Fixed Point Theorems Under Caristi’s Type Map on C*-Algebra Valued Fuzzy Soft Metric Space

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Abstract
In this paper, we present the extension of Caristi’s fixed point theorems for mappings defined on C*-algebra-valued Fuzzy soft metric spaces. We establish the existence of simple proof of caristi’s type fixed point theorems in C*-algebra-valued Fuzzy soft metric spaces and we give some examples which supports our main results.

Keywords: Bounded below function; Caristi’s mapping; C*-algebra-valued Fuzzy soft metric; completeness; fixed point; Lower semi continuity.

1. Introduction and Preliminaries

In 2001, Maji et al. [1] introduced the notion of a fuzzy soft set which unites a fuzzy set and a soft set and after that Thangaraj Beaula et al.[2] defined fuzzy soft metric space in terms of fuzzy soft point sets and established some results. Subsequently to improve many author established so many results on fuzzy soft metric spaces and its topological properties (see. e.g.[3]-[6]). The Caristi’s fixed point results is known as one of the very attractive and valuable generalization of the Banach fixed point results for self-mappings on a complete metric spaces. We establish the existence of simple proof of caristi’s type fixed point theorems in context of C*-algebra valued fuzzy soft metric spaces.

Throughout our discussion, U refers to an initial universe, E the set of all parameters for U, C subset of parameter set E and P (U) the set of all fuzzy set of U. (U, E) means the universal set U and parameter set E. C refer to a unital C*-algebra. Now we recollect some basic definitions, notations, and results on C*-algebras are available in ([17], [18]). The set \( \mathbb{C}_0 = \{ \hat{a} \in C : \hat{a} = a^* \} \). An element \( \hat{a} \in C \) is called a positive element, if \( \hat{a} = a^* \) and it is denoted by \( \hat{a} \leq \hat{a} \) where \( \hat{a} \) is the zero element in \( \mathbb{C}_0 \) and the spectrum of \( \hat{a} \) is \( \sigma(\hat{a}) \subseteq R(C)_+ \) is the set of fuzzy soft real numbers. A fuzzy soft real number is a fuzzy set on the set of all soft real set \( F:E \rightarrow B(R) \) where \( R \) is set of real numbers, \( \sigma(\hat{a}) \subseteq R(C)_+ \) is the set of fuzzy soft real numbers. A fuzzy soft real number is a fuzzy set on the set of all soft real set \( F:E \rightarrow B(R) \) where \( R \) is set of real numbers, \( \sigma(\hat{a}) \subseteq R(C)_+ \) is the set of fuzzy soft real numbers. A fuzzy soft real number is a fuzzy set on the set of all soft real set \( F:E \rightarrow B(R) \) where \( R \) is set of real numbers, \( \sigma(\hat{a}) \subseteq R(C)_+ \) is the set of fuzzy soft real numbers. A fuzzy soft real number is a fuzzy set on the set of all soft real set \( F:E \rightarrow B(R) \) where \( R \) is set of real numbers, \( \sigma(\hat{a}) \subseteq R(C)_+ \) is the set of fuzzy soft real numbers.

Definition 1.1 ([3]): A Fuzzy set \( A \) in \( U \) is characterized by a function with domain as \( U \) and values in \([0; 1]\). The collection of all fuzzy set \( U \) is \( P(\hat{U}) \).

Definition 1.2 ([1]): A pair \((F; E)\) is called a soft set (over \( U \)) if and only if \( F \) is a mapping of \( E \) into the set of all sub set of the set \( U \).

In other words, the soft set is a parametrized family of sub set of the set \( U \). Every set \( F(e) \), \( e \in E \), from this family may be considered as the set of \( \varepsilon \)-approximate elements of the soft set \((F; E)\), or as the set of \( \varepsilon \)-approximate elements of the soft set.

Definition 1.3 ([4]): Let \( C \subseteq E \) then the map \( F_{G}: C \rightarrow P(\hat{0}) \), defined by \( F_{G}(\hat{e}) = \mu^{K}F_{G} \) \((a \subseteq \mu_{F_{G}}(\varepsilon) \in C \) and \( \mu_{F_{G}} \neq 0 \) if \( e \in E \). The set of all fuzzy soft set over \((U; E)\) is denoted by \( FS(U, E) \).

