Equitable Power Domination Number of Mycielskian of Certain Graphs

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Abstract

Let \( G(V, E) \) be a simple graph with vertex set \( V \) and edge set \( E \). A set \( S \subseteq V \) is called a power dominating set (PDS), if every vertex \( u \in V - S \) is observed by some vertices in \( S \) by using the following rules: (i) if a vertex \( v \) in \( G \) is in PDS, then it dominates itself and all the adjacent vertices of \( v \) and (ii) if an observed vertex \( v \) in \( G \) has \( k > 1 \) adjacent vertices and if \( k - 1 \) of these vertices are already observed, then the remaining one non-observed vertex is also observed by \( v \) in \( G \). A power dominating set \( S \subseteq V \) in \( G(V, E) \) is said to be an equitable power dominating set (EPDS), if for every \( v \in V - S \) there exists an adjacent vertex \( u \in S \) such that the difference between the degree of \( u \) and degree of \( v \) is less than or equal to 1, i.e., \(|d(u) - d(v)| \leq 1\). The minimum cardinality of an equitable power dominating set of \( G \) is called the equitable power domination number of \( G \) and denoted by \( \gamma_{epd}(G) \). The Mycielskian of a graph \( G \) is the graph \( \mu(G) \) with vertex set \( V \cup V' \cup z \), where \( V' = \{v': v \in V\} \), and edge set \( E = \{uv': uv \in E\} \cup \{v'z : v' \in V'\} \). In this paper we investigate the equitable power domination number of Mycielskian of certain graphs.

Keywords: Dominating set; Equitable dominating set; Power dominating set; Equitable power dominating set; Equitable power domination number, Mycielskian graph.

1. Introduction

All the graphs considered in this paper are finite, connected, simple, and undirected. The notion of domination in graphs was introduced by Hedetniemi and Laskar [7] and the concept of equitable domination in the graphs was studied by Swaminathan et al. [9]. Haynes et al. introduced the concept of power domination in graphs and power domination number of graphs.

A dominating set [6, 7] of a graph \( G = (V, E) \) is a set \( S \) of vertices such that every vertex \( v \) in \( V - S \) has at least one neighbor in \( S \). The minimum cardinality of a dominating set of \( G \) is called the domination number of \( G \) and denoted by \( \gamma_d(G) \). The degree \( d(v) \) of a vertex \( v \) in \( G \) is the total number of edges of \( G \) incident with \( v \) and any two adjacent vertices \( u \) and \( v \) in \( G \) are said to hold equitable property if \(|d(u) - d(v)| \leq 1\). A dominating set \( S \subseteq V \) in \( G(V, E) \) is called equitable dominating set [1] if for every \( v \in V - S \) there exists an adjacent vertex \( u \in S \) such that the difference between degree of \( u \) and degree of \( v \) is less than or equal to 1, that is \(|d(u) - d(v)| \leq 1\). The minimum cardinality of an equitable dominating set of \( G \) is said to be equitable domination number of \( G \) and denoted by \( \gamma_{ed}(G) \).

A set \( S \subseteq V \) is called a power dominating set (PDS) [2, 4] of \( G \) if every vertex \( u \in V - S \) is observed by some vertices in \( S \) by using the following rules:

If a vertex \( v \) in \( G \) is in PDS, then it dominates itself and all the adjacent vertices of \( v \).

If an observed vertex \( v \) in \( G \) has \( k > 1 \) adjacent vertices and if \( k - 1 \) of these vertices are already observed, then the remaining one non-observed vertex is also observed by \( v \) in \( G \). The minimum cardinality of an power dominating set of \( G \) is called the power domination number of \( G \) and denoted by \( \gamma_{pd}(G) \).

Banu Priya et al. introduced the concept of equitable power domination in graphs [3]. A power dominating set \( S \subseteq V \) in \( G(V, E) \) is said to be an equitable power dominating set, if for every vertex \( v \in V - S \) there exists an adjacent vertex \( u \in S \) such that the difference between degree of \( u \) and degree of \( v \) is less than or equal to 1, that is \(|d(u) - d(v)| \leq 1\). The minimum cardinality of an equitable power dominating set of \( G \) is said to be the equitable power domination number of \( G \) and denoted by \( \gamma_{epd}(G) \). It is interesting to note that the equitable power dominating set of a graph \( G \) is not unique.

