Common Fixed Point Theorems in Bipolar Metric Spaces with Applications to Integral Equations

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Abstract

This paper establishes the existence of coincidence fixed-point and common fixed-point results for two mappings in a complete bipolar metric spaces. Some interesting consequences of our results is achieved. Finally, an illustration which presents the applicability of the results is achieved.

Keywords: Bipolar metric space; common fixed point; completeness; coincidence point; Covariant and contravariant maps; weakly compatible.

1. Introduction

In 1922, S. Banach, [4] made the introduction to the concept of Banach contraction principle. It is considered as the most fundamental tool in non-linear analysis. It explains that in complete metric spaces, each contractive mapping has a solitrary fixed point. It has been extended and generalization of various types of metric spaces (see [6] - [9], [13]). Jungck [11] has introduced the concept of common fixed point in metric spaces for commuting mappings in 1966. Afterwards Jungck [12] initiated concept of compatibility and established some results. Subsequently to improve many authors have established common and coincidence fixed point results for mappings (see [3], [5], [10]) and reference therein.

Very recently, Mutlu and Gürdal [2] introduced notion of bipolar metric spaces in 2016. Also, they investigated some fixed and coupled fixed point results on this space, [1] and reference therein.

In this paper, we will continue to study fixed points in the frame of bipolar metric-spaces. More squarely, some common fixed-point results for two covariant and contravariant mappings under various contractive conditions will be established. We have illustrated the validity and effectiveness of the hypotheses of the results. The present results extends and improves the concepts in some of the recent literatures [2].

**Definition 1.1:** [2] Let U and V be two non-empty sets. Suppose d: UxV → [0,∞) be a mapping satisfying the below properties:

(i) If d (u, v) = 0, then u=v for all (u, v) ∈ UxV,
(ii) If d (u, v) = d (v, u) for all (u, v) ∈ UxV,
(iii) If d (u, v) = d (u, a) + d (a, v) for all u, v ∈ UxV,
(iv) If d (u₁, v₂) ≤ d (u₁, v₁) + d (u₂, v₁) + d (u₂, v₂) for all u₁, v₁, u₂, v₂ ∈ UxV,

Then the mapping d is termed as Bipolar-metric of the pair (U, V) and the triple (U, V, d) is termed as Bipolar-metric space.

**Example 1.2:** [2]: Let U = (1, ∞) and V = [-1, 1]. Define d: UxV → [0,∞) as d (a, b) = |a² – b²|, for all (a, b) ∈ UxV. Then the triple (U, V, d) is a disjoint Bipolar-metric space.

**Definition 1.3:** [2] Assume (U₁, V₁) and (U₂, V₂) as two pairs of sets and a function as F: U₁ ∪ V₁ → U₂ ∪ V₂ is said to be a covariant map. If F (U₁) ⊆ U₂ and F (V₁) ⊆ V₂ and denote this with S: (U₁, V₁) → (U₂, V₂).

And the mapping S: U₁ ∪ V₁ → U₂ ∪ V₂ is said to be a contravariant map. If F (U₁) ⊆ V₂ and F (V₁) ⊆ U₂, and write F: (U₁, V₁) → (V₂, U₂). In particular, if d₁ and d₂ are bipolar metric on (U₁, V₁) and (U₂, V₂), respectively, we sometimes use the notation F: (U₁, V₁, d₁) → (U₂, V₂, d₂) and F: (U₁, V₁, d₁) → (V₂, U₂, d₂).

**Definition 1.4:** [2] Assume (U, V, d) as a bipolar metric space. A point v ∈ U∪V is termed as a left point if v ∈ U, a right point if v ∈ V and a central point if both. Similarly, a sequence {vₙ} on the set U and a sequence {vₙ} on the set V are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence is said to be convergent to a point v, if and only if {vₙ} is the left sequence, v is the right point and limₙ→∞ dₙ(v, vₙ) = 0; or {vₙ} is a right sequence, v is a left point and limₙ→∞ dₙ(v, vₙ) = 0. A bi-sequence (({vₙ₁}, {vₙ₂}) on (U, V, d) is a sequence on the set U∪V. If the sequence ({vₙ₁}) and {vₙ₂} are convergent, then the bi-sequence (({vₙ₁}, {vₙ₂})) is said to be convergent. (({vₙ₁}, {vₙ₂})) is Cauchy sequence. If limₙ→∞ dₙ(vₙ₁, vₙ₂) = 0 in a bipolar metric space, every convergent bi-sequence is convergent. A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent hence bi-convergent.