Definition 1.4 ([4]): The fuzzy soft set \( F_{G} \in FS(U, E) \) is called null fuzzy soft set and it is denoted by \( \Phi \). Here \( F_{G}(\varepsilon) = \hat{0} \) for every \( e \in E \).

Definition 1.5 ([4]): Let \( F_{G} \in FS(U, E) \) and \( F_{G}(\varepsilon) = \hat{1} \) for all \( e \in E \). Then \( F_{G} \) is called absolute fuzzy soft set and it is denoted by \( \hat{E} \).

2. Main Results

We begin this section by introducing the notion of lower semi continuity in the context of C*-algebra valued fuzzy soft metric space. And we proved that many of the known fixed point theorems can be deduced from caristi’s mapping.

Definition 2.1: Let \( C \subseteq E \) and \( \hat{E} \) be the absolute fuzzy soft set. Let \( \hat{C} \) denote the C*-algebra. The C*-algebra valued fuzzy soft metric using fuzzy soft points is defined as a mapping \( \hat{d}_{\hat{C},\hat{E}} : \hat{E} \times \hat{E} \rightarrow \hat{C} \) satisfying the following conditions

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Example 2.4: A sequence \( \{ F_{n} \} \) in a \( C^* \)-algebra valued fuzzy soft metric space \((\mathcal{E}, \mathcal{C}, d_{C^*})\) is said to converge to \( F_{e} \) as \( \mathcal{E} \rightarrow E \) if for every \( \bar{e} \in \mathcal{E} \), \( \forall \delta > 0 \), there exists \( N = N(\bar{e}, \delta) \) such that \( d_{C^*}(F_{n}; F_{e}) < \delta \) if \( d_{C^*}(F_{n}, F_{e}) < \delta \) for all \( n \in N \). Then \( d_{C^*}(F_{n}, F_{e}) \to 0 \) as \( n \to \infty \).

Definition 2.3: A sequence \( \{ F_{n} \} \) in a \( C^* \)-algebra valued fuzzy soft metric space \((\mathcal{E}, \mathcal{C}, d_{C^*})\) is said to be Cauchy sequence. If to every \( \bar{e} \in \mathcal{E} \), there exists \( \delta \) and \( \forall n, m \geq N \), \( d_{C^*}(F_{n}, F_{m}) < \delta \), \( \forall \bar{e} \in \mathcal{E} \) such that \( d_{C^*}(F_{n}, F_{m}) < \delta \) for all \( n, m, \delta \) then the sequence \( \{ F_{n} \} \) is complete. A complete \( C^* \)-algebra valued fuzzy soft metric space is called complete \( C^* \)-algebra valued fuzzy soft metric space.

Example 2.4: Let \( C \in C \subseteq R \) and \( E \subseteq R \), let \( \mathcal{E} \) be absolute fuzzy soft set and \( \mathcal{C} = M_{2}(R(C)) \), define \( d_{C^*} : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \) by \( d_{C^*}(F_{n}, F_{e}) = \left[ \begin{array}{c} 0 \\ \inf_{\bar{e}} \| (F_{n}(\bar{e})) \| \\ \sup_{\bar{e}} \| (F_{n}(\bar{e})) \| \end{array} \right] \) for all \( \bar{e} \in \mathcal{E} \). Then \((\mathcal{E}, \mathcal{C}, d_{C^*})\) is a complete \( C^* \)-algebra valued fuzzy soft metric space.

Definition 2.5: A sequence \( \{ F_{n} \} \) in a \( C^* \)-algebra valued fuzzy soft metric space \((\mathcal{E}, \mathcal{C}, d_{C^*})\) is said to be Cauchy sequence. If to every \( \mathcal{E} \to E \), there exists \( \delta \) and \( \forall n, m \geq N \), \( d_{C^*}(F_{n}, F_{m}) < \delta \) for all \( n, m, \delta \) then the sequence \( \{ F_{n} \} \) is complete. A complete \( C^* \)-algebra valued fuzzy soft metric space is called complete \( C^* \)-algebra valued fuzzy soft metric space.

Example 2.6: Let \( E = \{ e_{1}, e_{2}, e_{3} \} \), \( U = \{ a, b, c, d \} \) and \( C \subseteq D \subseteq E \) are two subsets of \( E \), where \( C = \{ e_{1}, e_{2}, e_{3} \} \) and \( D = \{ e_{1}, e_{2}, e_{3}, e_{4} \} \) define fuzzy soft set as \((F_{n}, C)\), \( n = 1, 2, 3, 4 \), \( e_{1} = \{ a \}, e_{2} = \{ b, c \}, e_{3} = \{ d \} \). Then \((\mathcal{E}, \mathcal{C}, d_{C^*})\) is a complete \( C^* \)-algebra valued fuzzy soft metric space.

