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Research Paper

A Branch-and-bound algorithm for concave minimization problem with upper bounded variables

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Abstract

This paper presents a branch-and-bound algorithm for solving the concave minimization problems with upper bounded variables. The algorithm uses simplex to construct the branching and the bounding procedure. The linear convex envelope (the objective function of the subproblem) is uniquely determined on the candidate simplex which contains the subset of the local minimal points. The optimal solution of the subproblem is a local optimum of the original concave problem and used in reducing the list of active subproblems. The branching process splits the candidate simplex into two subsimplices by fixing the selected branching variable at value 0 or upper bound. Then the subsimplices are one less dimensional than the candidate. It means that the size of the subproblems gradually decreases. Further research needs to be focused on the efficient determination method of the simplex. The algorithm of this paper can be applied to solving the concave minimization problems under knapsack type constraints.

Keywords: *Branch-and Bound Algorithm; Concave Minimization; Convex Envelope Simplex.*

1. Introduction

The global minimization of a linearly constrained concave function has attracted the attention of a number of researchers since Tui's fundamental work(Tui H, 1964). Since many practical applications can be formulated as concave minimization problems. The zero-one integer linear programming problem(Kalantari B and Bagchi A, 1990, More JJ and Vavasis SA, 1991, Rosen JB 1983, Sun XL et al.,2005), the linear fixed-charge problem(Thieu TV, 1978), economies of scale and strategic weapons planning(Falk JE and Hoffman KR, 1976), the facility location problem with concave costs(Soland RM, 1974) are included among them.

The local optimum for the convex function minimization problem must be the global one. But this property is not true for the concave function. The concave function problem may have many local solutions. Thus any special methods are required for solving the concave minimization problem. Since most of them repeatedly use any appropriate local optimization techniques, their computations are expensive. From the complexity point of view, the concave minimization problem is NP-hard. It is seen by the fact that the zero-one linear programming is a special case of the concave minimization problem(Murty KG, 1995). This is the reason that it is difficult to develop the encouraging algorithm.

The most general approach to the concave minimization problems is the branch and bound type search (Benson HP, 1996). Many authors have incorporated some useful schemes into branching and bounding strategies to design the efficient algorithms. These strategies are based on the exploitation of the underlying structure of the problem. The most important bounding strategy is the use of underestimating function. The cut which Tui(Tui H, 1964) suggested to exclude the part of feasible domain may be seen as its first form. Since then, it has developed into various types of the underestimating functions. A piecewise linear underestimating function has appeared, in the algorithm developed by Falk and Hoffman(Falk JE and Hoffman KR, 1976). Rosen(Rosen JB, 1983) developed the function which underestimates a smooth concave function over a polyhedron, and Kalantari and Rosen(Kalantari B and Rosen JB, 1987) considered the underestimator for the global minimization of a quadratic function over a polytope. Benson(Benson HP, 1985) showed that the underestimating function of the concave function over a simplex is linear and uniquely determined by solving linear equations.

The other strategy combined with the bounding is the branching. During the algorithm, the branching procedure partitions the set of feasible solutions into many subsets. Then these partitioned subsets may be the same form or be relaxed to be the same. Because the partition element of the same form makes it possible to consistently define the subproblem for bounding operation. Tui(Tui H, 1964) partitioned the feasible domain by cone in his algorithm. Kalantari and Rosen(Kalantari B and Rosen JB, 1987) considered the parallelpiped containing the feasible region as partition elements. Benson(Benson HP, 1985), Benson and Erenguc(Benson HP and Erenguc SS, 1990), Falk and Hoffman (Falk JE and Hoffman KR, 1986) and Horst(Horst R, 1986) developed algorithms in which the partition element is simplex.

