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Research paper

Complex Multi-Fuzzy Relation for Decision Making using Uncertain Periodic Data

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Abstract

We introduce a new type of relations called complex multi-fuzzy relation (CMFR). The novelty of CMFR lies in the ability of complex multi- membership functions to achieve more range of values while handling uncertainty of data that is periodic in nature. The application of complex multi-fuzzy sets is then discussed in determining: the influence of modern methods of education on student performance, and the time required for the former to affect the latter. A comparison between different existing relations and CMFR to show the ascendancy of our proposed CMFR is provided. Thereafter, a few related concepts such as complement, union, intersection and inverse along with several propositions are discussed, followed by the composition of CMFR along with some related theorems. Finally the notions of symmetric, transitive, reflexive, and equivalence complex multi-fuzzy relations are established in our work.

Keywords: *Complex Multi-Fuzzy Set, Complex Multi-Fuzzy Relation, Multi-Fuzzy Set, Complex Fuzzy Set, Decision Making.*

1. Introduction

The concept of fuzzy set (FS) was introduced for the first time by Zadeh [1] to handle uncertainty in many fields of everyday life. Fuzzy set theory generalised the range values of classical set theory from the integer 0 and 1 to the interval [0, 1]. It was henceforth extended to various forms of fuzzy sets [2]-[5] and has proven to be very useful in many different fields [6]-[9]. However, there are many problems like complete colour characterization of colour images, taste recognition of food items and decision making problems with multi aspects which cannot be characterized by a single membership function of Zadeh's fuzzy sets. To overcome this problem, Sabu and Ramakrishnan [10]-[11] proposed the concept of the multi-fuzzy sets theory as a mathematical tool to deal life problems that have multi dimensional characterization properties. The theory of multi-fuzzy sets is an extension of theories of L-fuzzy sets [12] and Atanassov intuitionistic fuzzy sets [13].
Ramot et al. [14] extended the range of the membership function in fuzzy set from the interval [0,1] to the unit circle in the complex plane and called it complex fuzzy set which had progressed rapidly to complex fuzzy logic [15]. The complex fuzzy sets are used to represent information with uncertainty and periodicity. Alkouri and Salleh [16] introduced the concept of complex intuitionistic fuzzy set (CIFS) which is generalized from the innovative concept of a complex fuzzy set . Recently, Al-Qudah and Hassan[17] developed a hybrid model of complex fuzzy sets and multi-fuzzy sets, called the complex multi-fuzzy set (CMFS). This model is useful for handling problems with multi dimensional characterization properties.

Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets. Since the definition of fuzzy relation by Zadeh [18], a great deal of researchers conducted intensive studies on fuzzy relation such as Dutta and Chakraborty [19], Kheniche et al [20] and Kocak [21]. Later, complex fuzzy relation [14] was introduced, where "the CFRs represent both the degree of presence or absence of association, interaction, or interconnectedness, and the phase of association, interaction, or interconnectedness between the elements of two or more sets". Since then, several researchers [22]-[23] have discussed more properties on complex fuzzy relation. Burillo and Bustince [24]-[25] introduced the definition of Atanassov's intuitionistic fuzzy relation and studied some of its properties. Thereafter, Thomas and John [26] introduced the concept of multi- fuzzy rough sets by combining the multi-fuzzy set and rough set models. They also defined a multi-fuzzy rough relation and studied its properties and operations. Alkouri and Saleh [27] introduced the concept of complex Atanassov's intuitionistic fuzzy relation and discussed many related concepts such as distance measure, Cartesian products, projections, and cylindric extensions.
We will first extend the discussion on multi-fuzzy relation further by proposing a new concept of complex multi-fuzzy

relation by adding a second dimension (phase term) to its multi-membership function. Complete characterization of many real life problems that have a periodic nature can then be handled by complex multi-fuzzy relation of the objects involved in these periodic relations. Then, the CMFS will be use together with a generalized algorithm to determine the degree and the total time of the influence of the modern methods on the students' performance and then deduce results that help to make decision in determine the most effective modern methods from these methods. A comparison between different existing relations and CMFR to show the ascendancy of our proposed CMFR will be provided. We will also define some opreations on CMFRs along with some theorems and propositions. Finally, we will define equivalence relation.
To facilitate our discussion, we first review some background on fuzzy sets, multi-fuzzy sets, complex fuzzy sets and complex multi-fuzzy sets. The notion of Cartesian product on CM^{k}FSs and complex multi-fuzzy relation are introduced. We will put forward a real-life application of complex multi-fuzzy relation to reveal its ability to describe and analyze real events. We introduce the basic set theoretic operations of complex multi-fuzzy relation such as complement, union, intersection and inverse operations along with a proposition. We introduce the composition of CMFRs along with some related theorems and properties. The notions of symmetric, transitive, reflexive, and equivalence complex multi-fuzzy relations are proposed. Finally, conclusions are presented. This enriches the study of vague fuzzy sets [28]-[31] and Q-fuzzy sets which are limited to certain data [32]-[35]. Decision making is no longer limited to deterministic forms of linear programing [36]-[40] and data envelopment analysis [41]-[43].

2. Preliminaries

In this section we will briefly recall the notions of fuzzy sets, multi-fuzzy sets and complex fuzzy sets which are relevant to our discussion.The complex multi-fuzzy set and its basic operations such as, complement, union and intersection is introduced.

First we shall recall the basic definitions of fuzzy sets developed by Zadeh in [1] followed by multi-fuzzy sets proposed by Sebastian and Ramakrishnan [10].

Definition 2.1. *[1] A fuzzy set A in a universe of discourse X is characterised by a membership function $\mu_A(x)$ that takes values in the interval $[0,1]$ for all $x \in X$.*

Definition 2.2. *[10] Let k be a positive integer and X be a non-empty set. A multi-fuzzy set A in X is a set of ordered sequences .*

$$A = \{\langle x, \mu_1(x), ..., \mu_k(x)\rangle : x \in X\},$$

wehre $\mu_i : X \longrightarrow L_i = [0,1]$, $i = 1,2,...,k$.

The function $\mu_A(x) = (\mu_1(x), ..., \mu_k(x))$ is called the multi-membership function of multi-fuzzy sets $\tilde{S}$, k is called a dimension of A. The set of all multi-fuzzy sets of dimension k in X is denoted by $M^k FS(x)$.

The concept of complex fuzzy sets was introduced by Ramot et al.[14] as a generalization of the concept of fuzzy sets by adding the phase term to the definition of fuzzy sets. In other words, they extended the range of membership function from the interval $[0,1]$ to unit circle in the complex plane.

In the following, we give some basic definitions and set theoretic operations of complex fuzzy sets.