Let \( G(V, E) \) be a graph. The Mycielskian of \( G \) [10] is the graph \( \mu(G) \) with vertex set \( V \cup V' \cup z \), where \( V' = \{v': v \in V\} \), and edge set \( E = \{uv': uv \in E\} \cup \{v'z : v' \in V'\} \). We call the vertices of \( V' \) as the corresponding vertices of \( V \) for convenience sake. In this paper we establish the equitable power domination number of Mycielskian of certain graphs.

2. The Equitable Power Domination Number of the Mycielskian of Cycle and Path

One can note that the equitable power domination number of the Mycielskian of a cycle \( C_n \), for \( 3 \leq n \leq 5 \) is 2. One such example is given in Figure 1.
Let $C_n$ be a cycle on $n$ vertices. Then $\gamma_{epd}(\mu(C_n)) = 3$, for $n \geq 6$.

**Proof**

Let $C_n$ be a given cycle on $n \geq 6$ vertices $u_1, u_2, \ldots, u_n$. Obtain the Mycielskian of a cycle $C_n$, $\mu(C_n)$ with $V(\mu(C_n)) = u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n'$. Note that the vertices $u_1, u_2, \ldots, u_n$ are of degree 4, and $u_1', u_2', \ldots, u_n'$ are of degree 3 and the degree of the root vertex $v$ is $n$ in $\mu(C_n)$. To obtain the equitable power dominating set $S$ of $\mu(C_n)$, one has to choose the root vertex $v$ to be in $S$. Because $d(u_i^{(j)}) - d(v); 1 \leq i \leq n \equiv 1$. There are $2n$ non-observed vertices and without loss of generality, let $u_4$ be in $S$.

Since the degree of $u_4$ is 4, it observes $u_{k-1}, u_{k+1}, u_{k-1}$, and $u_{k+1}$. One can also see that $u_{k-1}$ and $u_{k+1}$ have only one non-observed vertex, namely $u_{k-2}$ and $u_{k+2}$ respectively. So $u_{k-2}$ and $u_{k+2}$ are observed by $u_{k-1}$ and $u_{k+1}$, respectively by the definition of power domination in graphs. One can note that all the observed vertices are having at least two non-observed adjacent vertices, so the equitable power domination property fails. So one has to choose any one of the non-observed vertices $u_{k-3}, u_{k+3}, u_{k-2}, u_{k+2}$ to be in $S$ in order to get the minimum cardinality. We choose, without loss of generality $u_{k-3}$, for discussion sake. Now $u_{k-3}$ dominates $u_{k-4}, u_{k-4}, u_{k-2}$.

Now one can see that the remaining non-observed vertices are observed by the observed vertices by the equitable power domination property. Thus $\gamma_{epd}(\mu(C_n)) = 3$ for $n \geq 6$.

2.2. Definition [5]

A path $P_n$ is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n - 1$.

2.3. Lemma

Let $P_n$ be a path on $n$ vertices.

Then $\gamma_{epd}(\mu(P_n)) =$

1. for $n = 2$,
2. for $n = 3$,
3. for $n = 4, 5, 6$.

**Figure 2:** Equitable power domination number of a path $P_n$. $\gamma_{epd}(P_n) = 4$

2.4. Theorem

Let $P_n$ be a path. Then $\gamma_{epd}(\mu(P_n)) = 5$, for $n \geq 7$.