**Definition 1.5:** [2] Let (U₁, V₁, d₁) and (U₂, V₂, d₂) be a bipolar metric spaces.

(i) A map F is called continuous, if it left continuous at each point v ∈ U₁ and right continuous at each point v ∈ V₁.
(ii) A contravariant map F: (U₁, V₁, d₁) → (U₂, V₂, d₂) is continuous if and only if it is continuous as a covariant map F: (U₁, V₁, d₁) → (U₂, V₂, d₂).

It can be seen from the definition (1.4) that a covariant or a contravariant map F: (U₁, V₁, d₁) → (U₂, V₂, d₂) is continuous if and only if (uₙ₁) → v on (U₁, V₁, d₁) implies F(uₙ₁) → F(v) on (U₂, V₂, d₂).
2. Main Results

In this section, we will give some common fixed-point theorems for two covariant and contravariant mappings satisfying various contractive conditions in complete bipolar metric spaces.

Definition 2.1: The mappings $F$ and $G$ on bipolar metric-space $(U, V, d)$ are said to be compatible, if for arbitrary bisequences $\{(a_n), (b_n)\} \subseteq (U, V)$, such that $\lim n \to \infty F(a_n) = \lim n \to \infty G(b_n) = v \in U \cup V$, then $d(GF(a_n), FG(b_n)) \to 0$ as $n \to \infty$.

Definition 2.2: Assume that $F$ and $G$ are two covariant or contravariant mappings of the set $U \cup V$.

(i) If $v = Fv = Gv$ for some $v \in U \cup V$, then $v$ is named as a common fixed point of $F$ and $G$.

(ii) If $u = Fv = Gv$ for some $u \in U \cup V$, then $v$ is considered as a coincidence point of $F$ and $G$, and $u$ is termed as the point of coincidence of $F$ and $G$.

(iii) If $F$ and $G$ commute at all of their coincidence points; i.e., $FG = GF$ for all $v \in \{v \in U \cup V : Fv = Gv\}$, then $F$ and $G$ are called weakly Compatible. In the metric-space, if the mapping $F$ and $G$ are compatible, then they are weakly compatible, while the converse becomes untrue (12)). The same comes for the bipolar metric spaces.

Lemma 2.3: If the mapping $F$ and $G$ on the bipolar metric space $(U, V, d)$ are compatible, then they are weakly compatible.

Proof: Let $Fv = Gv$ for some $v \in U \cup V$. It is sufficient to show that $FGv = GFv$.

Putting $a_n = v$ and $b_n = v$ for every $n \in N$, we have

\[ \lim n \to \infty F(a_n) = \lim n \to \infty G(b_n) = v \]

and, hence $Fv = Gv$.

Now using the $d(Fv, Gv) = 0$, which means $FGv = GFv$.

But, the converse does not hold. For example, let $U = (0, \infty)$ and $V = \{1, 2\}$, define: $d : U \times U \to [0, \infty)$ as $d(v, w) = |v^2 - w^2|$, for all $(v, w) \in (U, U)$. Then $(U, U; d)$ is a Bipolar-metric space.

Set

\[ Fv = \{ v, if \ v \in (0, \frac{1}{3}) \} \]

and

\[ Gv = \{ 1 - 2v, if \ v \in (\frac{1}{3}, \infty) \} \]