Definition 2.3. *[14] A complex fuzzy set (CFS) A, defined on a universe of discourse X, is characterised by a membership function $\mu_A(x)$ that assigns to any element $x \in X$ a complex-valued grade of membership in A. By definition, the values of $\mu_A(x)$ may receive all lying within the unit circle in the complex plane and are thus of the form $\mu_A(x) = r_A(x).e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, $r_A(x)$ and ω_A are both real-valued, and $r_A(x) \in [0,1]$. The CFS A may be represented as the set of ordered pairs*

$$A = \{(x, \mu_A(x)) : x \in X\} = \{(x, r_A(x).e^{i\omega_A(x)}) : x \in X\}.$$

Definition 2.4. *[22] Let A_n, $n = 1,2,...,N$ be N complex fuzzy sets on X, and $\mu_{A_n}(x) = r_{A_n}(x).e^{i\omega_{A_n}(x)}$ their membership functions, respectively. The complex fuzzy Cartesian product of A_n, $n = 1,\ldots,N$, denoted $A_1 \times \cdots \times \tilde{A}_N$, is specified by a function*

$$\mu_{A_1\times\cdots\times A_N} = r_{A_1\times\cdots\times A_N}(x).e^{i\omega_{A_1\times\cdots\times A_N}(x)}$$

$$= min(r_{A_1}(x_1), \ldots, r_{A_N}(x_N)).e^{i\ \min(\omega_{A_1}(x_1),\ldots,\omega_{A_N}(x_N))},$$

where $x = (x_1, \ldots, x_N) \in \underbrace{X \times \cdots \times X}_{N}$.

Definition 2.5. *[23] Let X,Y and Z be universes, A a complex fuzzy relation of X and Y and B a complex fuzzy relation from Y and Z. Then we say a composition of A and B, denoted $A \circ B$, is a complex fuzzy relation of X and Z, is specified by a function*

$$\mu_{A\circ B}(x,z) = r_{A\circ B}(x,z).e^{i\omega_{A\circ B}(x,z)}$$

$$= \sup_{y\in Y} min(r_A(x,y), r_B(y,z)) \cdot e^{i \sup_{y\in Y} \min(\omega_A(x,y),\omega_B(y,z))}.$$

The novelty of the complex multi fuzzy sets introduced by Al-Qudah and Hassan [17] lies in the ability of complex multi-membership functions to achieve more range of values while handling uncertainty of data that is periodic in nature. Some of the basic concepts pertaining to complex multi-fuzzy sets are as follows.

Definition 2.6. *[17] Let k be a positive integer and X be a non-empty set. A complex multi-fuzzy set (CMFS) A, defined on a universe of discourse X, is characterised by a multi-membership function $\mu_A(x) = (\mu^j_A(x))_{j\in k}$, that assigns to any element $x \in X$ a complex-valued grade of multi- membership functions in A. $\mu_A(x)$ may all lie within the unit circle in the complex plane, and are thus of the form $\mu_A(x) = (r^j_A(x).e^{i\omega^j_A(x)})_{j\in k}$, $(i = \sqrt{-1})$, $\left(r^j_A(x)\right)_{j\in k}$ are real-valued function and $\left(r^j_A(x)\right)_{j\in k} \in [0,1]$. The CMFS A may be represented as the set of ordered sequence*

$$A = \{(x, \left(\mu^j{}_A(x) = a_j\right)_{j\in k}) : x \in X\}$$

$$= \{x, ((r^j_A(x).e^{i\omega^j_A(x)})_{j\in k}) : x \in X\}.$$

where $\mu^j{}_A : X \to L_j = \{a_j : a_j \in C, |a_j| \le 1\}$ for $j = 1,2,...,k$.

The function $(\mu_A(x) = r_A^j(x).e^{i\omega_A^j(x)})_{j\in k}$ *is called the complex multi-membership function of complex multi-fuzzy set A, k is called the dimension of A. In other words, a complex multi-fuzzy set can be regarded as a general complex fuzzy set using ordinary complex fuzzy sets as its building blocks. The set of all complex multi-fuzzy sets of dimension k in X is denoted by* $\mathcal{CM}^k\mathcal{FS}(X)$.

Definition 2.7. *[17] Let*

$$A = \{x, ((r_A^j(x).e^{i\omega_A^j(x)})_{j\in k}) : x \in X\}$$

and

$$B = \{x, ((r_B^j(x).e^{i\omega_B^j(x)})_{j\in k}) : x \in X\}$$

be two complex multi-fuzzy sets of dimension k in X. We define the following relations and operations.

1. $A \subset B$ if and only if $r_A^j(x) \le r_B^j(x)$ and $\omega_A^j(x) \le \omega_B^j(x)$, for all $x \in X$ and $j = 1,2,...,k$.
2. $A = B$ if and only if $r_A^j(x) = r_B^j(x)$ and $\omega_A^j(x) = \omega_B^j(x)$, for all $x \in X$ and $j = 1,2,...,k$.
3. $A \cup B = \{\langle x, r_{A\cup B}^j(x).e^{i\omega_{A\cup B}^j(x)}\rangle : x \in\}$ $= \{\langle x, \vee(r_A^j(x), r_B^j(x)).e^{i\max[\omega_A^j(x),\omega_B^j(x)]}\rangle : x \in X\}$, for all $j = 1,2,...,k$.
4. $A \cap B = \{\langle x, r_{A\cap B}^j(x).e^{i\omega_{A\cap B}^j(x)}\rangle : x \in\}$ $= \{\langle x, \wedge(r_A^j(x), r_B^j(x)).e^{i\min[\omega_A^j(x),\omega_B^j(x)]}\rangle : x \in X\}$, for all $j = 1,2,...,k$.
5. $A^c = \{x, [r_{A^c}^j(x).e^{i\omega_{A^c}^j(x)}]_{j\in k} : x \in X\}$ $= \{x, ([1 - r_A^j(x)].e^{i[2\pi - \omega_A^j(x)]})_{j\in k} : x \in X\}$,

where $\vee$ and $\wedge$ denote the max and min operator, respectively.

3. Complex Multi-Fuzzy Relation

We will propose the Cartesian product of two complex multi-fuzzy sets (CMFSs), define relations on complex multi-fuzzy sets and extend the concept of complex fuzzy relation [14] to complex multi-fuzzy relation. The notions of inverse relation, union relation and intersection relation are to be defined.

We begin by proposing the definition of Cartesian product between two CMFSs as follows.

Definition 3.1. *Let k be a positive integer and let* $\mathcal{A}$ *and* $\mathcal{B}$ *be complex multi-fuzzy sets of dimension k* ($\mathcal{CM}^k\mathcal{FS}s$) *on U and V, respectively. The Cartesian product* $A \times B$ *is a* $\mathcal{CM}^k\mathcal{FS}$ *defined by*

$$\mathcal{A}\times\mathcal{B} = \left\{ \left\langle (u,v), \mu^1_{\mathcal{A}\times\mathcal{B}}(u,v), \ldots, \mu^k_{\mathcal{A}\times\mathcal{B}}(u,v) \right\rangle : (u,v) \in U \times V \right\},$$

where
$\mu^j_{A\times B}(u,v) = min(r_A^j(u), r_B^j(v)) \cdot e^{i\min(\omega_A^j(u),\omega_B^j(v))}$, $\forall u \in U, v \in V$ and $j = 1,2,\ldots,k$.

Next, we present the concept of the complex multi-fuzzy relation (CMFR).