The algorithm of this paper uses the simplex as the partition element like the above author's algorithm. In order to initialize our algorithm, we introduce the simplex containing the feasible region. In fact, many researchers have suffered from the expensive computation for generating the simplices. Since most of the partition method endures the addition of the many constraints. But, in this paper the simplex is divided by being projected onto two half spaces. This operation is easily implemented by imposing two equality constraints on the candidate simplex. It means that two hyperplanes intersect the simplex respectively. Hence the dimension of two subsimplices is one less than the candidate simplex. Consequently, the subproblem size decreases one by one while iteration proceeds. This is the main advantage of our

algorithm. After solving the linear equations to determine the underestimating function on the candidate simplex, the subproblem is defined to be the linear programming problem. The optimum value of the linear function serves as the lower bound for the candidate problem. Since the optimal solution of the subproblem is locally optimal, the concave objective value at this point updates the incumbent value.
Section 2 shows the validity and embodiment of the bounding strategy in the algorithm. In section 3, we present the branching procedure. And Section 4 explains branching variable selection rule and describes the formal of the algorithm and gives a numerical example to illustrate the algorithm.
Finally, we present some concluding remarks.

2. Bounding strategy

2.1. Convex envelope

The following concave minimization problem is dealt with in this paper.

$$(P)\ \min_{x\in\Omega} f(x)$$

Where f(x) is any concave function and

$$\Omega = \{\, x \in R^n \mid 0 \le x_i \ \le u_i, i = 1,2, \dots n \,\}$$

The algorithm for solving the problem (P) performs the binary branching and the bounding operations. At each stage, the sub-problem, whose objective function underestimates the concave function over feasible region, is defined for bounding operation and two subsimplices are generated from the candidate simplex. After performing the bounding operation, our algorithm selects the branching variable and fixes that at 0 and the upper bound. The fixation of the branching variable results in two subsimplices. The motivation of our algorithm is based on the fact that the objective function of the subproblem underestimates the original concave function over feasible region and is uniquely determined over simplex. The important part of the bounding procedure is to determine the underestimating function. Thus the efficiency of the algorithm depends on what kind of the function is adopted. The following underestimating function is considered in this paper.

1) Definition 1(Falk JE and Hoffman KR, 1976). The convex envelope of a function f over a polytope Ω is a function $\Gamma(x)$ defined over Ω such that: $\Gamma(x)$ is convex over Ω.

2) $\Gamma(x) \le f(x), \forall x \in \Omega$.

3) If is g(x) any function satisfying (I) and (I) then $g(x) \le \Gamma(x), \forall x \in \Omega$.

The above function $\Gamma(x)$ is the supremum of all underestimating functions of f over a polytope Ω. If Ω is a general form of polytope, it is difficult to determine $\Gamma(x)$. Horst (Horst R, 1986) has shown that if Ω is a simplex, $\Gamma(x)$ is linear and agrees with f for each vertex of Ω. The following Fig. 1 illustrates the convex envelope of the concave function over 1-dimensional simplex.
Let us introduce the following convenient notations and the definitions to explain the algorithm.

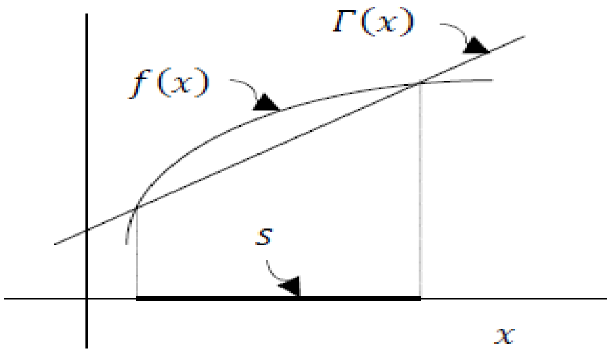

Fig. 1: The Convex Envelope Γ of F OnS.