Firstly, we can calculate that set of their coincidence point is singleton set $\{\frac{1}{3}\}$, and then we have $F$ and $G$ are compatible at this point. Hence $F$ and $G$ are weakly compatible. However, we can prove they are not compatible. In this purpose, we construct a bisequence $\{(a_n), (b_n)\} \subseteq (U, V)$ such that $a_n = 1 - \frac{1}{n} \in U$ and $b_n = 1 - \frac{1}{n} \in V$ for all $n \geq 3$. In this case, we have $F(a_n) = 1 - \frac{1}{n}$ and $G(b_n) = 1 - \frac{1}{n}$. Then $\lim n \to \infty F(a_n) = \lim n \to \infty G(b_n) = 1$. In fact we have

\[ d(F(a_n), 1) = d(1, \frac{1}{n}) = \left| 1 - \frac{1}{2} \right| - \left| 1 - \frac{1}{n} \right| \to 0 \]

as $n \to \infty$ and

\[ d(1, G(b_n)) = d(1, 1 - \frac{1}{n}) = \left| 1 - \frac{1}{2} \right| - \left| 1 - \frac{1}{n} \right| \to 0 \]

as $n \to \infty$.

But $d(GF(a_n), FG(b_n)) = d(G(1 - \frac{1}{n}), F(1 - \frac{1}{n})) = 2 - \frac{1}{n}$.

Therefore, $\lim n \to \infty d(GF(a_n), FG(b_n)) \to 0$.

Lemma 2.4: If the mappings $F$ and $G$ are weakly compatible mappings of a set $U \cup V$. If $F$ and $G$ have a unique coincidence point, then $F$ and $G$ have a unique common fixed point.

Proof: Since $u = Fv = Gv$ for some $v \in U \cup V$ and $F$ and $G$ are weakly compatible, we have $Fu = GFv = GFv = Gu$ is a point of coincidence of $F$ and $G$. But $u$ is the only coincidence point of $F$ and $G$, so $u = Fu = Gu$. Moreover, if $v = Fv = Gv$, then $v$ is coincidence point of $F$ and $G$, and hence $v = v$ by the uniqueness. Thus $v$ is a unique common fixed point of $F$ and $G$.

2.1. Common fixed point theorems on covariant maps

Theorem 2.5: Assume $(U, V, d)$ is a complete bipolar metric spaces and given contractions, $F, G : (U, V, d) \to (U, V)$ satisfies $d(Fu, Gu) \leq \mu d(u, v)$ for all $(u, v) \in U \cup V$, where $\mu \in (0, 1)$. (1)

Then the mappings $F, G : U \cup U \to U \cup V$ have a unique common fixed point.

Proof: Let $a_0 \in U$ and $b_0 \in V$ and we construct a bisequences $\{(a_n), (b_n)\} \subseteq (U, V)$ by the way: $a_{2n} = a_{2n+1}, Gb_{2n+1} = a_{2n+2}$ and $Fb_{2n} = b_{2n+1}, Gb_{2n+1} = b_{2n+2}$ for all $n \in N$. Let $\mu \in (0, 1)$, put $K = \mu d(a_0, b_0) + d(a_0, \beta_1)$ and $S_n = \frac{\mu^{n+1}}{1 - \mu} K$. Then for each positive integer $n$ and 1 from (1), we have

\[ d(a_{2n}, b_{2n+1}) = d(Fa_{2n-1}, GB_{2n+1}) \leq \mu d(a_{2n}, \beta_{2n+1}) \]

if $a_n \in N$ with $n > 1$

\[ d(a_n, \beta_n) = d(a_{n+1}, \beta_{n+1}) + d(a_{n+1}, \beta_n) = \mu d(a_{n+1}, \beta_n) + d(a_n, \beta_n) \leq \mu d(a_{n+1}, \beta_n) + \mu^{n+1} K \]

\[ d(a_{2n}, \beta_{2n+1})(d(a_{2n}, \beta_{n+1}) + d(a_{2n}, \beta_n)) \leq \mu d(a_{n+1}, \beta_n) + \mu^{n+1} K \]

\[ d(a_{2n+1}, \beta_{2n+2}) + d(a_{2n+2}, \beta_{2n+1}) \leq \mu d(a_{2n}, \beta_{2n+1}) + \mu^{n+2} K \]

\[ d(\beta_n, \beta_{n+1}) \leq \mu d(a_{n+1}, \beta_n) + \mu^{n+1} K \]

\[ \mu^{n+1} + \mu^{n+2} \leq \mu^{n+1} + \mu^{n+2} + \mu^{n+3} K \]

Therefore, $\mu^{n+1} K, K = S_n$

And similarly, $d(a_n, \beta_n) \leq S_n$

Let $\epsilon > 0$ and $0 < \mu < 1$, there exist a positive integer $n_0 \in N$ such that $S_n = (1 - \mu^n) \leq \frac{\epsilon}{\mu^{n+1}}$.

