A Note on Boolean Like Algebras

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Abstract

In this paper we develop on abstract system: viz Boolean-like algebra and prove that every Boolean algebra is a Boolean-like algebra. A necessary and sufficient condition for a Boolean-like algebra to be a Boolean algebra has been obtained. As in the case of Boolean ring and Boolean algebra, it is established that under suitable binary operations the Boolean-like ring and Boolean-like algebra are equivalent abstract structures.

Keywords: Boolean algebra; Boolean like algebra; Boolean like ring; Boolean ring;

1. Preliminaries

Following A.L.Foster’s, the concept of Boolean-like ring is as follows:

Definition 1.1: A Boolean-like ring R is a commutative ring with unity which satisfies the following conditions.

(1) a + a = 0, and
(2) a (1+a) b (1+b) = 0 for all a,b ∈ B.

we give some examples of Boolean-like rings.

Example 1.2: Every Boolean ring is a Boolean -like ring.

Proof: If B is a Boolean ring, then for all a ∈ B.

(a + a)2 = a + a. whence a2 + a + a2 + a2 = a + a
and so a + a + a + a = a + a. Thus a + a = 0.

Further a(1+a) = a + a = a + a.

Hence (a+ab+b) = 0, for all a,b ∈ B.

By a remark, B is a commutative ring with unity.

Thus B is a Boolean-like ring.

But the converse need not be true (For this refer example 1.4).

Example 1.3: Let R be a ring with unity and characteristic 2. Let B be the set of all central idempotent of R . Then B is a Boolean subring of R. Further B × R is a Boolean -like ring with addition and multiplication defined as follows:

(b1, r1) + (b2, r2) = (b1 + b2, r1 + r2)
(b1, r1)b2, r2) = (b1b2, b1r2+b2r1)

for all b1, b2 ∈ B and r1, r2 ∈ R

Proof: we first prove that B is a Boolean subring of R.

Let b1, b2 ∈ B.

We show that b1 - b2 ∈ B and

b1 - b2 ∈ B

(b1 - b2)2 = b1 - b2. (Since R has characteristic 2)

For a ∈ R . (b1 - a) = b1a - b1a

Hence b1 - b2 ∈ B.

Also, (b1 b2)2 = (b1 b2)(b1 b2) = b1(b2b1) = b1 (b1 b2) = b1 b2

Further (b1b2) = b1(b2 = b1( ab)) = (b1 a ) b2 = a(b b2).

Hence b1 , b2 ∈ B.

Trivially 1 ∈ B and e2 = e for all e ∈ B.

Therefore B is a Boolean subring of R.

We now verify that B × R is a Boolean –like ring. For b1, b2 ∈ B and t1, t2, r1 ∈ R,

[(b1, t1) + (b2, t2)] + (b3, r3) = (b1, r1) + [(b2, t2) + (b3, r3)]

Hence ‘+’ is associative.

Now (0, 0) ∈ B × R and (b1, r1) + (0, 0) = (b1 +0 , r1+0 ) s = (b1 , r1)

Therefore (0, 0) is additive identity of B × R. For (b1 , r1) ∈ B × R

There exists (-b1 , -r1) ∈ B × R such that

(b1, t1) + (-b1 , -r1) = (b1 - b1 , t1 - r1) = (0 , 0)

Hence (-b1, b1) is the additive inverse of (b1, t1) (b1, r1)

For (b1, r1) + (b2, r2) = (b1+b2, r1+r2)

Therefore ‘+’ is commutative.

Thus (B×R, +) is an abelian group.

Now [(b1, r1), (b2, r2)] = (b1, t2) . (b1, t2)

= (b1b2, b2r1+ r2b1, r2r1)

= (b1, t2)(b2, r2)]

Hence ‘.’ is associative.

Also (1, 0) ∈ B × R and (b1, r1) (1, 0) = (b1, r1)

Further (b1, t1, b2, r2) = (b1, t1+b2, r1) = (b1, t1+b2, r2)

( b1, r1) . (b2, r2)

To prove the distributive law.