Definition 2.7: Let \((\mathcal{E}, \mathcal{C}, d_{C^*})\) be a \( C^* \)-algebra valued fuzzy soft metric space. Let \( T : \mathcal{E} \to \mathcal{E} \) be a self-mapping, we say that \( T \) is \( C^* \)-algebra valued fuzzy soft contractive mapping on \( \mathcal{E} \). If there exists \( \bar{e} \in \mathcal{E} \), \( \forall \| T(F_{n}) \| < \| F_{n} \| \) such that \( \| T(F_{n}) \| < \| F_{n} \| \) for all \( n \to \infty \), then \( \mathcal{E} \to \mathcal{E} \) is complete. Then \( \mathcal{E} \to \mathcal{E} \) is complete.

Theorem 2.8: Let \((\mathcal{E}, \mathcal{C}, d_{C^*})\) be a complete \( C^* \)-algebra valued fuzzy soft metric space and suppose \( \mathcal{E} \to \mathcal{E} \) be lower semi continuous and bounded below function. Let the self-mapping \( T : \mathcal{E} \to \mathcal{E} \) satisfies for all \( F_{n} \in \mathcal{E} \), \( d_{C^*}(T(F_{n}), F_{n}) \leq \Gamma(F_{n}) - \Gamma(T(F_{n})) \).

Then \( T \) has a fixed point.
There exists $F_n \in \hat{E}$ such that $F_n \to F_n'$ as $n \to \infty$. Since $\Gamma$ is lower semi continuous, by (4), for all $k \in N$, we get
\[ \|\Gamma(F_n')\| \leq \liminf_{n \to \infty} \|\Gamma(F_n)\| = \inf_{n \in \mathbb{N}} \|\Gamma(F_n)\| \leq \|\Gamma(F_n)\| \]

Next, we prove $\hat{\Gamma}(F_n) = \hat{F}_n'$. For $m \geq n$, $m \in N$ by (3) and (5), we get that
\[ \|\hat{\Gamma}(F_n) - \hat{\Gamma}(F_m)\| \leq \sum_{i=n}^{m-1} \|\hat{\Gamma}(F_{i+1}) - \hat{\Gamma}(F_i)\| \leq \|\hat{\Gamma}(F_n) - \hat{\Gamma}(F_m)\| \]

Since $F_n \to F_n'$ as $m \to \infty$. The inequality (6) implies
\[ \|\hat{\Gamma}(F_n) - \hat{\Gamma}(F_m)\| \leq \|\hat{\Gamma}(F_n) - \hat{\Gamma}(F_m)\|^{1/n} \text{ for all } \in N \]

Therefore, $\lim_{n \to \infty} \|\hat{\Gamma}(F_n) - \hat{\Gamma}(F_m)\| = 0$, or equivalently, $F_n \to F_n'$ as $n \to \infty$. By the uniqueness of limit of a sequence, we have $F_n = F_n'$. Hence we show $T(F_n) = \hat{\Gamma}(F_n)$. Since $S(F_n) \neq 0$ and $S(F_n') \subseteq S(F_n)$, we get $S(F_n') = \{F_n'\}$. We get $S(F_n') = \{F_n'\}$. On the other hand, by (1), we know $T(F_n') \in S(F_n')$. Hence it must be $T(F_n') = F_n'$, that is $T$ has a fixed point.

**Example 2.9:** Let $E=C=[0, 1]$ and $\hat{E}$ be a absolute fuzzy soft set, that $\hat{E}$ is $\hat{E}$ for all $e \in \hat{E}$, and $\hat{C} = M_{1}(R(C)+)$ be a $C_a$-algebra with partial order as given in Example 2.6. Defined $\hat{d}_C: \hat{E} \times \hat{E} \to \hat{C}$ by
\[ \hat{d}_C(F_n, F_n') = \{0, 1\} \text{ where } F_n, F_n' \in \hat{E} \] and
\[ i = \inf \{ \mu_{d}(F_n(s)) \to \mu_{d}(F_n')(s) \} \text{ for all } s \in \hat{C} \] and $\hat{E} \to \hat{C}$ by
\[ \Gamma(F_n) = \left\{ \begin{array}{ll} F_n & 0 \\ \{F_n' \} & 1 \end{array} \right.$
be a continuous mapping and $T: \hat{E} \to \hat{E}$ be given as $T(F_n) = F_n'$. Then it is easy to see that all the conditions of Theorem 2.8 are satisfied and $T$ has a fixed point.