Definition 3.2. *Let k be a positive integer. A CMFRs* $R(U,V)$ *from U to V is a complex multi-fuzzy subset of the product space* $U \times V$. *The relation* $R(U,V)$ *is characterised by the complex multi-membership function* $\mu_R(u,v) = (\mu_R^j(u,v))_{j\in k}$ *, where* $u \in U$, $v \in V$ *and* $\mu_R(u,v)$ *assigns each pair* (u,v) *a complex-valued grade of multi-membership functions to the set* $R(U,V)$ *which may be represented as the set of ordered sequence*

$$R(U,V) = \left\{((u,v), (\mu_R^1(u,v), ..., \mu_R^k(u,v)) | (u,v) \in U \times V\right\}.$$

where
$$\mu_R^j(u,v) = r_R^j(u,v).e^{i\omega_R^j(u,v)} = min(r_{\mathcal{A}}^j(u), r_{\mathcal{B}}^j(v)).e^{i\ min(\omega_{\mathcal{A}}^j(u),\omega_{\mathcal{B}}^j(v))}.$$

$\forall u \in U, v \in V$ and $j = 1,...,k$.

The values $(\mu_R^j(u,v))_{j\in k}$ *may lie within the unit circle in the complex plane and both the amplitude terms* $r_R^j(u,v)$ *and the phase terms* $\omega_R^j(u,v)$ *are real-valued such that* $r_R^j(u,v) \in [0,1]$, $\forall j = 1,...,k$.

The complex multi-membership function $\mu_R^j(u,v)$ for $j = 1,...,k$ is to be interpreted in the following manner.

1. The amplitude terms $(r_R^j(u,v))_{j\in k}$ represent a degree of presence or absence of association, interaction, or interconnectedness between the elements of U and V.
2. The phase terms $(\omega_R^j(u,v))_{j\in k}$ represent the phase of association, interaction, or interconnectedness between the elements of U and V.

We now present an example of the relation between two complex multi-fuzzy sets. We will show the importance of the time frame in studying the relation between modern methods in education to students' performance.

Example 3.3. *Let U be the set of modern methods in education applied to a certain group of students, where* $U = \{u_1 =$ *using technology,* $u_2 =$ *using games,* $u_3 =$ *social media* $\}$. *Let V be the set of indicators that measure students' achievement and thire interaction, where* $V = \{v_1 =$ *academic achievement,* $v_2 =$ *emotional interaction,* $v_3 =$ *social interaction*$\}$.
Let $\mathcal{A}$ *and* $\mathcal{B}$ *be two complex multi-fuzzy sets of three dimensions over U and V respectively, defined as follows:*

$$\mathcal{A} = \left\{ \frac{\left\langle 1.0e^{i2\pi(11/12)}, 0.5e^{i2\pi(7/12)}, 0.2e^{i2\pi(2/12)}\right\rangle}{u_1}, \frac{\left\langle 0.3e^{i2\pi(4/12)}, 0.6e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)}\right\rangle}{u_2}, \frac{\left\langle 0.8e^{i2\pi(1/12)}, 0.6e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)}\right\rangle}{u_3} \right\},$$

$$\mathcal{B} = \left\{ \frac{\left\langle 0.9e^{i2\pi(12/12)}, 0.8e^{i2\pi(5/12)}, 0.1e^{i2\pi(1/12)}\right\rangle}{v_1}, \frac{\left\langle 0.4e^{i2\pi(5/12)}, 0.5e^{i2\pi(4/12)}, 0.3e^{i2\pi(8/12)}\right\rangle}{v_2}, \frac{\left\langle 0.1e^{i2\pi(12/12)}, 0.5e^{i2\pi(9/12)}, 0.9e^{i2\pi(2/12)}\right\rangle}{v_3} \right\}.$$

For any u in U and v in V, the membership values μ_1, μ_2 and μ_3 represent strong influence, average influence and minimal influence.

Suppose that the relation between $\mathcal{A}$ and $\mathcal{B}$ are measured over the limited time frame of 12 months. Let the complex multi-fuzzy relation be denoted as $R(\mathcal{A},\mathcal{B})$ is a subset of the product space $\mathcal{A}\times\mathcal{B}$. Thus the relation $R(\mathcal{A},\mathcal{B})$ measures the influence of modern methods in education to students' performance over the limited time of 12 months. Suppose the complex multi-fuzzy relations is

$$R(\mathcal{A},\mathcal{B}) = \Big\{ \frac{\langle 0.9e^{i2\pi(\frac{11}{12})}, 0.5e^{i2\pi(\frac{5}{12})},\ 0.1e^{i2\pi(\frac{1}{12})}\rangle}{(u_1,v_1)}, \frac{\langle 0.4e^{i2\pi(\frac{5}{12})},\ 0.5e^{i2\pi(\frac{4}{12})},\ 0.2e^{i2\pi(\frac{2}{12})}\rangle}{(u_1,v_2)}, \frac{\langle 0.1e^{i2\pi(\frac{11}{12})},\ 0.5e^{i2\pi(\frac{7}{12})},\ 0.2e^{i2\pi(\frac{2}{12})}\rangle}{(u_1,v_3)}, \frac{\langle 0.3e^{i2\pi(\frac{4}{12})},\ 0.8e^{i2\pi(\frac{4}{12})},\ 0.1e^{i2\pi(\frac{1}{12})}\rangle}{(u_2,v_1)}, \frac{\langle 0.3e^{i2\pi(\frac{4}{12})},\ 0.5e^{i2\pi(\frac{4}{12})},\ 0.1e^{i2\pi(\frac{4}{12})}\rangle}{(u_2,v_2)}, \frac{\langle 0.1e^{i2\pi(\frac{4}{12})},\ 0.5e^{i2\pi(\frac{4}{12})},\ 0.1e^{i2\pi(\frac{2}{12})}\rangle}{(u_2,v_3)}, \frac{\langle 0.8e^{i2\pi(\frac{1}{12})},\ 0.6e^{i2\pi(\frac{4}{12})}, 0.1e^{i2\pi(\frac{1}{12})}\rangle}{(u_3,v_1)}, \frac{\langle 0.4e^{i2\pi(\frac{1}{12})}, 0.5e^{i2\pi(\frac{4}{12})},\ 0.1e^{i2\pi(\frac{4}{12})}\rangle}{(u_3,v_2)}, \frac{\langle 0.1e^{i2\pi(\frac{1}{12})},\ 0.5e^{i2\pi(\frac{4}{12})},\ 0.1e^{i2\pi(\frac{2}{12})}\rangle}{(u_3,v_3)} \Big\}.$$

In our example, the amplitude terms of the membership values represent the degree of influence of the modern methods in education on student's performance, whereas the phase terms represent the time that elapses before the influence of modern methods.

To illustrate what we meant, we provide an example of scenarios that could possibly occur in this context, instead of providing the complete set for $R(A,B)$. For example, in the complex multi-fuzzy value $\frac{\langle 0.9e^{i(11/12)\pi},\ 0.5e^{i(5/12)\pi},\ 0.1e^{i(1/12)\pi}\rangle}{(u_1,v_1)}$, the first membership value $0.9e^{i(11/12)\pi}$ indicates that using technology in the classroom strongly influences students' academic achievement with degree 0.9 and this influence need 11 month which is a very long time (phase term with value very close to one) to become evident in the students' academic performance. The second membership value $0.5e^{i(5/12)\pi}$ indicates that there is moderate influence of using technology on the academic achievement of the students' with degree 0.5 and the time required for this effect is 5 months which is a medium time. For the three membership value $0.1e^{i(1/12)\pi}$ reveals a weak influence of using technology on the academic achievement of the students', since the amplitude term 0.1 is very close to zero and this effect 1 is not clear because it is a period of very little influence.