Consider (b1, t1) (b2, r2) + (b3, r3)]

= (b1, r1) (b2 +b2, r2 + r3)

=F(b1b2+b3, b1r2+b3r1)

Furthermore, (b1, t1) (b2, r2) + (b1, r1)(b2, r2) = (b1b2, b2r1+b3, r2) + (b1b2, b2r1+b3, r2)

= (b1b2+b1b2, b2r1+b2r1+b3, r2)

Therefore (B×R, +, ·) is a commutative ring with unity.

Suppose (b1, r1) ∈ B × R.

Since R is a ring of characteristic 2,

(b1, r1) + (b1, r1) = (b1, r1, r1) = (0, 0)

Also, (b1, r1) (1, 0) + (b2, r2) [(1, 0) + (b1, r1)] (b2, r2) [(1, 0) + (b2, r2)]
(0, r₁) (0, r₂) = (0, 0)

Hence B×R is a case of Example 1.3, we have the following

**Example 1.4:** Let Z₂ = [0,1] be the ring of integers modulo 2. Then Z₂ is a commutative ring with unity and its characteristic is 2. Obviously Z₂ is a Boolean ring. Hence Z₂×Z₂ is a Boolean-like ring under the operations of addition and multiplication defined as in example 1.3 above. This Boolean-like ring is denoted by H₂. Write 0 = (0,0), 1 = (1,0), p = (0,1) and q = (1,1).

Thus H₂ = [0, 1, p, q] and addition and multiplication tab les of H₂ are as follows

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Obviously H₂ is not a Boolean ring.

**Theorem 1.5:** Each element ‘a’ of a Boolean-like ring B satisfies a.e = a.

**Proof:** We have that a(1+a)b = 1

By taking a = b in (i), we get that a(1+a) = 0

⇒ a = a*, since the characteristic of B is 2.

**2. Boolean Like Algebras**

We now give the following definition:

**Definition 2.1:** An algebraic structure (A, ∧, ∨,0,1) where ∧ and ∨ are binary operations, 0 and 1 are elements of A, is called a Boolean-like algebra if the following conditions are satisfied

(1) ∧, ∨ are associative and commutative
(2) (a ∨ b) = a∧b ; (a) = a ; 0 = 1
(3) a ∧ 0 = 0 ; a ∧ 1 = 1
(4) a ∧ = a ∧ (b ∨ c)
(5) a ∧ a ∧ b ∧ c = 0
(6) (a ∧ a) = 0
(7) a ∨ b = a ∨ (a ∧ b)
(8) [(a ∨ b) ∧ (a ∨ b)] = (a ∨ b)
(9) (a ∧ b) = (a ∧ b)

The following result gives the most important elementary properties of elements in a Boolean-like algebra.

**Lemma 2.2:** In any Boolean-like algebra A, we have the following

(i) a = 0
(ii) a = 1
(iii) a = a
(iv) a = a
(v) a(va) = (ava) = 0
(vi) (a(a) = (a(aa) = 1

**Proof:** (i) By (2) and (3) of definition 2.1
(a ∨ 0) = a ∨ 0 = a ∨ 1 = a. Hence a ∨ 0 = (a ∨ 0) = (a) = a
By (2) of definition 2.1, we have that
(ii) 0 = (0) = 1
(iii) a = [a ∨ 1] = (a ∨ 0) = 0 = 1
(iv) a = (a ∨ b) = (a ∨ b) = a ∨ b
(v) By taking b = a in (7), we get that
(a ∨ a) = (a ∨ a) = (a ∨ a)

**Remark 2.3:** Every complemented distributive lattice is a Boolean-like algebra.

**Proof:** Let (L, ∧, ∨,0,1) be a complemented distributive lattice. By the definition of a complemented distributive lattice the conditions (1)∧ (6) of a Boolean-like algebra are satisfied.

Therefore 1 = 0, 1 = 0, a ∧ b = a ∧ b, a ∨ b = a ∨ b, a ∨ b = a ∨ b.

Hence L is a Boolean-like algebra.

**Theorem 2.4:** A Boolean-like algebra (A, ∨,0,1) is a Boolean algebra if and only if a ∧ a = a for all a ∈ A.

**Proof:** Suppose a ∧ a = a = a for all a ∈ A. Then (A, ∧) is a semilattice. By (5) of definition 1.1 × ∨ x = 0, for all x ∈ A. Also, By (iv) of lemma 2.2,

i = 0, x = 0, and a = a = a = a = a = a

Conversely, if a ∧ b = a ∧ b, then a ∧ b = a ∧ b = a ∧ b = 0, 0 = 0.
Thus, (A, ∧,0) is a Boolean algebra. Conversely, if (A, ∧,1) is a Boolean algebra, then a ∧ a = a for all a ∈ A.

