Analysis of nonlinear oscillator arising in the microelectromechanical system by using the parameter expansion and equivalent linearization methods

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Abstract

In this paper, the nonlinear oscillator arising in the microbeam-based micro-electromechanical system (MEMS) is described. The motion equation of a microbeam is simplified into an ordinary differential equation by using the Galerkin method. The nonlinear ordinary differential equation is solved by using two methods including the Parameter-Expansion and Equivalent Linearization Methods. To verify the accuracy of the present methods, illustrative examples are provided and compared with other analytical, exact and numerical solutions.

Keywords: Parameter Expansion Method; Equivalent Linearization Method; Nonlinear Oscillator; MEMS.

1. Introduction

Micro-electromechanical systems (MEMS) is a process technology used to create tiny integrated devices or systems that combine mechanical and electrical components. They are fabricated using integrated-circuit (IC) batch processing techniques and can range in size from a few micrometers to millimetres. These devices (or systems) have the ability to sense, control and actuate on the micro scale, and generate effects on the macro scale. MEMS are used in many engineering applications such as: microswitches, transistors, accelerometers, etc. While working, electrostatic actuation, large deflections and damping caused by different sources raises nonlinear behavior. Nonlinearity in MEMS may cause some difficulties in computations. Until now, several techniques have been used to find numerical solutions of this nonlinear problem, for example the shooting method [1], the differential quadrature method [2] and the Adomian decomposition method [3]. Although it is difficult to get analytic approximations for different phenomena in MEMS, there are some analytic techniques for nonlinear problems of MEMS such as perturbation techniques [4], the energy balance method (EBM) [5], the homotopy analysis method (HAM) [6] and the He’s Variational Approach (VA) [7].

The Equivalent Linearization method (ELM) is one of the common approaches to approximate analysis of nonlinear dynamical systems. The original linearization for deterministic systems was proposed by Krylov and Bogoliubov [20]. Then Caughey [21] expanded the method for stochastic systems. To date, there have been some extended versions of the Equivalent Linearization method [22, 23]. It has been shown that the Equivalent Linearization method is presently the simplest tool widely used for analyzing nonlinear stochastic problems.

The Parameter-Expansion Method proposed by He [8, 10] is an effective approach to analytical investigation of nonlinear problems. This approach are investigated in different works [9, 11-17]. In this work, the Parameter-Expansion Method (PEM) and the Equivalent Linearization Method (ELM) are applied to analyse the nonlinear oscillator arising in the microbeam-based micro-electromechanical system. The analytical results achieved by two methods are compared with the previous analytical results, the exact results and the numerical results.

2. Basic of parameter-expansion method

In order to introduce the PEM, we consider the general form of Duffing equation in the following form:

\[ \ddot{u} + \omega^2 u + \lambda N(u,t) = 0 \quad (1) \]

Where \( N(u, t) \) includes the nonlinear term.

Expanding the solution \( u, \varepsilon \) as a coefficient of \( u \), and \( \lambda \) as a coefficient of \( N(u, t) \), the series of \( p \) can be introduced as follows [8, 10]:

\[ u = \sum_{i=0}^{\infty} \lambda^i u_i(t) = u_0(t) + \lambda u_1(t) + \lambda^2 u_2(t) + \cdots \quad (2) \]

\[ \varepsilon = \varepsilon_0 + \sum_{i=0}^{\infty} \lambda^i \varepsilon_i = \varepsilon_0 + \lambda \varepsilon_1 + \lambda^2 \varepsilon_2 + \cdots \quad (3) \]

\[ \lambda = \sum_{i=0}^{\infty} \lambda^i \lambda_i = \lambda_0 + \lambda \lambda_1 + \lambda^2 \lambda_2 + \cdots \quad (4) \]

Substituting Eqs. (2)-(4) into Eq. (1) and equating the terms with the identical powers of \( \lambda \), we have:

\[ p': \quad \ddot{u}_i + \omega^2 u_i = 0 \quad (5) \]

\[ p': \quad \ddot{u}_i + \omega^2 u_i + \lambda N(u, t) = 0 \quad (6) \]
Considering the initial conditions \( u_0(0) = A \) and \( \dot{u}_0(0) = 0 \), the solution of Eq. (5) is \( u_i = A \cos(\alpha t) \). Substituting \( u_0 \) into Eq. (6), we obtain:

\[
p' \ : \ \ddot{u} + \dot{\omega} u + \omega^2 A \cos(\alpha t) + a_i N (A \cos(\alpha t), t) = 0
\]  