Suppose that experts in the educational field would like to know the extent of the effect of modern methods of education by using technology, games or social media on the students' performance. Thus, this case study deals with three different methods; using technology, games and social media, each method will be studied separately. For example, the first method will evaluate whether the use of technology in the classroom has any effect on one of the indicators that measure students' achievement and their interaction.

Next, the CMFR $R(\mathcal{A},\mathcal{B})$ is used together with a generalized algorithm to solve the decision-making problem. In this algorithm, the first three steps will be applied to all cases. Step 4 to Step 7 deal with each case separately. The algorithm steps are given as follows.

1. Input the CMFSs $\mathcal{A}$ and $\mathcal{B}$.
2. Calculate the CMFRs $R(\mathcal{A},\mathcal{B})$ of $\mathcal{A}\times\mathcal{B}$.
3. Convert the CMFR $R(\mathcal{A},\mathcal{B})$, which is actually a CMFS to the MFS $\vec{R}(\mathcal{A},\mathcal{B})$ by obtaining the weighted aggregation values of $\mu^j_R(u,v)$, $\forall j = 1,2,..,k$ and $(u,v)\in U\times V$ as in the following formulas:
$$\mu^j_{\vec{R}}(u,v) = \nu_1 r^j_R(u,v) + \nu_2(1/2\pi)\omega^j_R(u,v),$$
where $r^j_R(u,v)$ and $\omega^j_R(u,v)$ (for $j = 1,2,...,k$) are the amplitude and phase terms in the CMFS $R(\mathcal{A},\mathcal{B})$, respectively. $\mu^j_{\vec{R}}(u,v)$ is the multi-membership function in the MFS $\vec{R}(\mathcal{A},\mathcal{B})$ and ν_1, ν_2 are the weights for the amplitude terms (degrees of influence) and the phase terms (times of influence), respectively, where ν_1 and $\nu_2 \in [0,1]$ and $\nu_1 + \nu_2 = 1$.
4. Compute the comparison table for each ordered pair of the elements $u\in U$ and $v_j \in V$ (for $j = 1,2,3$).
5. Compute the score of each element $(u,v)\in U\times V$ by taking the sum of the row and column of each element, by $\mathcal{R}_\ell$ and $\mathcal{C}_\ell$, respectively.
6. Find the values of the score $s_\ell = \mathcal{R}_\ell - \mathcal{C}_\ell$ for each element $(u,v)\in U\times V$
7. Determine the value of the highest score $h = max_{(u,v)\in U}\{s_\ell\}$.

Then the decision is to choose element (u,v) as the optimal or best solution to the problem. If there are more than one element with the highest s_ℓ score, then any one of those elements can be chosen as the optimal solution.

Definition 3.4. *(Comparison table). It is a square table in which number of rows and number of columns are equal and both are labeled by the object name of the universe such as* $((u_1,v_1),...,(u_1,v_j)),((u_2,v_1),...,(u_2,v_j)),((u_i,v_1),...,(u_i,v_j))$ *and the entries* d_{nm} *where* d_{ij} *= the number of parameters for which the value of* d_n *exceeds or equal to the value of* d_m*.*

To execute the above steps, we will use the CMFR $R(\mathcal{A},\mathcal{B})$ to apply Step 3 to Step 7. To illustration step 3, we assume that the weight for the amplitude term is $\nu_1 = 0.6$ and the weight for the phase term is $\nu_2 = 0.4$.

Now, to convert the CMFS $R(\mathcal{A},\mathcal{B})$ to MFS $\vec{R}(\mathcal{A},\mathcal{B})$, to obtain the weighted aggregation values of $\mu^j_R(u,v)$, $\forall(u,v)\in U\times V$ and $j = 1,2,...,k$. To illustrate this step, we calculate $\mu^j_R(u,v)$, when $(u,v) = (u_1,v_1)$ as shown below.

$$\mu^1_{\vec{R}}(u_1,v_1) = \nu_1 r^1_R(u_1,v_1) + \nu_2(1/2\pi)\omega^1_R(u_1,v_1),$$
$$= (0.6)(0.9) + (0.4)(1/2\pi)(2\pi)(11/12) = 0.907.$$
$$\mu^2_{\vec{R}}(u_1,v_1) = \nu_1 r^2_R(u_1,v_1) + \nu_2(1/2\pi)\omega^2_R(u_1,v_1),$$
$$= (0.6)(0.5) + (0.4)(1/2\pi)(2\pi)(5/12) = 0.467.$$
$$\mu^3_{\vec{R}}(u_1,v_1) = \nu_1 r^3_R(u_1,v_1) + \nu_2(1/2\pi)\omega^3_R(u_1,v_1),$$
$$= (0.6)(0.1) + (0.4)(1/2\pi)(2\pi)(1/12) = 0.093.$$

Then, for $(u,v) = (u_1,v_1)$, the multi-fuzzy values is $(\mu^1_{\vec{R}}(u_1,v_1),\mu^2_{\vec{R}}(u_1,v_1),\mu^3_{\vec{R}}(u_1,v_1)) = (0.907,0.467,0.093)$.

In the same manner, we calculate the other multi-fuzzy values , $\forall(u,v)\in U\times V$ and the results are displayed in Table 1.

Table 1: Values of $\vec{R}(\mathcal{A},\mathcal{B})$.

$\vec{R}$	v_1	v_2	v_3
u_1	$(0.907,0.467,0.093)$	$(0.407,0.433,0.187)$	$(0.427,0.533,0.187)$
u_2	$(0.313,0.613,0.093)$	$(0.313,0.433,0.193)$	$(0.193,0.433,0.127)$
u_3	$(0.513,0.493,0.093)$	$(0.273,0.433,0.193)$	$(0.093,0.433,0.127)$

Now, we apply Step 4 to Step 7 to each method separately. We begin by testing the first method to find out the extent of the relationship between the use of technology in education and indicators that measure students' achievement and their interaction.

Table 2 represent the relation between using technology in education (u_1) and indicators that measure students' achievement and their interaction of v_1, v_2 and v_3, respectively.

Table 2: Tabular representation of the relation between (u_1) with (v_1,v_2,v_3).

$\vec{R}_{(u_1,V)}$	$\mu^1_{\vec{R}}(u,v)$	$\mu^2_{\vec{R}}(u,v)$	$\mu^3_{\vec{R}}(u,v)$
(u_1,v_1)	0.907	0.467	0.093
(u_1,v_2)	0.407	0.433	0.187
(u_1,v_3)	0.427	0.533	0.187

Computing the comparison table. The results are display in Table 3.

Table 3: Comparison table.

(u_1,V)	(u_1,v_1)	(u_1,v_2)	(u_1,v_3)
(u_1,v_1)	3	2	1
(u_1,v_2)	1	3	1
(u_1,v_3)	2	3	3

Let $\mathcal{R}_\ell$ and $\mathcal{C}_\ell$ represent the score of the each sum of the rows and columns, respectively. These values are given in Table 4.

Table 4: The score $r_\ell = \mathcal{C}_\ell - \mathcal{K}_\ell$.