Notation
S_φ : The candidate simplex
$(C.P_\varphi)$: The subproblem generated from S_φ
$\Omega_\varphi (= \Omega \cap S_\varphi)$: The feasible region of $(C.P_\varphi)$
υ^j_φ , $j = 0,1, \dots n_\varphi$: Vertices of S_φ
n_φ: dimension of S_φ
i_φ: Index of the branching variable selected at $(C.P_\varphi)$
$(\upsilon^j_\varphi)_i$: i th component of vertex $\upsilon^j_\varphi \in R^n$
$L.B_\varphi$: Lower bound of $(C.P_\varphi)$
$\overline{x}^\varphi$: The optimal solution of$(C.P_\varphi)$
Ψ_φ: The index set of unfixed variables in $(C.P_\varphi)$
$U.B_\varphi$: The current incumbent value after bounding operation over S_φ

Definition 2: *Let $v^0, v^1, \dots, v^n$ be (n+1) affinely independent points in R^n . The convex hull of $\{ v^0, v^1, \dots, v^n \}$ denoted conv($\{ v^0, v^1, \dots, v^n \}$) is called a n-dimensional simplex, and the points $v^0, v^1, \dots, v^n$ are called vertices of the simplex.*

The convex envelope on the given simplex is uniquely determined by solving the following (n+1) linear equations (Benson HP, 1985).

$$< \alpha, \upsilon^i > +\gamma = f(\upsilon^i), i = 0,1,2, \dots n$$

For the unknowns $\alpha \in R^n, \gamma \in R$.
Let

$$\Gamma(x) =< \alpha, x > +\gamma$$

Then the above linear convex envelope underestimates the concave function over the given simplex.
In order to initiate the algorithm, it is necessary to take an simplex S_0 as tightly as possible that contains the feasible region Ω. Without loss of generality, a vertex $\upsilon^0 = (0,0, \dots ,0)$ of Ω i.e., the origin of $R^n$is chosen. The n nonnegative variable constraints binding at the origin form a cone. Let us consider the hyperplane whose normal vector is $\left(\frac{1}{u_1}, \frac{1}{u_2}, \dots, \frac{1}{u_n}\right)$and which contains the feasible point $(u_1, u_2, \dots, u_n)$. This hyperplane intersects the n coordinate axes at $(0,0, \dots ,0, n \cdot u_i ,0, \dots ,0), i = 1,2, \dots n$ and the intersecting points are n affinely independent. Let us take a simplex S_0 as follows:

$$S_0 = \left\{ x \in R^n \middle| \sum_{i=1}^{n} \frac{x_i}{u_i} \le n, x_i \ge 0, i = 1,2, \dots n \right\}$$

The simplex S_0 can be identified by its set of vertices as follows:

$$\upsilon^0_0 = (0,0, \dots ,0),$$

$$\Upsilon^i_0 = (0,0, \dots ,0, \xi_i ,0, \dots ,0) \text{ , for } i = 2, \dots , n$$

Where$\xi_i = n \cdot u_i, i \ne 1$

Then $\upsilon_0^0, \upsilon_0^1, \ldots, \upsilon_0^n$ are the vertices of an n-dimensional simplex S_0.

2.2. Sub problem generation and bounding operation

If the convex envelope is determined on S_0, the first subproblem $(C.P_0)$ is defined as follows for bounding operation.

$$〚(C.P〛 \_0) \ \min\nolimits_{(x \in \Omega\_0)} \Gamma(x)$$

$$\Omega_0 = \{ x \in R^n | \ 0 \le x_i \le u_i, i = 1,2, \ldots n \} \cap S_0$$

The optimal value of $(C.P_0)$ serves as a lower bound of all children subproblems generated from it by the following theorem.

Theorem 1: $\Gamma(x)$ underestimates $f(x)$ on Ω_0.

Proof

$$f(x) \ge \Gamma(x) \ for \ x \in S_0$$

Since $S_0 \supseteq \Omega_0, f(x) \ge \Gamma(x) \ for \ x \in \Omega_0$

When any one is selected among the candidate simplices, the bounding operation performs the following procedure.