(7)

For achieving the secular term, we use Fourier expansion series as follows:

\[
N (A \cos(\alpha t), t) = \sum_{n=1}^\infty b_n \cos(2n - 1) \alpha t
\]  

(8)

Substituting Eq. (8) into Eq. (7) yields:

\[
p' \ : \ \ddot{u} + \dot{\omega} u + (\omega^2 A + a_1 b_1) \cos(\alpha t) = 0
\]  

(9)

For avoiding secular term, we consider:

\[
a_i A + a_1 b_1 = 0
\]  

(10)

Setting \( p=1 \) in Eqs. (3) and (4), we have:

\[
a_i = e - \omega_i
\]  

(11)

\[
a_1 = 1
\]  

(12)

Substituting Eqs. (11) and (12) into Eq. (10), we will achieve the first-order approximation frequency of this oscillator (1). Note that, from Eqs. (4) and (12), we can find that \( a_1 = 0 \) for all \( i=1,2,3,4,\ldots \).

3. Basic of equivalent linearization method

To introduce an overview of the Equivalent Linearization method, we consider the oscillation described by following nonlinear differential equation:

\[
X + g(X) = 0, \quad X(0) = A, X(0) = 0
\]  

(13)

where \( g(X) \) is a nonlinear function of \( X \) and \( A \) is the initial amplitude.

The idea of the Equivalent Linearization method is to replace the nonlinear term \( g(X) \) in Eq. (13) by the linear term as follows [21]:

\[
g(X) \rightarrow \alpha X
\]  

(14)

By this manner, the linearized equation of Eq. (13) is given by:

\[
X + \alpha X = 0
\]  

(15)

where the coefficient \( \alpha \) of the linear term is determined by using the mean-square criterion [21-23]:

\[
e'(X) \ := \ \left[ g(X) - \alpha X \right] \rightarrow M_{in}
\]  

(16)

Thus, from:

\[
\frac{\partial g'(X)}{\partial a} = 0
\]

yields:

\[
\alpha = \frac{g(X)X}{X^2}
\]  

(17)

In Eq. (17), the symbol \( \langle \cdot \rangle \) denotes the time-averaging operator in classical meaning [20]:

\[
\langle f(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^{T} f(t) \, dt
\]  

(18)

For a \( \omega \)-frequency function \( f(\omega t) \), the averaging process is taken during one period \( T \), i.e.

\[
\langle f(\omega t) \rangle = \frac{1}{T} \int_{0}^{T} f(\omega t) \, dt = \frac{1}{2\pi} \int_{0}^{2\pi} f(\tau) \, d\tau, \quad \tau = \omega t
\]  

(19)

Using the solution of the linearized equation (15), we can calculate averaging values in Eq. (17), and thus the frequency-amplitude relationship of the oscillation is obtained in closed form.

4. Modeling and formulation

Consider a fixed-fixed microbeam placed between two stationary electrodes with length \( l \), width band thickness \( h \), whose sketch is shown in Fig. 1 together with coordinate \( \text{oxy} \), where \( g_0 \) is the initial gap and \( v \) the electrostatic load. The equation of motion that governs the transverse deflection \( w(x,t) \) is written as [5]:

\[
E \frac{\partial^2 w}{\partial x^2} + \rho \omega \frac{\partial^2 w}{\partial x^2} = \left[ K + \frac{E \omega}{2l} \right] \frac{\partial^2 w}{\partial x^2} + q(x,t)
\]  

(20)