(u_1,V)	Row sum ($\mathcal{C}_\ell$)	Column sum ($\mathcal{K}_\ell$)	Final score (r_ℓ)
(u_1,v_1)	6	6	0
(u_1,v_2)	5	8	-3
(u_1,v_3)	8	5	3

From Table 4, $max_{(u,v)\in U}\{r_\ell\} = r_3$. Thus, the optimal decision is to select (u_1,v_3). Therefore, we conclude that the use of technology in the classroom strongly influences students' social interaction.

Same calculations mentioned above will be utilized to explore the effect of games and social media on students' achievement and their interaction. For the games method, we get (u_2,v_2). as the optimal solution, which means that the use of games in the classroom strongly influences students' emotional interaction. In the meantime, (u_3,v_3) as the optimal solution was obtained for social media effect. This means that the use of social media in the classroom has the most effects on the students' academic performance.

3.1. Comparison Between CMFR and the Existing Methods

In this section, we will compare the CMFR with two existing methods of complex fuzzy relation [14] and multi-fuzzy relation [26].
As an extension of fuzzy relation [18] and intuitionistic fuzzy relation [24], multi-fuzzy relation [26] was developed to measure the degree of the interaction between two multi-fuzzy sets, with the ability to handle all types of uncertainties which is beyond the scope of fuzzy and intuitionistic fuzzy relations. However, multi-fuzzy relation fails to deal with problems that involve two-dimensional information/date i.e., two different types of information/data pertaining to the problem parameters.

Complex fuzzy relation can deal with problems using single complex membership function, but many real problems in our life such as complete colour characterization of colour images, taste recognition of food items, decision making problems with multi aspects and others cannot completely be characterized by a single membership function. However, the structure of CMFR can completely characterize these problems. On the other hand, multi-fuzzy relation cannot handle problems that have a periodic nature, as its structure lacking the phase term, whereas CMFR has an ability to represent the problems with uncertainty and periodicity using the phase term features.

From Example 3.3, it is clear that it is not possible to apply multi-fuzzy relation [26] to describe the effect of the modern methods in education on the students' performance at a certain period of time, since it is unable to represent variables in two dimensions. In other words multi-fuzzy set cannot represent both the degree and phase of the influence simultaneously. However, the structure of complex multi-fuzzy relation provides the ability to describe these two variables simultaneously.

4. Operations on Complex Multi-Fuzzy Relation

We will now introduce some basic operations on CMFR such as the inverse, complement,union and interaction operations.

In the following, we introduce the concept of the inverse complex multi-fuzzy relation and give a proposition on the inverse complex multi-fuzzy relation.

Definition 4.1. *If $\mathcal{A},\mathcal{B} \in \mathcal{CM}^k\mathcal{FS}(U,V)$ and R is a $\mathcal{CM}^k\mathcal{FR}$ from $\mathcal{A}$ to $\mathcal{B}$, then R^{-1} is the inverse complex multi-fuzzy relation R from $\mathcal{B}$ to $\mathcal{A}$, defined as follows:*

$$R^{-1} = \left\{((v,u),(\mu^1_{R^{-1}}(u,v),...,\mu^k_{R^{-1}}(u,v))|(u,v)\in U\times V\right\}.$$

where $\mu^j_{R^{-1}}(u,v) = \mu^j_R(v,u)$ *for* $j = 1,...,k$.

It is clear from the above definition that the inverse of R is *defined by reversing the order of every pair belonging to R.*

Proposition 4.2. *If $\mathcal{A},\mathcal{B} \in \mathcal{CM}^k\mathcal{FS}(U,V)$ and suppose that R and S are two $\mathcal{CM}^k\mathcal{FR}$ from $\mathcal{A}$ to $\mathcal{B}$, then the following results hold true.*

1. $(R^{-1})^{-1} = R$
2. $R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}$

Proof
$\forall (u,v) \in U \times V$ and $j = 1,...,k$,

1. since R^{-1} is a $\mathcal{CM}^k\mathcal{FS}$ from $\mathcal{B}$ to $\mathcal{A}$, we have $\mu^j_{R^{-1}}(v,u) = \mu^j_R(u,v)$.
   Thus, $\mu^j_{(R^{-1})^{-1}}(u,v) = \mu^j_{R^{-1}}(v,u) = \mu^j_R(u,v)$.
   Therefore $(R^{-1})^{-1} = R$.

2. $R \subseteq S \Rightarrow \mu^j_R(u,v) \le \mu^j_S(u,v)$
   $\Rightarrow \mu^j_R(u,v) = \mu^j_{R^{-1}}(v,u) \le \mu^j_S(u,v) = \mu^j_{S^{-1}}(v,u)$
   $\Rightarrow \mu^j_{R^{-1}}(v,u) \le \mu^j_{S^{-1}}(v,u)$
   $\Rightarrow R^{-1} \subseteq S^{-1}$.

We will now define the complement, union and interaction operations on CMFRs.

Definition 4.3. *Let $j \in \{1,...,k\}$ and let $\mathcal{A},\mathcal{B} \in \mathcal{CM}^k\mathcal{FS}(U,V)$ and R and S is a $\mathcal{CMFR}$ from $\mathcal{A}$ to $\mathcal{B}$, where:*

$$R = \left\{ \left\langle (u,v), \mu^j_R(u,v) = r^j_R(u,v).e^{i\omega^j_R(u,v)} \right\rangle : (u,v) \in U \times V \right\},$$
$$S = \left\{ \left\langle (u,v), \mu^j_S(u,v) = r^j_S(u,v).e^{i\omega^j_S(u,v)} \right\rangle : (u,v) \in U \times V \right\}.$$

1. *The complex multi-fuzzy complement relations of R, denoted by A^c, is defined as*
$$R^c = \{ \langle (u,v), \mu^j_{R^c}(u,v) \rangle : (u,v) \in U \times V \},$$
   where
   $\mu^j_{R^c}(u,v) = r^j_{R^c}(u,v).e^{i\omega^j_{R^c}(u,v)}$
   $= [1 - r^j_R(u,v)].e^{i[2\pi - \omega^j_R(u,v)]}$,
   for all $u \in U, v \in V$ and $j \in \{1,...,k\}$.

2. *The complex multi-fuzzy union relations of R and S, denoted by $R \cup S$, is defined as*
$$R \cup S = \{ \langle (u,v), \mu^j_{R\cup S}(u,v) \rangle : (u,v) \in U \times V \},$$
   where
   $\mu^j_{R\cup S}(u,v) = r^j_{R\cup S}(u,v).e^{i\omega^j_{R\cup S}(u,v)}$
   $= max(r^j_{\mathcal{A}}(u), r^j_{\mathcal{B}}(v)).e^{i\max(\omega^j_{\mathcal{A}}(u),\omega^j_{\mathcal{B}}(v))}$,
   for all $u \in v \in V$ and $j \in \{1,...,k\}$.