Then $d(a_n, \beta_n) \leq S_n \leq \frac{\epsilon}{\mu^{n+1}}$, hence $d(a_n, \beta_n) \to 0$ as $n \to \infty$. Hence $\{a_n\}$ is a Cauchy sequence. $(U, V, d)$ is complete, the bisequence $\{(a_n), (\beta_n)\}$ converges, and thus biconverges to point v in $U \cup V$ such that $\lim n \to \infty \lim a_n = v$.

Then there exist $n_1 \in N$ with $d(a_{n_1}, v) < \frac{\epsilon}{2} \leq d(\beta_{n_1}, v) < \frac{\epsilon}{2}$ for all $n \geq n_1$ and $\epsilon > 0$. Since $\{a_n\}$ is a Cauchy sequence, we get $d(a_{n_1}, v) < \frac{\epsilon}{2}$. Now using the $(B_1)$ and from (1), we have

\[ d(Fv, v) \leq d(Fv, \beta_{n_1}) + d(\beta_{n_1}, \alpha_{n_1}) + d(\alpha_{n_1}, v) \]

\[ \leq d(Fv, \beta_{n_1}) + d(\alpha_{n_1}, \beta_{n_1}) + d(\alpha_{n_1}, v) \]

\[ \leq \mu \cdot \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \]

For each $n \in N$ and $0 < \mu < 1$. Then $d(Fv, v) = 0$, and hence $Fv = v$.

But $d(v, Gu) \leq d(Fv, Gu) \leq \mu d(v, v) \leq \mu d(v, v) = 0$.

Therefore, we have $d(v, Gu) = 0$, which implies that $Gu = v$.

Thus $v$ is a common fixed point of $F$ and $G$. 

In the following, we will prove the uniqueness of common fixed point in $U \cup V$. For this purpose, let $\psi \in U \cup V$ be another fixed point of $F$ and $G$ such that $F \psi = G \psi = \psi$ implies $\psi \in U \cap V$. From (1), we have

$$d(\psi, \psi) = d(F(\psi), G(\psi)) \leq \mu d(\psi, \psi) < \epsilon$$

Thus, it’s holds only when $d(\psi, \psi) = 0$ which gives that $\psi = \psi$. Hence $F$ and $G$ have a unique common fixed point in $U \cup V$.

**Remark 2.6**: In theorem 2.5, if $F=G_1(1)$ becomes $d(Fu, Fv) \leq \mu d(u, v)$ for all $(u, v) \in U \cup V$, where $\mu \in (0, 1)$. (2) In this case, we have the following corollary, which can also be found in [2].

**Corollary 1**: Assume $(U, V, d)$ be a complete bipolar metric spaces and given contractions, $F, \ G: (U, V, d) \rightarrow (U, V, d)$ satisfies (2).

Then the mappings $F: U \cup V \rightarrow U \cup V$ has a unique fixed point.

**Theorem 2.7**: Assume $(U, V, d)$ be a complete bipolar metric spaces and given contractions, $F, \ G: (U, V, d) \rightarrow (U, V, d)$ satisfies $d(Fu, Fv) \leq \mu d(Gu, Gv)$ for all $(u, v) \in U \cup V$, where $\mu \in (0, 1)$. (3) If $R(F) \subseteq R(G)$ and $R(G)$ is complete in $U \cup V$.