**Proof**

Let $P_n$ be a path on $n$ vertices $u_1, u_2, \ldots, u_n$. Obtain the Mycielskian of a path $P_n$, $\mu(P_n)$ with the vertex set of $\mu(P_n)) = u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n'$, where $v$ is the root vertex of $\mu(P_n)$. It is interesting to see that the vertices $u_1, u_1', u_n$, and $u_n$, are of degree 2. The vertices $u_2, \ldots, u_{n-1}$ are of degree 4, vertices $u_1, u_2, \ldots, u_n$, and $d(v) = n$ in $\mu(P_n)$. In order to obtain an equitable power dominating set $S$ of $\mu(P_n)$, one has to choose the vertex $v$, as $d(u_1^{(j)}) - d(v); 1 \leq i \leq n \equiv 1$. And also, one has to put the vertices $u_4$ and $u_9$ in $S$ since $u_4$ and $u_9$ have the neighbors whose degrees are more than two that of itself, which clearly violates the equitable property. Now there are $2(n - 1)$ non-observed vertices in $\mu(P_n)$. Without loss of generality, choose $u_k$ to be in $S$. Now $u_k$ equitably power dominates $u_{k-1}, u_{k+1}, u_{k-1}$, and $u_{k+1}$. And also the observed vertices $u_{k-1}$ and $u_{k+1}$ eventually equitably power dominate $u_{k-2}$ and $u_{k+2}$, respectively as they are the only non-observed vertices of $u_{k-1}$ and $u_{k+1}$. Now the observed vertices $u_j$’s have more than one non-observed vertices and also the observed vertices $u_j$’s have no adjacent vertices to equitably power dominate. Therefore one has to choose any one of the vertices $u_{k-3}, u_{k+3}, u_{k-2}, u_{k+2}$ to be in $S$ in obtaining the optimum cardinality. So select $u_{k-3}$ to be in $S$.

Next $u_{k-3}$ equitably power dominates its adjacent non-observed vertices. Consequently the observed vertices $u_4$ and $u_9$ equitably power dominate the remaining non-observed vertices alternatively in both the directions. Thus $S = \{v, u_1, u_n, u_k, u_{k-3}\}$ and $|S| = 5$.

3. The Equitable Power Domination Number of the Mycielskian of Complete Bipartite Graphs

3.1. Definition [5]

A complete bipartite graph, denoted $K_{m,n}$, is a simple bipartite graph with bipartition $(X, Y)$ in which each vertex of $X$ is joined to each vertex of $Y$.

3.2. Theorem [3]

Let $K_{m,n}$, $m,n \geq 2$ be a complete bipartite graph. Then $\gamma_{epd}(K_{m,n}) = \begin{cases} m + n & \text{if } |m - n| \geq 2 \\ 2 & \text{if } |m - n| < 2. \end{cases}$

3.3. Theorem

Let $K_{m,n}$ be a complete bipartite graph. Then $\gamma_{epd}(\mu(K_{m,n})) = \begin{cases} 2n + 3 & \text{for } m = 2, n \geq 5, \\ m + n + 3 & \text{for } |2m - n - 1| < 2, \\ 2(m + n) + 1 & \text{for } |2m - n - 1| \geq 2. \end{cases}$

**Proof**

Let $K_{m,n}$ be a complete bipartite graph with partitions $V_1 = \{a_1, a_2, \ldots, a_m\}$ and $V_2 = \{b_1, b_2, \ldots, b_n\}$. The Mycielskian of complete bipartite graph $K_{m,n}$, denoted $\mu(K_{m,n})$ is obtained as follows: $V(\mu(K_{m,n})) = V_1 \cup V_2 \cup V_3 \cup w$, where $V_1 = V(K_{m,n}) = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$, $V_2 = V(K_{m,n}) = \{a_1', a_2', \ldots, a_m', b_1', b_2', \ldots, b_n'\}$ and $V_3 = \{w\}$, $E(\mu(K_{m,n})) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(K_{m,n}), E_2 = \{a_i b_j; 1 \leq i \leq m, 1 \leq j \leq n\}$, $E_3 = \{a_i b_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E_4 = \{w a_i, w b_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. It is interesting to see that the degree of vertices in $V(\mu(K_{m,n}))$ are as follows. The vertices in the partition $V_1$ are of degree 2m, the vertices in the partition $V_2$ are of degree $2n$, the corresponding vertices of the partition $V_1$ are of degree $n + 1$, the corresponding vertices of the partition $V_2$ are of degree $m + 1$ and the root vertex is of degree
When obtaining the equitable power dominating set $S$ of $\mu(G_{i,j})$, the following three cases arise.

**Case 1:** When $m = 2$ and $n \geq 5$

In this case one has to select all the vertices except duplicate vertices of the partition $\{i\}$ to be in the equitable dominating set $S$, since the root vertex is of degree 2 and the remaining three vertices are of degree 2. Moreover, when the newly added vertex $i$ is of degree 2 which clearly violates the required equitable property with all other vertices in $\mu(G_{i,j})$. Now one also has to choose the remaining vertices in $G_{i,j}$ to be in $S$ as they are of degree 2 and the remaining three vertices are of degree greater than 3, which violates the equitably dominating condition.