3. *The complex multi-fuzzy interaction relations of R and S, denoted by $R \cap S$, is defined as*
$$R \cap S = \{ \langle (u,v), \mu^j_{R\cap S}(u,v) \rangle : (u,v) \in U \times V \},$$
   where
   $\mu^j_{R\cap S}(u,v) = r^j_{R\cap S}(x,y).e^{i\omega^j_{R\cap S}(u,v)}$
   $= min(r^j_{\mathcal{A}}(u), r^j_{\mathcal{B}}(v)).e^{i\min(\omega^j_{\mathcal{A}}(u),\omega^j_{\mathcal{B}}(v))}$,
   for all $u \in v \in V$ and $j \in \{1,...,k\}$.

Proposition 4.4. *If $\mathcal{A},\mathcal{B} \in \mathcal{CM}^k\mathcal{FS}(U \times V)$ and suppose that R and S are two $\mathcal{CM}^k\mathcal{FR}s$ from $\mathcal{A}$ to $\mathcal{B}$, then the following results hold true.*

1. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.
2. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Proof
$\forall (u,v) \in U \times V$ and $j = 1,...,k$.

1. $\mu^j_{(R\cup S)^{-1}}(v,u) = \mu^j_{(R\cup S)}(u,v)$
   $= \mu^j_R(u,v) \cup \mu^j_S(u,v)$
   $= \mu^j_{R^{-1}}(v,u) \cup \mu^j_{S^{-1}}(v,u)$
   Therefore $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

2. The proof is similar to that in part (1) and therefore is omitted.

5. Composition of Complex Multi-Fuzzy Relations

In this section, we introduce the definition of composition of CMFRs and define four types of relations on CMFRs. These are reflexive, symmetric, transitive and equivalence relations.

First, we will propose the axiomatic definition of the composition of CMFR along with an illustrative example, followed by two associated theorems.

Definition 5.1. *If $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{CM}^k\mathcal{FS}s$ over universes U,V and W, respectively. Let R and S are $\mathcal{CM}^k\mathcal{FR}s$ from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{B}$ to $\mathcal{C}$ respectively, where $R \subseteq \mathcal{A} \times \mathcal{B}$ and $S \subseteq \mathcal{B} \times \mathcal{C}$, then the composition of the $\mathcal{CM}^k\mathcal{FR}s$ R and S denoted by $R \circ S$ from $\mathcal{A}$ to $\mathcal{C}$ is defined as:*

$$R \circ S = \{ \langle (u,w), \mu^1_{R\circ S}(u,w), \ldots, \mu^k_{R\circ S}(u,w) \rangle : (u,w) \in U \times W \},$$

where
$$\mu^j_{R\circ S}(u,w) = \sup_{v\in V} min(r^j_R(u,v), r^j_S(v,w)) .e^{i\sup_{v\in V}\min(\omega^j_R(u,v),\omega^j_S(v,w))}$$
for every $(u,w) \in U \times W$, for every $v \in V$ and $j = 1,2,\ldots,k$.

Example 5.2. *Let c_1,c_2,c_3,c_4,c_5 and c_6 be a set of airline companies, and let $\mathcal{A} = \{c_1,c_2\}$, $\mathcal{B} = \{c_3,c_4\}$ and $\mathcal{C} = \{c_5,c_6\}$ be three complex multi-fuzzy sets of three dimensions, where the first dimension, second dimension and third dimension represent the economic class, business class and first class, respectively. The Cartesian product of $\mathcal{A}$ and $\mathcal{B}$ will be the set $\mathcal{A} \times \mathcal{B} = \{(c_1,c_3),(c_1,c_4),(c_2,c_3),(c_2,c_4)\}$, and the Cartesian product of $\mathcal{B}$ and $\mathcal{C}$ is the set $\mathcal{B} \times \mathcal{C} = \{(c_3,c_5),(c_3,c_6),(c_4,c_5),(c_4,c_6)\}$. For example, let $R(\mathcal{A},\mathcal{B})$ be a relation named "the first element (airline) is more expensive than the second element (airline) in winter, summer, autumn, or spring," and let $S(\mathcal{B},\mathcal{C})$ be a relation called "the first element (airline) is more attractive than the second element (airline) in winter, summer, autumn or spring."*

The complex multi-fuzzy relations $R(\mathcal{A},\mathcal{B})$ and $S(\mathcal{B},\mathcal{C})$ can be presented by the following relational matrices:

$$R(\mathcal{A},\mathcal{B}) = \begin{array}{cc} & \begin{array}{cc} \qquad\qquad c_3 \qquad\qquad & \qquad\qquad c_4 \qquad\qquad \end{array} \\ \begin{array}{c} c_1 \\ c_2 \end{array} & \begin{bmatrix} (0.3e^{i(\frac{4}{12})\pi}, 0.6e^{i(\frac{1.5}{12})\pi}, 0.1e^{i(\frac{2}{12})\pi}) & (0e^{i(\frac{9}{12})\pi}, 0.2e^{i(\frac{4.5}{12})\pi}, 0.3e^{i(\frac{3}{12})\pi}) \\ (0.4e^{i(\frac{3}{12})\pi}, 0.1e^{i(\frac{1.8}{12})\pi}, 0.2e^{i(\frac{5}{12})\pi}) & (0.1e^{i(\frac{7}{12})\pi}, 0.8e^{i(\frac{3}{12})\pi}, 0e^{i(\frac{4}{12})\pi}) \end{bmatrix} \end{array}$$

$$S(\mathcal{B},\mathcal{C}) = \begin{array}{cc} & \begin{array}{cc} \qquad\qquad c_5 \qquad\qquad & \qquad\qquad c_6 \qquad\qquad \end{array} \\ \begin{array}{c} c_3 \\ c_4 \end{array} & \begin{bmatrix} (0.2e^{i(\frac{6}{12})\pi}, 0.7e^{i(\frac{5}{12})\pi}, 0.1e^{i(\frac{1}{12})\pi}) & (0.1e^{i(\frac{8}{12})\pi}, 0.2e^{i(\frac{4}{12})\pi}, 0.3e^{i(\frac{1}{12})\pi}) \\ (\ 0.4e^{i(\frac{4}{12})\pi}, 0.1e^{i(\frac{1.8}{12})\pi}, 0.3e^{i(\frac{11}{12})\pi}) & (0.3e^{i(\frac{9}{12})\pi}, 0.5e^{i(\frac{3}{12})\pi}, 0e^{i(\frac{1.5}{12})\pi}) \end{bmatrix} \end{array}$$