Then for $F$ and $G$ have a unique point of coincidence in $U \cup V$. Furthermore, if $F$ and $G$ are weakly compatible, then the mappings $F, \ G: U \cup V \rightarrow U \cup V$ have a unique common fixed point. **Proof**: Let $\alpha_1 \in U$ and $\beta_1 \in V$ such that $F \alpha_1 = G \beta_1$ and $F \beta_1 = G \alpha_1$ which can be done $R(F) \subseteq R(G)$. Let $\alpha_2 \in U$ and $\beta_2 \in V$ such that $F \alpha_2 = G \beta_2$ and $F \beta_2 = G \alpha_2$. Repeating the process, we get a bi-sequences $(\alpha_n, \beta_n) \subseteq (U, V)$ satisfying $F \alpha_{n+1} = G \beta_n$ and $F \beta_{n+1} = G \alpha_n$ for all $\in \mathbb{N}$. Then $\mu(0, 1, 1)$, put $K = (G \alpha_n, G \beta_n) + d(G \alpha_n, G \beta_n)$ and $S_n = \frac{\epsilon^n}{\mu^n}$.

Then for each positive integer $n$ and $1$ from (3), we have $d(F \alpha_n, G \beta_n) = d(F(G \alpha_n)) < \epsilon$.

**Theorem 2.9**: Let $(U, V, d)$ be a complete bipolar metric spaces and given contravariant contractions, $F, \ G: (U, V, d) \rightarrow (U, V, d)$ satisfies $d(Fu, Fv) \leq \mu d(Gu, Gv)$ for all $(u, v) \in U \cup V$, where $\mu \in (0, 1)$. (4) Then the mappings $F, \ G: U \cup V \rightarrow U \cup V$ have a unique common fixed point. **Proof**: Let $\alpha_0 \in U$ and $\beta_0 \in V$ and we construct a bi-sequences $(\alpha_n, \beta_n) \subseteq (U, V)$ by the way: $F \alpha_0 = \beta_0$, $G \beta_0 = \beta_0$ and $F \beta_0 = G \alpha_0$, $G \alpha_0 = \alpha_0$. Then for each positive integer $n$ and $1$ from (4), we have $d(F \alpha_n, \beta_n) = d(F \beta_n, G \alpha_n) \leq \mu d(\alpha_n, \beta_n)$.

**Example 2.8**: In Theorem 2.7, the condition that $R(G)$ is complete in $U \cup V$ is essential. For example, let $U = \{u \in U \cup V \}$ and $V = \{v \in U \cup V \}$ is lower triangular matrices over $R$ and $\mathcal{M}_n(R)$ be another fixed point of $U \cup V$. Then it follows from Lemma 2.4 that $F$ and $G$ have a unique common fixed point.

**Remark 2.6**: In theorem 2.5, if $F=G_1(1)$ becomes $d(Fu, Fv) \leq \mu d(Gu, Gv)$ for all $(u, v) \in U \cup V$, where $\mu \in (0, 1)$. (2) In this case, we have the following corollary, which can also be found in [2].

**Corollary 1**: Assume $(U, V, d)$ be a complete bipolar metric spaces and given contractions, $F, \ G: (U, V, d) \rightarrow (U, V, d)$ satisfies (2).

Then the mappings $F: U \cup V \rightarrow U \cup V$ has a unique fixed point. **Proof**: Let $\alpha_1 \in U$ and $\beta_1 \in V$ such that $F \alpha_1 = G \beta_1$ and $F \beta_1 = G \alpha_1$ which can be done $R(F) \subseteq R(G)$. Let $\alpha_2 \in U$ and $\beta_2 \in V$ such that $F \alpha_1 = G \beta_2$ and $F \beta_2 = G \alpha_2$.

Then for each positive integer $n$ and $1$ from (3), we have $d(F \alpha_n, G \beta_n) = d(F(G \alpha_n)) < \epsilon$.

**2.2. Common fixed point theorems on contravariant maps**

**Theorem 2.9**: Let $(U, V, d)$ be a complete bipolar metric spaces and given contravariant contractions, $F, \ G: (U, V, d) \rightarrow (U, V, d)$ satisfies $d(Fu, Gu) \leq \mu d(u, v)$ for all $(u, v) \in U \cup V$, where $\mu \in (0, 1)$. (4) Then the mappings $F, \ G: U \cup V \rightarrow U \cup V$ have a unique common fixed point. **Proof**: Let $\alpha_0 \in U$ and $\beta_0 \in V$ and we construct a bi-sequences $(\alpha_n, \beta_n) \subseteq (U, V)$ by the way: $F \alpha_0 = \beta_0$, $G \beta_0 = \beta_0$ and $F \beta_0 = G \alpha_0$, $G \alpha_0 = \alpha_0$. Then for each positive integer $n$ and $1$ from (4), we have $d(F \alpha_n, \beta_n) = d(F \beta_n, G \alpha_n) \leq \mu d(\alpha_n, \beta_n)$.