Finally one has a choice of choosing either $i$ or the adjacent root vertex $j$ to be in $S$ as they are of same degree. So we choose $j$ to be in $S$ for $i = j$.

**Proof**

Let $G_{i,j}$ be a star graph with pendant vertices $p, p_1, p_2, \ldots, p_n$. To obtain the equitable power dominating set $\mu(G_{i,j})$, the vertex $i$ must be in $S$, since $i$ is of degree 2 and the pendant vertices $p_1, p_2, \ldots, p_n$ must be in $S$ because they are of degree 2 and the remaining three vertices are of degree greater than 3.

Moreover, it is easy to see that $i$ is of degree 2 and the root vertex $j$ must be in $S$ as it is of degree 2 which clearly violates the required equitable property with all other vertices in $\mu(G_{i,j})$. Now one also has to choose the remaining vertices in $G_{i,j}$ to be in $S$ as they are of degree 2 and the remaining three vertices are of degree greater than 3, which violates the equitably dominating condition. Finally one has a choice of choosing either $i$ or the adjacent root vertex $j$ to be in $S$ as they are of same degree. So we choose $j$ to be in $S$ for $i = j$.

Thus $|S| = 2(i + j)$.

### 4. Equitable Power Domination Number of the Mycielskian of Comb, Complete, and Fan Graphs

#### 3.6. Definition

Comb is a graph obtained by joining a single pendant edge to each vertex of a path. One can see that the equitable power domination number of the Mycielskian of a comb graph $G_{i,j}$ is 4 for $i = j = 0$, as given in Figure 3. We have the following theorem when $i \geq 3$.

**Theorem**

Let $G_{i,j}$ be a comb graph. Then $\mu(G_{i,j}) = 2i + 2j$.

**Proof**

Let $G_{i,j}$ be a comb graph with vertex set $\{p_1, p_2, \ldots, p_n\}$. The Mycielskian of a comb graph $G_{i,j}$ is obtained as follows. $\mu(G_{i,j}) = \{p_1, p_2, \ldots, p_n, i, j\}$, where $i$ is the root vertex of $G_{i,j}$ and $j$ is the terminal vertex of $G_{i,j}$. The Mycielskian of a comb graph $G_{i,j}$ is obtained as follows. $\mu(G_{i,j}) = \{p_1, p_2, \ldots, p_n, i, j\}$, where $i$ is the root vertex of $G_{i,j}$ and $j$ is the terminal vertex of $G_{i,j}$. The Mycielskian of a comb graph $G_{i,j}$ is obtained as follows. $\mu(G_{i,j}) = \{p_1, p_2, \ldots, p_n, i, j\}$, where $i$ is the root vertex of $G_{i,j}$ and $j$ is the terminal vertex of $G_{i,j}$.

Finally, it is easy to see that $i = j$ as $i$ is of degree 2 and the root vertex $j$ must be in $S$ as it is of degree 2 which clearly violates the required equitable property with all other vertices in $\mu(G_{i,j})$. Now one also has to choose the remaining vertices in $G_{i,j}$ to be in $S$ as they are of degree 2 and the remaining three vertices are of degree greater than 3, which violates the equitably dominating condition. Finally one has a choice of choosing either $i$ or the adjacent root vertex $j$ to be in $S$ as they are of same degree. So we choose $j$ to be in $S$ for $i = j$.

Thus $|S| = 2(i + j)$.
equitable power dominating set of \( \square \), one must choose \( \square \) which is of maximum degree \( 2 \). One should also choose \( \square, \square, \ldots, \square \) for \( 1 \leq \square \leq 2 \) and \( \square, \square \) for \( 2 \leq \square \leq \square - 1 \) since there are no adjacent vertices satisfying the equitable property. Now \( \square \) and \( \square \) observe \( \square \) and \( \square \) respectively. Finally, it suffices to choose any one of the vertices from \( \square_2 \) to \( \square_2 \) to be in \( \square \), since they are of same degree and also all its adjacent neighbors are already observed.