We now compute the composite relational matrix, denoted by $R \circ S(\mathcal{A},\mathcal{C})$. We should note that $\mathcal{A} \times \mathcal{C}$ has four elements: $(c_1,c_5),(c_1,c_6),(c_2,c_5)$ and (c_2,c_6). Thus, our task is to determine the membership values of $\mu_{R\circ S} = (\mu^1_{R\circ S},\ \mu^2_{R\circ S},\ \mu^3_{R\circ S})$ of these four elements above. We use Definition 15 to determine the membership $R \circ S(c_1,c_5) = (\mu^1_{R\circ S}(c_1,c_5),\ \mu^2_{R\circ S}(c_1,c_5),\ \mu^3_{R\circ S}(c_1,c_5))$. Thus, we have

$$\mu^1_{R\circ S}(c_1,c_5) = \sup_{v\in V} \{min\,[r^1_R(c_1,c_3), r^1_S(c_3,c_5)], min\,[r^1_R(c_1,c_4), r^1_S(c_4,c_5)]\,\} \cdot e^{i \sup_{v\in V} \{min\,[\omega^1_R(c_1,c_3),\omega^1_S(c_3,c_5)], min\,[\omega^1_R(c_1,c_4),\omega^1_S(c_4,c_5)]\,\}}$$

$$= \sup_{v\in V} \{min\,[0.3, 0.2], min\,[0, 0.4]\,\} \cdot e^{i \sup_{v\in V} \{min\,[(\frac{4}{12})\pi,(\frac{6}{12})\pi], min\,[(\frac{9}{12})\pi,(\frac{4}{12})\pi]\,\}}$$

$$= \sup_{v\in V} \{0.2, 0\,\} \cdot e^{i \sup_{v\in V} \{(\frac{4}{12})\pi,(\frac{4}{12})\pi\,\}} = 0.2e^{i(\frac{4}{12})\pi}.$$

$$\mu^2_{A\circ B}(c_1,c_5) = \sup_{v\in V} \{min\,[0.6, 0.7], min\,[0.2, 0.1]\,\} \cdot e^{i \sup_{v\in V} \{min\,[(\frac{1.5}{12})\pi,(\frac{5}{12})\pi], min\,[(\frac{4.5}{12})\pi,(\frac{1.8}{12})\pi]\,\}}$$

$$= \sup_{v\in V} \{0.6, 0.1\,\} \cdot e^{i \sup_{v\in V} \{(\frac{1.5}{12})\pi,(\frac{1.8}{12})\pi\,\}} = 0.6e^{i(\frac{1.8}{12})\pi}$$

$$\mu^3_{A\circ B}(c_1,c_5) = \sup_{v\in V} \{min\,[0.1, 0.1], min\,[0.3, 0.3\,\} \cdot e^{i \sup_{v\in V} \{min\,[(\frac{2}{12})\pi,(\frac{1}{12})\pi], min\,[(\frac{3}{12})\pi,(\frac{11}{12})\pi]\,\}}$$

$$= \sup_{v\in V} \{0.1, 0.3\,\} \cdot e^{i \sup_{v\in V} \{(\frac{1}{12})\pi,(\frac{3}{12})\pi\,\}} = 0.3e^{i(\frac{3}{12})\pi}$$

Therefore, $A \circ B(c_1,c_5) = (0.2e^{i(\frac{4}{12})\pi}, 0.6e^{i(\frac{1.8}{12})\pi}, 0.3e^{i(\frac{3}{12})\pi})$

Similarly, we determine the elements $(c_1,c_6),(c_2,c_5)$ and (c_2,c_6). Thus, the final matrix of $R \circ S$ is

$$R \circ S = \begin{array}{cc} & \begin{array}{cc} \qquad\qquad c_5 \qquad\qquad & \qquad\qquad c_6 \qquad\qquad \end{array} \\ \begin{array}{c} c_1 \\ c_2 \end{array} & \begin{bmatrix} (0.2e^{i(\frac{4}{12})\pi}, 0.6e^{i(\frac{1.5}{12})\pi}, 0.3e^{i(\frac{3}{12})\pi}) & (0.1e^{i(\frac{9}{12})\pi}, 0.2e^{i(\frac{4}{12})\pi}, 0.1e^{i(\frac{1.5}{12})\pi}) \\ (0.2e^{i(\frac{4}{12})\pi}, 0.1e^{i(\frac{1.8}{12})\pi}, 0.1e^{i(\frac{4}{12})\pi}) & (0.1e^{i(\frac{7}{12})\pi}, 0.5e^{i(\frac{3}{12})\pi}, 0.2e^{i(\frac{1.8}{12})\pi}) \end{bmatrix} \end{array}.$$

In Example 5.2 the amplitude terms represent the degrees of belongingness to the set of expensive items and attractive items and the phase terms represent the degrees of belongingness to the phase of seasons.

Theorem 5.3. *If $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{CM}^k\mathcal{FS}$s over universes T,U,V and W respectively. Let R, S and P are $\mathcal{CM}^k\mathcal{FR}$s from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{C}$ to $\mathcal{D}$ respectively, where $R \subseteq \mathcal{A}\times\mathcal{B}$, $S \subseteq \mathcal{B}\times\mathcal{C}$ and $P \subseteq \mathcal{C}\times\mathcal{D}$, then $R\circ(S\circ P) = (R\circ S)\circ P$.*

Proof

Let $R \circ (S \circ P) = \{\,\langle (t,w), \mu^j_{R\circ(S\circ P)}\ (t,w)\ \rangle : j = 1,...,k,\ (t,w) \in T\times W\,\}$,

and

$(R\circ S)\circ P = \{\,\langle (t,w), \mu^j_{(R\circ S)\circ P}\ (t,w)\ \rangle : j = 1,...,k,\ (t,w) \in T\times W\,\}$,

To prove the equality $R\circ(S\circ P) = (R\circ S)\circ P$, we have to show that $\mu^j_{R\circ(S\circ P)} = \mu^j_{(R\circ S)\circ P}$ for $j = 1,...,k$. By Definition 5.1, we have

$$\mu^j_{R\circ(S\circ P)}(t,w) = \sup_{u\in U} min\,[r^j_R(t,u), (r^j_{(S\circ P)}(u,w))] \cdot e^{i \sup_{u\in U} min\,[\omega^j_R(t,u), \omega^j_{(S\circ P)}(u,w)]}$$

$$= \sup_{u\in U} min \left[r^j_R(t,u), [\sup_{v\in V} min(r^j_S(u,v), r^j_P(v,w))] \right] \cdot e^{i \sup_{u\in U} min \left[\omega^j_R(t,u), [\sup_{v\in V} min\,(\omega^j_S(u,v),\omega^j_P(v,w)\,)] \right]}$$

$$= \sup_{u\in U, v\in v} min\left[r^j_R(t,u), r^j_S(u,v), r^j_P(v,w))\right]$$

$$\cdot e^{i \sup_{u\in U, v\in v} min[\omega^j_R(t,u), \omega^j_S(u,v), \omega^j_P(v,w)]}$$

$$= \sup_{v\in v} min\left[[\sup_{u\in u} min\,(r^j_R(t,u), r^j_S(u,v))], r^j_P(v,w)\right]$$

$$\cdot e^{i \sup_{v\in V} min\left[[\sup_{u\in u} min\,(\omega^j_R(t,u), \omega^j_S(u,v))], \omega^j_P(v,w)\right]}$$

$$= \sup_{v\in V} min\,[r^j_{(R\circ S)}(t,v), r^j_P(v,w)]$$

$$\cdot e^{i \sup_{v\in V} min\,[\omega^j_{(R\circ S)}(t,v), \omega^j_{(P)}(v,w)]}$$

$$= \mu^j_{(R\circ S)\circ P}(t,w)$$

Hence $\mu^j_{R\circ(S\circ P)}(t,w) = \mu^j_{(R\circ S)\circ P}(t,w)$ for $j = 1,...,k$, thus $R\circ(S\circ P) = (R\circ S)\circ P$.