Where \( E \) is the effective modulus, should be noted that, when a wide microbeam \( (b \geq 5h) \) is considered, \( E \) becomes the plate modulus \( E = E / (1-\nu^2) \) where \( E \) is the Young’s modulus and \( \nu \) is the Poisson ratio, while a narrow microbeam is considered, \( E \) simply becomes the Young’s modulus \( E = S/Eh \) with \( S=bh \) and \( l=abh/12 \) are the area and moment of inertia of the cross-section, respectively. \( K \) denotes the tensile or compressive axial load created by the mismatch of both thermal expansion coefficient and crystal lattice period between substrate and the thinfilm (microbeam). The second term on the right-hand side, \( q(x,t) \), denotes the driving force per unit length, resulting from electrostatic excitation [20]:

\[
q(x,t) = \varepsilon_0 \frac{1}{2} \left[ \frac{1}{(g_x-w)^2} - \frac{1}{(g_x+w)^2} \right]
\]  

(21)

Where \( \varepsilon_0 \) is the dielectric constant of the gap medium which is usually taken as \( \varepsilon_0 = 8.85 \times 10^{-12} \text{F/m} \).

As fixed-fixed microbeams are of primary interest here, the boundary conditions are given by:

\[
x = 0, \quad l: \quad w = \frac{\partial w}{\partial x} = 0
\]  

(22)

![Fig. 1: Schematics of A Double-Sided Driven Clamped-Clamped Microbeam-Based Electromechanical Resonator.](image-url)
\[ \xi = \frac{x}{l}, \quad W = \frac{w}{g}, \quad \tau = t \sqrt{\frac{Et}{24EI}}, \quad \alpha = \left( \frac{g}{h} \right)^2, \]
\[ \mathbf{N} = \frac{N_0}{l}, \quad \nu = 24\nu_0 \sqrt{\frac{Et}{EIg}}, \]

The normalized form of the governing equation is given by:
\[ \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 \nu}{\partial \tau^2} = \frac{N}{W(1-W)} \left[ \frac{1}{1-W^2} \right] + \frac{\nu}{4(1-W^2)} \]
(24)

And the corresponding nondimensional boundary conditions are:
\[ \xi = 0, 1: \quad W = \frac{\partial W}{\partial \xi} = 0 \]
(25)

The deflection \( W(\xi, \tau) \) in Eq. (24) is expressed as a sum of spatial shapes that, a priori, satisfy the imposed boundary conditions:
\[ W(\xi, \tau) = \sum_{\xi=0}^{\infty} \phi(\xi) u(\tau) \]
(26)

Where \( n \) is the number of degrees of freedom, \( \phi(\xi) \) is the ith eigenfunction of the beam and \( u(\tau) \) is the ith time-dependent deflection parameter of the beam. Based on a single degree-of-freedom model of the beams (\( n=1 \)), Eq. (24) can be solved with appropriate accuracy [18]. Hence, the solution is constructed by expressing the deflection function \( W(\xi, \tau) \) as the product of two separate functions:
\[ W(\xi, \tau) = \phi(\xi) u(\tau) \]
(27)

As earlier work suggested, here the trial function is \( \phi(\xi) = 16\xi(1-\xi)^2 \) [19]. Obviously, Eq. (27) satisfies all the boundary conditions listed in Eq. (25). In order to avoid division by zero in the electrostatic force term, we multiply Eq. (24) by \( (1-W)^2 \). Substituting Eq. (27) into the resulting equation, multiplying by \( \phi(\xi) \), and integrating from 0 to 1, we obtain:
\[ (ap'u' + a\nu' + a\mu' + a\mu' + a\mu') = 0 \]
(28)

where
\[ a_1 = \left[ \phi^2 \phi \right] d\xi; \quad a_2 = -2[\phi' \phi] d\xi; \quad a_3 = [\phi' \phi] d\xi; \]
\[ a_4 = \left[ \phi^2 \phi' - 2N \phi^2 \phi' + a\phi \phi' \right] d\xi; \]
\[ a_5 = \left[ \phi^2 \phi' - N \phi^2 \phi' + 2a\phi \phi' \right] \phi' d\xi; \]
\[ a_6 = \left[ \phi^2 \phi' - 2N \phi^2 \phi' + a\phi \phi' \right] \phi' d\xi; \]
\[ a_7 = \left[ \phi^2 \phi' - N \phi^2 \phi' + 2a\phi \phi' \right] \phi' d\xi; \]
\[ a_8 = \left[ \phi^2 \phi' - 2N \phi^2 \phi' + a\phi \phi' \right] \phi' d\xi; \]

Here a dot (.) denotes differentiation with respect to the time variable \( \tau \), while a prime (') indicates the partial differentiation with respect to the coordinate variable \( \xi \). Eq. (28) is a nonlinear ordinary differential equation. In the next sections, the parameter-expansion and equivalent linearization methods are utilized to study this nonlinear oscillator.