Bounding-Algorithm

1) Select the simplex S_φ in the list.
2) Identify the vertices of the selected simplex.
3) Determine the convex envelope by solving equation (1).
4) Set up the subproblem $(C.P_\varphi)$ as follows:
5) $〚(C.P〛 \_\varphi) \ (min)_{(x \in \Omega\_\varphi)} \ 〚\Gamma\_\varphi \ (x)〛 \ \ \Omega\_\varphi = \{ x \in R\hat{}n┤| \ 0 \le x\_i \ \le u\_i, for \ i \in \Psi\_\varphi \ \} \cap S\_\varphi$
6) Seek the optimal solution $\overline{x}^\varphi$ of $(C.P_\varphi)$
7) Calculate $L.B_\varphi = \Gamma_\varphi(\overline{x}^\varphi)$, update the incumbent $U.B_\varphi$

Since $\overline{x}^\varphi$ is the local optimum, $f(\overline{x}^\varphi)$ updates the incumbent. If $L.B_\varphi$=$f(\overline{x}^\varphi)$, the simplex $S_\varphi$need not be considered any further. It is fathomed. And any candidate problems whose lower bounds are greater than the current incumbent are removed.

3. Branching of simplex

The branching begins by selecting a branching variable after bounding operation. The candidate simplex is divided into two subsimplices of similar figure by fixing a branching variable x_{i_φ} at 0 and u_{i_φ}. At this time, feasible vertices contained in the simplex are exclusively and exhaustively partitioned into two groups. Consequently, two new simplices which contain two groups of feasible vertices respectively may be registered in the candidate simplex list. The following theorem guarantees the validity of our approach.

Theorem 2: *Assume that the n-dimensional simplex $S_0(\subset R^n)$ contain Ω.*

Let us consider two hyperplanes $H^0 = \{x \in R^n | \ x_i = 0\}, H^1 = \{x \in R^n | \ x_i = u_i\} \ for \ some \ i$.Then $S_{01} = H^0 \cap S_0$and $S_{02} = H^1 \cap S_0$ are the (n-1)-dimensional simplices that partitions the vertices set of Ω into two subsets.

Proof.

S_{01} is a facet of S_0. Hence it is a (n-1)-dimensional simplex. The vertices of S_{02} are the points on the segments between the vertex $(0,0, \ldots ,0, n \cdot u_i, 0, \ldots ,0)$ of S_0 and the vertices of S_{01}. These n points are n affinely independent in R^n. S_{02} is a (n-1)-dimensional simplex . The vertices of S_0 are placed on S_{01} or S_{02}.

Hence S_{01} and S_{02} partition vertices set of Ω into two subsets.

S_{01} and S_{02} are one less dimensional than S_0. It means that whenever the branching occurs, the size of the subproblem decreases. Assume x_1 be the branching variable. Then by making the hyperplane $\{x \in R^n | \ x_1 = 0\}$ intersect S_0, a (n-1)-dimensional simplex is obtained as follows:

S_{01}=conv ($\{ \upsilon_{01}^0, \upsilon_{01}^2, \ldots , \upsilon_{01}^n \}$),

$\upsilon_{01}^0 = (0,0, \ldots ,0)$,

$\Upsilon_{01}^i = (0,0, \ldots ,0, \xi_i ,0, \ldots ,0)$, for $i = 2, \ldots , n$

$Where \xi_i = n \cdot u_i, i \neq 1$

The above simplex S_{01} is a facet of S_0 . And conv ($\{ \upsilon_{01}^2, \upsilon_{01}^3, \ldots , \upsilon_{01}^n \}$) is a facet of S_{01}. Even if it is translated to the point $(0, u_2, u_3, u_4, \ldots , u_n)$, the compressed S_{01} still contains all vertices of $S_{01} \cap \Omega$. Hence it guarantees tighter underestimating function. Thus S_{01} may be redefined as follows:

S_{01}=conv ($\{ \upsilon_{01}^0, \upsilon_{01}^2, \ldots , \upsilon_{01}^n \}$),

$\upsilon_{01}^0 = (0,0, \ldots ,0)$,

$\Upsilon_{01}^i = (0,0, \ldots ,0, \xi_i ,0, \ldots ,0)$, for $i = 2, \ldots , n$

$Where \xi_i = (n-1) \cdot u_i, i \neq 1$

The other simplex is generated by translating S_0 to the vertex $(u_1, 0,0,0, \ldots ,0)$as follows:

S_{02}=conv ($\{ \upsilon_{02}^0, \upsilon_{02}^2, \ldots , \upsilon_{02}^n \}$),