4.2. Definition [5]

Any two distinct vertices of a graph \( \square \) are adjacent, then \( \square \) is said to be a complete graph and it is denoted by \( \square \).

One can see the equitable power domination number of the Mycielskian of a complete graph \( \square \), for \( \square = 3 \), as given in Figure 4. We have the following theorem when \( \square \geq 3 \).

Figure 4: Equitable power domination number of the complete graph \( \square \).

4.3. Theorem

Let \( \mu(\square, \square) \) be a Mycielskian graph of a complete graph \( \square \). Then \( \square \) and \( \square \) are denoted by \( \mu(\square, \square) = 2 \).

Proof

Let \( \square \) be a complete graph with vertex set \( \square = \{ \square_1, \square_2, \ldots, \square_n \} \). The Mycielskian of a complete graph \( \mu(\square, \square) \) is obtained with the vertex set \( \mu(\mu(\square, \square)) = \{ \square_1, \square_2, \ldots, \square_n, \square_1, \square_2, \ldots, \square_n \} \), where \( \square \) is the root vertex of \( \mu(\square, \square) \). It is interesting to see that the degree of the vertices \( \square_1, \square_2, \ldots, \square_n \) in the graph \( \mu(\square, \square) \) is \( \square \) and the vertices \( \square_1, \square_2, \ldots, \square_n \) are of degree \( 2(\square - 1) \). In order to obtain an equitable power dominating set \( \square \), let us first choose \( \square \) which equitably power dominates \( \square_1, \square_2, \ldots, \square_n \) and from the remaining vertices, it is enough to choose at least any one vertex from \( \square_1, \square_2, \ldots, \square_n \) of \( \mu(\square, \square) \) say \( \square \). Then \( \square \) equitably power dominates \( \square_1, \square_2, \ldots, \square_n \). Thus \( \square \) is \( \square \) and hence \( \square \) is \( \square \).

4.4. Definition

A fan graph \( \square_{(\square, \square)} \) is defined as the graph \( \square_{(\square, \square)} \) where \( \square \) is the empty graph on one vertex and \( \square_1 \) is a path on \( \square \) vertices.

4.5. Theorem

Let \( \square_{(\square, \square)} \) be a fan graph. Then \( \square \) and \( \square \) are denoted by \( \mu(\square_{(\square, \square)}) = \square + 3 \) for \( \square \geq 3 \).

Proof

Let \( \square_{(\square, \square)} \) be the given fan graph with common vertex \( \square_0 \) and the remaining vertices \( \square_1, \square_2, \ldots, \square_n \). The Mycielskian of a fan \( \square_{(\square, \square)} \), denoted by \( \mu(\square_{(\square, \square)}) \) is obtained with the vertex set \( \mu(\mu(\square_{(\square, \square)})) = \{ \square_0, \square_1, \ldots, \square_n, \square_0, \square_1, \ldots, \square_n \} \). It is interesting to see that there are \( 2(\square + 3) \) vertices, namely \( \square_0, \square_1, \ldots, \square_n, \square_1, \square_2, \ldots, \square_n \). When we construct \( \mu(\square_{(\square, \square)}) \), the degree of vertices are as follows:

\[
\begin{align*}
\mu(\square_0) &= \mu(\square_1) = \square + 1, \quad \mu(\square_2) = \mu(\square_3) = \square + 2, \quad \mu(\square_4) = \square + 3, \quad \mu(\square_5) = \square + 4, \quad \mu(\square_6) = \square + 5, \quad \mu(\square_7) = \square + 6.
\end{align*}
\]

\[
\begin{align*}
\mu(\square_{(\square, \square)}) &= \{ \square_0, \square_1, \ldots, \square_n, \square_1, \square_2, \ldots, \square_n \} \quad \text{where } \square_
\end{align*}
\]

4.5. Remark

1. \( \square \) and \( \square \) for \( \square = 3, 4 \).
2. \( \square \) and \( \square \) for \( \square = 5, 6 \).

5. Conclusion

In this paper the equitable power domination in the Mycielskian of certain graphs has been studied and the equitable power domination number of the Mycielskian of various classes of graphs has been determined. Establishing the equitable power domination number of other classes of graphs is open and this is for future work.

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