**Example 2.8**: In Theorem 2.7, the condition that $R(G)$ is complete in $U \cup V$ is essential. For example, let $U = \{u \in U \cup V \}$ and $V = \{v \in U \cup V \}$ is lower triangular matrices over $R$ and $\mathcal{M}_n(R)$ be another fixed point of $U \cup V$. Then it follows from Lemma 2.4 that $F$ and $G$ have a unique common fixed point.
\[ d(Fv, Fu) \leq \mu d(Gu, Fu) \leq d(Fv, Gu) \] (7)

for all \( (u, v) \in U \times V \), where \( \mu \in (0, \frac{1}{2}) \). If \( R(F) \subseteq R(G) \) and \( R(G) \) is complete in \( U \times V \). Then \( F \) and \( G \) have a unique point of coincidence in \( U \times V \).

**Proof:** Let \( x_0 \in U \) and \( \beta_0 \in V \), for each nonnegative integer \( n \), we construct a bi-sequence \( \{ x_n \} \subseteq U \) as \( F \alpha_n = G \beta_n \) and \( F \beta_n = G \alpha_n \), for all \( n \). Then each sequence \( x_n \) and \( \beta_n \) have a common fixed point in \( U \times V \). Hence \( F \) and \( G \) have a unique common fixed point in \( U \times V \). (4)

**Remark 2.10:** In theorem 2.9, if \( F \subseteq G \) becomes \( d(Fv, Fu) \leq \mu d(u, v) \) for all \( (u, v) \in U \times V \), where \( \mu \in (0, 1) \). (5)

In this case, we have the following corollary, which can also be found in [2].

**Corollary 2:** Assume \( (U, V, d) \) be a complete bipolar metric spaces and contravariant contractions, \( F: (U, V, d) \rightarrow (U, V, d) \) satisfies (5). Then the mappings \( F: U \times V \rightarrow U \times V \) has a unique fixed point.

**Theorem 2.11:** Assume \( (U, V, d) \) be a complete bipolar metric spaces and given contractions, \( F, G: (U, V, d) \rightarrow (U, V, d) \) satisfies \( d(Fv, Fu) \leq \mu d(Gu, Fu) \) for all \( (u, v) \in U \times V \), where \( \mu \in (0, 1) \). (6)

If \( R(F) \subseteq R(G) \) and \( R(G) \) is complete in \( U \times V \). Then \( F \) and \( G \) have a unique point of coincidence in \( U \times V \).

Furthermore, if \( F \) and \( G \) are weakly compatible, then the mappings \( F, G: U \times V \rightarrow U \times V \) have a unique common fixed point.

**Proof:** Let \( x_0 \in U \) and \( \beta_0 \in V \), for each nonnegative integer \( n \), we construct a bi-sequence \( \{ x_n \} \subseteq U \) as \( F \alpha_n = G \beta_n \) and \( F \beta_n = G \alpha_n \), for all \( n \). Then each sequence \( x_n \) and \( \beta_n \) have a unique common fixed point in \( U \times V \). Hence \( F \) and \( G \) have a unique common fixed point in \( U \times V \). (4)

**Remark 2.10:** In theorem 2.9, if \( F \subseteq G \) becomes \( d(Fv, Fu) \leq \mu d(u, v) \) for all \( (u, v) \in U \times V \), where \( \mu \in (0, 1) \). (5)

In this case, we have the following corollary, which can also be found in [2].

