize the inequality uponour quantum logic copulas as follow

$$\overline{W}(a,b) \leq Q(a,b) \leq \overline{M}(a,b) \quad (2)$$

We emphasize that $\overline{W}(a,b)$ is the lower bound QL-copula which means that no any other QL-copula can be found to be less than it and $\overline{M}(a,b)$ is the upper bound QL-copula that no other QL-copula can be found to be greater than it. Indeed, equation (2)is a generalization of classical lower and upper bounds of copulas, respectively, and it leads us to the following results

1. $Q(a,b) \leq Q(a,I) = a$;
2. $Q(a,b) \leq Q(I,b) = b$;
3. $\overline{W}(a,b) \leq a$, or $\overline{W}(a,b) \leq b$;
4. $\overline{M}(I,I) \geq Q(a,b)$.

Furthermore, we could extended our QL-copula bounds from Bivariate to multivariate, so we have proposed them by the following ways:

Proposition3:- Let Q be a QL-copula. Let $\overline{W}(a_i) = \max(m(\vee_{i=1}^{n} a_i) - m(I), m(O))$ is a function defined on $\mathcal{L}$. If $\overline{W}$ is QL-copula, then it is ***the lower bound QL-copula***.

Proposition4:- Let Q be a QL-copula. Let $\overline{M}(a_i) = \min(m(a_i))$, $\mathrm{i} = 1, \ldots, \mathrm{n}$ is a function defined on $\mathcal{L}$. If $\overline{M}$ is QL-copula then it is ***the upper bound QL-copula***.

4. Conclusion

It is clear that a generalization of copula function on quantum logic gives several different properties from the classical properties. The boundaries that we have proposed in ***proposition1***, ***proposition2*** are good examples of generalization of copulas. We have examine that no lower quantum logic copula than $\overline{\mathrm{W}}$, and also no upper quantum logic copula than $\overline{\mathrm{M}}$ depending on the prove of the extremes of copula in [4]. Finally, one could have a more general forms that are need to be investigated with respect to other algebraic systems that have homomorphism properties with orthomodular lattice.

References

[1] Aladilee A & Nanasiova O, "Copula and s-map on quantum logic", *inf. sci.*, Vol.179, (2009), pp.4199-4207.

[2] Fodor J.C, "t-norms and generalized copula", *Computing center, Eotvos Lor and Univ.*, Vol.152, (2005), pp.2203-2208.

[3] Kalmbach G, *Orthomodular Lattices*, Academic Press, London, (1983).

[4] Nelsen RB, *An introduction to copulas*, Springer-New York, (2006).

[5] Nelsen RB, "On measures of association as measures of positive dependence", *Statist. Probab. Lett.*, Vol.14, (1992), pp.269-274.