5. Application of the parameter-expansion method

To apply PEM, first we rewrite Eq. (28) in the form:

\[ \ddot{u} + \frac{a_m}{a_1} u + 1 + a_m' u + a_m' u + a_m' u + a_m' u = 0, \]
(30)

Assume that the solution can be expressed as a power series in an artificial parameter \( p \):
\[ u = \sum_{i=0}^{n} p_i u_i(\tau) + pu_i(\tau) + p' u_i(\tau) + \cdots \]
(31)

where \( p \) is a booking parameter. We assume that the coefficients \( a_i/p \) and 1 on the left of Eq. (30) can be respectively expanded into a series of \( p \):
\[ a_i/p = a_i' + \sum_{n=1}^{\infty} p=a_i + p_0 a_i + p_1 a_i + \cdots \]
(32)

\[ 1 = \sum_{n=0}^{\infty} pb_n = pb_0 + pb_1 + \cdots \]
(33)

Substituting Eqs. (31), (32) and (33) into Eq. (30) yields:
\[ \left( a_i' + p_i a_i + p_0 a_i + p_1 a_i + \cdots \right) + \cdots \]

Equating the terms with the identical powers of \( p \) yields:
\[ p^n: \quad u_n + a_i u_n = 0, \quad u_n(0) = A, \quad u_n(0) = 0 \]
(35)

\[ p^m: \quad u_n + a_i u_n + a_0 a_i = b_n a_i u_n + b_n a_i u_n + b_n a_i u_n + b_n a_i u_n + b_n a_i u_n = 0 \]
(36)

The solution of Eq. (35) can be easily obtained
\[ u_n = A \cos(\alpha r) \]
(37)

Substituting the result (37) into Eq. (36), we have:
\[ u_n + a_i u_n + a_i A \cos(\alpha r) = b_n a_i u_n + b_n a_i u_n + b_n a_i u_n + b_n a_i u_n + b_n a_i u_n = 0 \]
(38)

Note that:
\[ \cos(\alpha r) = \frac{3 \cos(\alpha r) + \cos(3\alpha r)}{4} \]
(39)
\[
\cos'(ar) = \frac{10\cos(ar) + 5\cos(3ar) + \cos(5ar)}{16}
\]

\[
\cos'(ar) = \frac{35\cos(ar) + 21\cos(3ar) + 7\cos(5ar) + \cos(7ar)}{64}
\]

Substituting Eqs. (39), (40) and (41) into Eq. (38) yields:

\[
\ddot{u} + \omega^2 u = \left(\frac{15}{32} a_1 a_2 + \frac{9}{16} b_1 a_4 + \frac{3}{4} b_2 a_4 + \frac{35}{64} b_4 a_4\right) \cos(\omega t)
\]

\[
+ \left(\frac{5}{16} b_1 a_4 A^3 - \frac{1}{4} b_2 a_4 A^3 + \frac{1}{4} b_2 a_4 A^3 - \frac{1}{16} b_4 a_4 A^3\right) \cos(3\omega t)
\]

\[
+ \left(\frac{1}{6} b_1 a_4 A^3 + \frac{1}{4} b_2 a_4 A^3 + \frac{7}{64} b_4 a_4 A^3\right) \cos(5\omega t)
\]

\[
+ \frac{1}{64} \frac{a_4}{a_1} A^3 \cos(7\omega t) = 0
\]

No secular terms in \(u_1\) requires:

\[
\frac{a_4}{a_1} = \omega^2 + \omega^4
\]

\[
\omega_{1\text{st}} = \sqrt{\frac{64a_1 + 48a_2 A^3 + 40a_4 A^3 + 35a_5 A^3}{4(16a_1 + 10a_4 A^3 + 12a_5 A^3)}}
\]