$\upsilon_{02}^0 = (u_1, 0, \ldots ,0)$,

$\Upsilon_{02}^i = (u_1, 0, \ldots ,0, \xi_i ,0, \ldots ,0)$, for $i = 2, \ldots , n$

$Where \xi_i = (n-1) \cdot u_i, i \neq 1$

Generally, at the kth iteration, assume S_φ be splitted and x_{i_φ} be selected by branching variable selection rule. Then S_φ = conv($\{ \upsilon_\varphi^0 , \upsilon_\varphi^j , j \in \Psi_\varphi \}$). Consequently

$S_{\varphi 1}$=conv($\{ \upsilon_{\varphi 1}^0, \upsilon_{\varphi 1}^j , j \in \Psi_\varphi \backslash i_\varphi \}$),

Where

$\upsilon_{\varphi 1}^0 = \upsilon_\varphi^0$,

$(\upsilon_{\varphi 1}^j)_i = (n_\varphi\text{-}1) \cdot u_i$

For

$j \in \Psi_\varphi \backslash i_\varphi$

At the kth iteration, subsimplices generation

Procedure is as follows:

Simplex splitting algorithm

1) Choose the branching variable $x_{i_\varphi}$by selection rule.
2) Identify the vertices of S_φ s.t. S_φ= conv($\{ \upsilon_\varphi^0 , \upsilon_\varphi^j , j \in \Psi_\varphi \}$).
3) Generate the vertices of $S_{\varphi 1}$

Delete $\upsilon_\varphi^{i_\varphi}$

$(v_{\varphi 1}^{0}) \leftarrow (v_{\varphi}^{0})$

For $j \in \Psi_{\varphi} \backslash i_{\varphi}$

If $\left(i \in \Psi_{\varphi} \backslash i_{\varphi}\right)$

$(v_{\varphi 1}^{j})_i = \left(n_{\varphi} - 1\right) \cdot u_i$

Else If

$(v_{\varphi 1}^{j})_i = (v_{\varphi}^{j})_i$

End If

End For

$\Psi_{\varphi 1} \leftarrow \Psi_{\varphi} - i_{\varphi}$

Generate the vertices of $S_{\varphi 2}$

$(v_{\varphi 2}^{0}) \leftarrow (v_{\varphi}^{0})$

For j $\in \{0, \Psi_{\varphi 1}\}$

If $(i = i_{\varphi})$

$(v_{\varphi 2}^{j})_i = (v_{\varphi 1}^{j})_i + u_i$

$Else\ If\ (v_{\varphi 2}^{j})_i = (v_{\varphi 1}^{j})_i$

End If

End For

$\Psi_{\varphi 2} \leftarrow \Psi_{\varphi} - i_{\varphi}$

$S\_\varphi 1 = conv$

$S_{\varphi 2} = conv\left(\{ v_{\varphi 2}^{j}, j \in \Psi_{\varphi 2} \}\right)$

4. Algorithm and numerical example

4.1. Branching variable selection and candidate simplex selection rule

The branching variable selection can be carried out arbitrarily. The first variable among the unfixed variables may be selected as the branching variable. In this paper, a heuristic rule for selecting the branching variable is suggested such as:
Selecting the variable x_{i_φ} such that

$$\left|\frac{\partial f}{\partial x_{i_\varphi}}(\overline{x}^{\varphi})\right| = \max_{j \in \Psi_\varphi}\left|\frac{\partial f}{\partial x_j}(\overline{x}^{\varphi})\right|$$

This rule is based on the fact that the direction of x_{i_φ}-axis is the steepest ascent or descent direction of the concave function value at $\overline{x}^{\varphi}$.
Over new surplices $S_{\varphi 1}$, $S_{\varphi 2}$, $(C.P_{\varphi 1})$, $(C.P_{\varphi 2})$ are define respectively as follows and two bounding operations are performed:

$$(C.P_{\varphi 1})\ \min_{x \in \Omega_{\varphi 1}} f(x)$$

Where

$$\Omega_{\varphi 1} = \left\{ x \in R^{|\Psi_{\varphi 1}|} \middle| \, 0 \le x_i \le u_i, i \in \Psi_{\varphi 1} \right\} \cap S_{\varphi 1}$$

$$(C.P_{\varphi 2})\ \min_{x \in \Omega_{\varphi 2}} f(x)$$

Where

$$\Omega_{\varphi 2} = \left\{ x \in R^{|\Psi_{\varphi 2}|} \middle| \, 0 \le x_i \le u_i, i \in \Psi_{\varphi 2} \right\} \cap S_{\varphi 2}$$