**Corollary 2.5:** A Boolean-like algebra is a complemented distributive lattice 0 ∧ 0 = a, for all a.

**Proof:** Let B be a Boolean-like algebra. If B is a complemented distributive lattice, then evidently, a ∧ a = a = a. Conversely suppose that a ∧ a = a for all a ∈ B.

By the above theorem B is a Boolean algebra. Then, by the theorem [1], B is a complemented distributive lattice.

We now prove that every Boolean-like algebra is a Boolean-like ring under some binary operations.

**Theorem 2.6:** Let (A, ∨,0,1) be a Boolean-like algebra. Define binary operations +, ∨ by a + b = (a ∨ b) = (a ∨ b) = a + b for all a,b,c ∈ A. Then (A, +,0) is a Boolean-like ring.

**Proof:** In order to prove that (A, +,0) is a Boolean-like ring, we have to prove that
1) (A,+ is an abelian group with identity 0
2) (A,·) is a commutative semi group with identity 1.
3) Distributive law a(b + c) = ab + ac for all a,b,c ∈ A
4) a + a = 0 for all a ∈ A,
5) a(1+a) (1+b) = 0 for all a, b ∈ A
Now, $a+b = (a\land b)\lor (a\lor b) = (b\land a) = b + a$
Therefore $+\lor$ is commutative.

$(a + b) + c = (a + c) + b$

From (A) and (B), $(a + b) + c = a + (b + c)$. Further $a + 0 = (a\land 0) = (a\lor 0) = a = a\land 1 = a$.
Therefore 0 is the additive identity in A.
Also $a + a = (a\land a) + (a\lor a) = 0$.

Thus inverse of a is itself.

Therefore, $(+,\land)$ is an abelian group with identity 0. Further $a(b\land c) = a(b\land c) = (ab\land c) = (a\land b)\land c$ if $a \neq b$ and $a = b$.

Thus $a(b\land c) = a\land (b\land c) = (a\land b)\land c$. Therefore $(A,\land,\lor)$ is a Boolean algebra.

**Theorem 2.7:** Let $(A,\land,\lor,0,1)$ be a Boolean algebra.

Define the binary operations $A$ and $V$ and complementation $\overline{a}$ by $aVb = a + b + ab$ and $a = a\land 1$ for all $a, b \in A$.

Thus the algebraic system $(A,A,V,1,0)$ is a Boolean algebra.

**Proof:** In order to prove that $A$ is a Boolean algebra, we need to verify the following.

1. $V$ and $A$ are associative and commutative.
   
   Now $aV b = a + b + ab = a + b + b = aV b$, and $A a + a = a = a\land b = b + a$.

2. Also, $aV(b\land c) = a + (b + c + bc) + a(b + c + bc) = a + b + c + ab + ac + bc$.

Further, $(A,A\land V) = (ab) = (bc) = (c = (a\land b)\land c) = (a\land b)\land c$.

Therefore $V$ and $A$ are associative and commutative.

3. $(A\land b)\land c = (a\land (b + c)) = (a + b + ac) = (a + b + ac)$.

4. $(A\land b)\land c = (a\land b)\land c$.

5. $a = a\land 0 = a\land 0$ and $A a = a\land 1 = a\land 1 = a$.

6. $(A\land b)\land c = (a\land b)\land c$.

7. $a = a\land 1 = a\land 1 = a$.

Let $b = c$.

Then $a\land (b\land c) = (a\land b\land c) = ab + ac + abc$.

8. $(A\land b)\land c = (a\land b)\land c$.

9. $a = a\land 1 = a\land 1 = a$.

10. $(A\land b)\land c = (a\land b)\land c$.

11. $a = a\land 0 = a\land 0$ and $A a = a\land 1 = a\land 1 = a$.

Therefore $A = A$ and $V = V$.

This completes the proof.

Thus, the newly obtained Boolean algebra is same as the originally given Boolean algebra.

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**References:**