Substituting Eq. (46) into Eq. (36), and noting Eq. (43), yields:

\[
\ddot{u} + \omega^4 u = -B, \cos(3\omega t) + B, \cos(5\omega t) + B, \cos(7\omega t)
\]

where:

\[
B_i = \begin{cases}
\frac{5}{16} \frac{a_4}{a_1} A^3 - \frac{1}{4} \frac{b_2}{a_1} A^3 \\
+ \frac{1}{4} \frac{b_2}{a_1} A^3 + \frac{5}{16} \frac{b_4}{a_1} A^3 + \frac{21}{64} \frac{b_4}{a_1} A^3
\end{cases}
\]

\[
B_i = \begin{cases}
- \frac{1}{64} \frac{b_2}{a_1} A^3 + \frac{1}{16} \frac{b_4}{a_1} A^3 \\
+ \frac{7}{64} \frac{b_4}{a_1} A^3
\end{cases}
\]

Solving Eq. (47), we obtain:
where \( \omega \) is given in Eq. (46).
The higher-order approximations for \( \omega \) and \( u(t) \) can be established in a similar manner.

6. Application of the equivalent linearization method

Now, we will apply the equivalent linearization method to find the solution of Eq. (28). The linearized equation of Eq. (28) has the form:

\[ \ddot{u} + \omega^2 u = 0 \]  

The error equation between the two Eqs. (28) and (57) is:

\[ e(\omega) = (\omega u' + \omega' u + \omega^2 u) - \omega^2 u \]  

The unknown coefficient \( \omega^2 \) is determined from the mean square error criterion:

\[ \frac{\partial}{\partial \omega^2} \langle e^2(\omega) \rangle = 0 \]

It follows that:

\[ \omega^2 = \frac{\langle uu' \rangle + \langle uu'' \rangle - \langle u' \rangle^2}{\langle u' \rangle^2} \]  

The periodic solution of Eq. (57) is:

\[ u(t) = A \cos(\omega t) \]  

Using (60), we calculate averaging operators in Eq. (59):

\[ \langle u' \rangle = \langle A^{+} \cos(\omega t) \rangle = \frac{A^{+} \cos(\omega t)}{2\pi} \int \cos(\omega t) dt = \frac{A^{+}}{2} \]  

Substituting Eqs. (61) - (67) into Eq. (59), we get:

\[ a \omega^2 = \frac{1}{2} A^+ + \frac{1}{2} A^- \]  

From Eq. (68), we obtain the approximate frequency of this oscillator:

\[ \omega_{ne} = \sqrt{\frac{6k_{d} + 48k_{A} A + 40k_{A} A + 35k_{A} A}{128}} \]  

According to Eqs. (60) and (69), we can obtain the approximate solution of this oscillator:

\[ u(t) = A \cos \left( \frac{\sqrt{6k_{d} + 48k_{A} A + 40k_{A} A + 35k_{A} A}}{\sqrt{128}} \right) \]  

Note that the frequency in Eq. (69) obtained by ELM is the same one in Eq. (46) obtained by PEM.

7. Discussions

In order to validate the effectiveness of the current approaches, Tab. 1 compares the frequencies obtained by the Energy Balance Method (EBM) [5], Variational Approach (VA) [7], Parameter-Expansion Method (in Eq. (46)), Equivalent Linearization Method (in Eq. (60)), and the exact solution \( \omega_{ex} \) [6] for various parameters  \( N, a, V \) and initial amplitudes \( A \). The values of dimensionless parameters \( N, a, V \) in Tab. 1 correspond to different forces and voltages acting on the microbeam.