**Corollary 2.1.1:** Let $(\hat{E}, \hat{C}, \hat{d}_C)$ be a complete $C_a$-algebra valued fuzzy soft metric space and suppose. Suppose the self-mapping $T: \hat{E} \to \hat{E}$ satisfies for all $F_n, F_n' \in \hat{E}$
\[ \hat{d}_C(F_n, F_n') \leq \Gamma(F_n'') - \Gamma(T(F_n'), T(F_n'')). \]

Where $\hat{E} \times \hat{E} \to \hat{C}$ is lower semi continuous function with respect to first variable. Then $T$ has a unique fixed point in $\hat{E}$.

**Proof:** for any $F_n \in \hat{E}$, we define $F_n = TF_n$ and $\Gamma(T(F_n)) = \hat{\Gamma}(F_n')$, then for each $F_n \in \hat{E}$, we have $\hat{d}_C(F_n, T(F_n)) \leq \|\Gamma(T(F_n)) - \Gamma(F_n')\|$. Since $\hat{E}$ is lower semi continuous function. Thus, we can applying Theorem 2.8 lead us to conclude the appropriate result.

To see the uniqueness of fixed point, Suppose $F_n', F_n''$ are two distinct fixed points of $T$. Then we have
\[ \hat{\bar{d}} \leq \hat{d}_C(F_n', F_n'') \leq \|\Gamma(F_n'') - \Gamma(T(F_n'), T(F_n''))\| \leq \Gamma(F_n', F_n'') - \|\Gamma(F_n', F_n'')\| \leq \hat{\bar{d}} \]

Therefore, we get $F_n' = F_n''$.

**Corollary 2.1.2:** Let $(\hat{E}, \hat{C}, \hat{d}_C)$ be a complete $C_a$-algebra valued fuzzy soft metric space and suppose. Suppose the self-mapping $T: \hat{E} \to \hat{E}$ satisfies for all $F_n, F_n' \in \hat{E}$
\[ \hat{d}_C(T(F_n'), T(F_n'')) \leq \Gamma(T(F_n'), T(F_n'')) \]

Where $\hat{E} \subseteq \hat{F}_n \in \hat{E}$, then $T$ has a unique fixed point in $\hat{E}$.

**Proof:** Let $\Gamma(T(F_n'), T(F_n'')) = \hat{d}_C(F_n, F_n')$. Then (9) show that
\[ \|\Gamma(T(F_n'), T(F_n''))\| \leq (1 - \alpha) \hat{d}_C(F_n, F_n') \]

Therefore, by applying Corollary 2.1.1 one can conclude that $T$ has a unique fixed point in $\hat{E}$.

**Corollary 2.1.3:** Let $(\hat{E}, \hat{C}, \hat{d}_C)$ be a complete $C_a$-algebra valued fuzzy soft metric space and suppose. Suppose the self-mapping $T: \hat{E} \to \hat{E}$ satisfies for all $F_n, F_n' \in \hat{E}$
\[ \hat{d}_C(F_n, F_n') \leq \Gamma(F_n', F_n'') - \Gamma(T(F_n'), T(F_n'')) \]

Where $\hat{E} \times \hat{E} \to \hat{C}$ is lower semi continuous function with respect to first variable. Then $T$ has a unique fixed point $\hat{E}$.

**Proof:** for each $F_n \in \hat{E}$, we define $F_n = TF_n$, then $F_n = T^2F_n$ and $\Gamma(T(F_n')) = \Gamma(F_n')$, $T(F_n')$. Then for each $F_n \in \hat{E}$
\[ \hat{d}_C(F_n, T(F_n')) \leq \|\Gamma(T(F_n')) - \Gamma(F_n')\| \]

Since $\hat{E}$ is lower semi continuous function. Thus, we can applying Theorem 2.8 lead us to conclude the appropriate result.

**3. Conclusion**

In the present research, we have presented unique fixed point results on various Caristi’s type contractive conditions defined on $\hat{E}$-algebra valued fuzzy soft metric spaces and provided suitable examples that supports our main results.

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**References**