Next candidate simplex selection rule is as follow:
Selecting the simplex that is associated with the least lower bound among all terminal nodes
If the tie occurs, it is broken arbitrarily.

4.2. Branch and bound algorithm description

Branch & Bound Algorithm
Iteration 0: Initialization
0-1: Determine the initial simplex S_0
0-2: Perform the bounding-Algorithm
0-3: Calculate $L.B_0$, the incumbent
0-4: Call branching-algorithm and Register generated subsimplices in the list
Iteration k: Repeat until registered subsimplices in the list do not exist
k-1: Select the candidate simplex S_{φ} of the subproblem $(C.P_{\varphi})$ whose $L.B_{\varphi}$ is the least
k-2: Call the simplex splitting algorithm and Register generated two subsimplices in the list
k-3: Perform the bounding-algorithm and Calculate $L.B_{\varphi 1}, L.B_{\varphi 2}$ respectively for two children
k-4: Update the incumbent
Delete the simplices in the candidate list whose lower bound is greater than the incumbent

4.3. Numerical example and branch & bound tree consider the problem below

$Min \quad f(x) = -2x_1^2 - 2x_2^2 - 3x_3^2 - 4x_4^2 - 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3$

$-2x_2x_4 - 2x_3x_4 +$

$30x_1 + 28x_2 + 30x_3 + 34x_4$

$s.t. \, 0 \le x_1 \le 9$

$0 \le x_2 \le 7$

$0 \le x_3 \le 5$

$0 \le x_4 \le 3$

In Fig. 2, the search tree is drawn to illustrate how the search is processing.

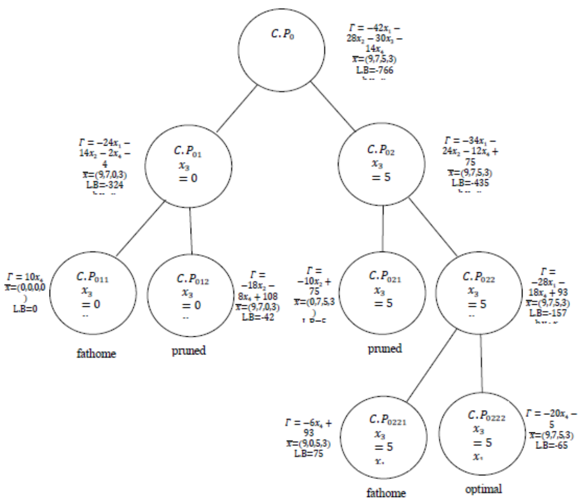

Fig. 2: The Branching Tree of the Example.

5. Conclusions

The branch and bound method for solving the concave minimization problem was investigated. The algorithm in this paper was based on the followings:

a) The bounding procedure uses the convex envelope to underestimate the concave function.
b) The convex envelope is uniquely determined on the given simplex.
c) It splits the candidate simplex by restricting the branching variable to be 0 and the upper bound.
d) The local points i.e., the vertices of the feasible region are partitioned when subsimplices are generated.
e) The simplex is identified by the vertices to obtain linear equations for calculating the convex envelope.
f) It solves the simple linear programming problem for bounding operation.
g) Gradually the size of the subproblem decreases at every stage.

The problem size of the candidate subproblem gradually decreases. It means that the computatinal effort also diminishes. The more the feasible region which does not contain the local solutions is excluded from consideration, the tighter convex underestimators for the bounding operation is. Thus further research needs to be focused on the efficient determination method of the simplex and the introduction of the strong valid cut. The algorithm of this paper can be applied to solving the concave minimization problems under knapsack type costraints and the box-type constrained concave minimization proble

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