Figs. 2(A), 2(B), 2(C) and 2(D) represent comparisons of the analytical solutions of \( u(t) \) based on time with the Energy Balance Method, Variational Approach and numerical solutions. It could be obtained from the figures that the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions.
Table 1: Comparison of Frequencies Corresponding to Various Parameters of System

<table>
<thead>
<tr>
<th>case</th>
<th>A</th>
<th>N</th>
<th>α</th>
<th>V</th>
<th>(\omega_\text{A} \text{[6]})</th>
<th>(\omega_\text{EBM} \text{[5]})</th>
<th>(\omega_\text{VA} \text{[7]})</th>
<th>(\omega_\text{PEM} \text{and PEM})</th>
</tr>
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<tr>
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<td>10</td>
<td>24</td>
<td>0</td>
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<td>26.3867</td>
<td>26.3644</td>
<td>26.3304</td>
</tr>
<tr>
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<td>10</td>
<td>24</td>
<td>20</td>
<td>16.6486</td>
<td>16.3829</td>
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<td>16.6445</td>
</tr>
<tr>
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<td>0.6</td>
<td>10</td>
<td>24</td>
<td>10</td>
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<td>28.9227</td>
<td>28.5579</td>
<td>28.4030</td>
</tr>
<tr>
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<td>0.6</td>
<td>10</td>
<td>24</td>
<td>20</td>
<td>18.5902</td>
<td>17.5017</td>
<td>17.0940</td>
<td>18.5409</td>
</tr>
</tbody>
</table>

Fig. 2: Comparison Of Solution Corresponding to Various Parameters of System: (A) A=0.3, N=10, \(\alpha=24, V=0\); (B) A=0.3, N=10, \(\alpha=24, V=20\); (C) A=0.6, N=10, \(\alpha=24, V=10\); (D) A=0.6, N=10, \(\alpha=24, V=20\);

Figs. 3(A), 3(B), 3(C) and 3(D) show the comparisons of the analytical solution of \(du(t)/dt\) based on \(u(t)\) with the energy balance method, variational approach and numerical solutions.
Fig. 3: Comparison Of Phase Solution Corresponding to Various Parameters of System: (A) \(A=0.3, N=10, \ A=24, \ V=0\); (B) \(A=0.3, N=10, \ A=24, \ V=20\); (C) \(A=0.6, N=10, \ A=24, \ V=10\); (D) \(A=0.6, N=10, \ A=24, \ V=20\);

Fig. 4 is presented to show the effect of amplitude \(A\) on the phase plan of the problem. And the effects of \(V, \alpha\) and \(N\) on the phase plan of the problem are presented in Figs. 5, 6 and 7, respectively. Figs. 8-10 are presented the effects of \(V, N, \alpha\) parameters on non-linear frequency base on amplitude, respectively. From Fig. 8, we see that when the voltage \(V\) acting on the microbeam increases then the frequency decreases. But from Figs. 9 and 10, the frequency increases when the axial force and the ratio \(g/h\) increase.

Fig. 4: Effect of Amplitude \(A\) on the Phase Plan of the Problem for \(N=10, \ A=24, \ V=5\).

Fig. 5: Effect of \(V\) on the Phase Plan of the Problem for \(N=10, \ A=24, \ A=0.4\).

Fig. 6: Effect of \(\alpha\) on the Phase Plan of the Problem for \(N=10, \ V=10, \ A=0.5\).

Fig. 7: Effect of \(N\) on the Phase Plan of the Problem for \(A=24, \ V=10, \ A=0.5\).

Fig. 8: Effect of \(V\) Parameter on Nonlinear Frequency Base on Amplitude for \(A=24, \ N=10\).

Fig. 9: Effect of \(N\) Parameter on Nonlinear Frequency Base on Amplitude for \(A=24, \ V=10\).
Fig. 10: Effect of α Parameter on Nonlinear Frequency Base on Amplitude for N=10, V=10.

8. Conclusions

In this paper, the Parameter-Expansion Method and the Equivalent Linearization Method were employed to solve the nonlinear governing equation of the microbeam-based micro-electromechanical system where the midplane stretching effect and distributed electrostatic force are both considered. The frequency – amplitude relationships achieved by two methods are obtained in the closed forms. Excellent agreement between approximate frequencies can be seen. The high accuracy of two methods is presented by comparing the present solutions with the previous solutions and the exact solutions as well as the numerical solutions. The effect of the initial amplitude A and the parameters V, α and N on the phase plane of the problem are investigated. And the effects of parameters V, N and α on nonlinear frequency base on amplitude A are also presented. The results show that the two current methods are very useful in analyzing nonlinear oscillations.

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