Theorem 5.4. *If $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{CM}^k\mathcal{FS}$s over universes U,V and W, respectively. Let R and S are $\mathcal{CM}^k\mathcal{FR}$s from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{B}$ to $\mathcal{C}$ respectively, where $R \subseteq \mathcal{A}\times\mathcal{B}$ and $S \subseteq \mathcal{B}\times\mathcal{C}$, then $(R\circ S)^{-1} = (S^{-1}\circ R^{-1})$.*

Proof

$\forall(u,v) \in \mathcal{A}\times\mathcal{B}$, $(v,w) \in \mathcal{B}\times\mathcal{C}$ and $j = 1,...,k$.

If the composition $R\circ S$ is a complex multi-fuzzy relation from $\mathcal{A}$ to $\mathcal{C}$, then the composition $S\circ R$ is a complex multi-fuzzy relation from $\mathcal{C}$ to $\mathcal{A}$. Then,

$$\mu^j_{(R\circ S)^{-1}}(w,u) = \mu^j_{(R\circ S)}(u,w)$$

$$= \sup_{v\in V} min\,[r^j_R(u,v), r^j_S(v,w)]$$

$$\cdot e^{i \sup_{u\in U} min\,[\omega^j_R(u,v), \omega^j_S(v,w)]}$$

$$= \sup_{v\in V} min\,[r^j_{R^{-1}}(v,u), r^j_{S^{-1}}(w,v)]$$

$$\cdot e^{i \sup_{u\in U} min\,[\omega^j_{R^{-1}}(v,u), \omega^j_{S^{-1}}(w,v)]}$$

$$= \sup_{v\in V} min\,[r^j_{S^{-1}}(w,v), r^j_{R^{-1}}(v,u)]$$

$$\cdot e^{i \sup_{u\in U} min\,[\omega^j_{S^{-1}}(w,v), \omega^j_{R^{-1}}(v,u)]}$$

$$= \mu^j_{S^{-1}\circ R^{-1}}(w,u)$$

Therefore $(R\circ S)^{-1} = S^{-1}\circ R^{-1}$.

Now, we introduce the notion of symmetric, transitive and reflexive relations between $\mathcal{CMFS}$s.

Definition 5.5. *Let $\mathcal{A} \in \mathcal{CM}^k\mathcal{FS}(U)$ and R is a $\mathcal{CM}^k\mathcal{FR}$ on $\mathcal{A}$, then*

1. *R is a reflexive $\mathcal{CM}^k\mathcal{FR}$ if $\mu^j_R(u,u) = 1$, for all $u \in U$ and $j = 1,...,k$.*
2. *R is a symmetric $\mathcal{CM}^k\mathcal{FR}$ if $\mu^j_R(u,v) = \mu^j_R(v,u)$, for all $(u,v) \in U\times U$ and $j = 1,...,k$.*
3. *R is a transitive $\mathcal{CM}^k\mathcal{FR}$ if $R\circ R \subseteq R$.*
4. *R is a equivalence $\mathcal{CM}^k\mathcal{FR}$ if the relation R satisfies symmetric, transitive and reflexive.*

Proposition 5.6. *Let $\mathcal{A} \in \mathcal{CM}^k\mathcal{FS}(U)$ and R is a $\mathcal{CM}^k\mathcal{FR}$ on $\mathcal{A}$, then the following assertions hold true.*

1. *R is a symmetric $\mathcal{CM}^k\mathcal{FR}$ if and only if R^{-1} is a symmetric $\mathcal{CM}^k\mathcal{FR}$.*
2. *R is a symmetric $\mathcal{CM}^k\mathcal{FR}$ if and only if $R = R^{-1}$.*
3. *If R is a transitive $\mathcal{CM}^k\mathcal{FR}$, then R^{-1} is also a transitive $\mathcal{CM}^k\mathcal{FR}$.*
4. *If R is a reflexive $\mathcal{CM}^k\mathcal{FR}$, then R^{-1} is also a reflexive $\mathcal{CM}^k\mathcal{FR}$.*

Proof

1. ($\Rightarrow$) Let R be symmetric, then we have

$$\left[\mu^j_R(u,v) = r^j_R(u,v).e^{i\omega^j_R(u,v)}\right]$$

$$= \left[\mu^j_R(v,u) = r^j_R(v,u).e^{i\omega^j_R(v,u)}\right]$$

and since R^{-1} is an inverse relation, we will have

$$\left[\mu^j_{R^{-1}}(u,v) = r^j_{R^{-1}}(u,v).e^{i\omega^j_{R^{-1}}(u,v)}\right]$$

$$= \left[\mu^j_R(v,u) = r^j_R(v,u).e^{i\omega^j_R(v,u)}\right],$$

for all $(u,v) \in U\times U$ and $j = 1,...,k$.

To prove R^{-1} is symmetric, it is enough to show

$$\mu^j_{R^{-1}}(u,v) = \mu^j_{R^{-1}}(v,u),$$

for all $(u,v) \in U\times V$ and $j = 1,...,k$.

Therefore

$\mu^j_{R^{-1}}(u,v) = \mu^j_R(v,u) = \mu^j_R(u,v) = \mu^j_{R^{-1}}(v,u)$.

($\Leftarrow$) Let R^{-1} be symmetric, then we have

$$[\mu^j_{R^{-1}}(u,v) = r^j_{R^{-1}}(u,v).e^{i\omega^j_{R^{-1}}(u,v)}]$$

$$= [\mu^j_{R^{-1}}(v,u) = r^j_{R^{-1}}(v,u).e^{i\omega^j_{R^{-1}}(v,u)}]$$

and since $(R^{-1})^{-1}$ is an inverse relation, we will have

$$[\mu^j_{(R^{-1})^{-1}}(u,v) = r^j_{(R^{-1})^{-1}}(u,v).e^{i\omega^j_{(R^{-1})^{-1}}(u,v)}]$$

$$= [\mu^j_{(R^{-1})}(v,u) = r^j_{(R^{-1})}(v,u).e^{i\omega^j_{(R^{-1})}(v,u)}],$$

for all $(u,v) \in U\times U$ and $j = 1,...,k$.

To prove R is symmetric, it is enough to show

$$\mu^j_R(u,v) = \mu^j_R(v,u),$$

for all $(u,v) \in U\times U$ and $j = 1,...,k$

Therefore

$$\mu^j_R(u,v) = \mu^j_{(R^{-1})^{-1}}(u,v)$$

$$= \mu^j_{R^{-1}}(v,u) = \mu^j_{R^{-1}}(u,v) = \mu^j_R(v,u).$$

(2)-(4). The proofs are straightforward using Definition 5.5.

6. Conclusion

A new structure of relation, named complex multi-fuzzy relation is established by incorporating the features of both complex fuzzy relation and multi-fuzzy relation. An application in education has been discussed using properties of complex multi-fuzzy relation, followed by a comparison between our proposed method and existing methods to reveal the dominancy of our method. The fundamental operations on CMFR such as complement, union, intersection, inverse and composition operation along with associated propositions and theorems have been discussed. Finally the notions of symmetric, transitive, reflexive, and equivalence complex multi-fuzzy relations have been established. CMFR is a promising new concept, paving the way toward numerous possibilities for future research. In future research, we may introduce δ-equalities of CMFRs. It will be meaningful to introduce the concept of CMF logics and systems based on CMF relation, which will enable the decision making processes to real-life applications, such as in medical, physics, engineering, automobiles, internet and computer security, and other fields.

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