**Corollary 2:** Assume \( (U, V, d) \) be a complete bipolar metric spaces and contravariant contractions, \( F: (U, V, d) \rightarrow (U, V, d) \) satisfies (5). Then the mappings \( F: U \times V \rightarrow U \times V \) has a unique fixed point.
Example 2.13: In Theorem 2.12, the condition that \( R(G) \) is complete in \( U \cup V \) is essential. For example, let \( U = \{ u_m \} \) be upper triangular matrices over \( R \) and \( V = \{ v_m \} \) be lower triangular matrices over \( R \). Define \( d: \{ u_m \} \times \{ v_m \} \rightarrow [0, \infty) \) by \( d(P, Q) = \sum_{i=1}^{\infty} |p_{ij} - q_{ij}| \) for all \( P = (p_{ij}) \in \{ u_m \} \) and \( Q = (q_{ij}) \in \{ v_m \} \). Then obviously, \( (U, V, d) \) is a complete bipolar metric space. Define two mappings \( F, G: (U, V) \rightarrow (U, V) \) by the following way:

\[
F(P) = \begin{cases} 3P_{m \times m}, & m \leq 3P_{m \times m} \\ 1_{m \times m}, & (p_{ij})_{m \times m} = 0 \\ \end{cases}
\]

\[
G(P) = \begin{cases} 2P_{m \times m}, & m \leq 2P_{m \times m} \\ 3I_{m \times m}, & (p_{ij})_{m \times m} = 0 \\ \end{cases}
\]

Then we have

\[
d(FQ, ), F(P) = d\left(\frac{1}{3}(q_{ij})_{m \times m} \right) \leq \frac{1}{3} \sum_{i=1}^{\infty} |q_{ij}| \leq \frac{1}{3} \sum_{i=1}^{\infty} |p_{ij}| \leq \mu(dGP, F) + d(FQ, GQ)
\]

where \( \mu = \frac{1}{3} \) in \((0,1)\) and \( R(F) \subseteq R(G) \) but \( R(G) \) is not complete in \( U \cup V \). We can compute that \( F \) and \( G \) do not have a point of coincidence in \( U \cup V \).

3. Applications

3.1. Application to the existence of solutions of integral equations

Theorem 3.1: Let us consider the integral equation

\[
y(x) = \int f(x, y(x)) \, dx, \quad \forall x \in E_1 \cup E_2
\]

where \( E_1 \cup E_2 \) is Lebesgue measurable set with \( m(E_1 \cup E_2) < \infty \). Suppose that \( f: E_1 \times E_2 \times [0, \infty) \rightarrow [0, \infty) \) and \( S_2 \) is continuous function.

Then, the equation admits unique solution in \( L^p(E_1) \cup L^p(E_2) \).

Proof: Let \( U = L^p(E_1) \) and \( V = L^p(E_2) \) be two normed linear spaces, where \( E_1 \cup E_2 \) are two Lebesgue measurable sets with \( m(E_1 \cup E_2) < \infty \). Consider \( d: u \in U \cup V \rightarrow [0, \infty) \) be defined by \( d(f, g) = \|f - g\|_p \) for all \((f, g) \in U \times V \). Then \((U, V, d)\) is complete bipolar metric space. Define covariant map \( F, G: L^p(E_1) \cup L^p(E_2) \rightarrow L^p(E_1) \cup L^p(E_2) \) by

\[
F(y(x)) = \int S_2(x, y(x)) \, dx, \quad \forall x \in E_1 \cup E_2
\]

\[
G(y(x)) = \int S_2(x, y(x)) \, dx, \quad \forall x \in E_1 \cup E_2
\]

Notice that

\[
d(F(f(x)), G(g(x))) = \|f(x) - g(x)\|_p \]

\[
= \|S_2(x, f(x)) - S_2(x, g(x))\|_p \quad \forall x \in E_1 \cup E_2
\]

\[
\leq \mu \|f(x) - g(x)\|_p \quad \forall x \in E_1 \cup E_2
\]

Thus, it is verified that the functions \( F \) and \( G \) satisfy all the conditions of Theorem 2.5, and then \( F \) and \( G \) have a unique common fixed point in \( U \cup V \).

4. Conclusion

In the present research, we have presented unique common fixed point results on various contractive conditions defined on bipolar metric spaces, suitable examples that supports our main results. Also, applications to integral equations are provided.

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References