The Impulsive Neutral Integro-Differential Equations with Infinite Delay and Non-Instantaneous Impulses

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Abstract

In this manuscript, we work to accomplish the Krasnoselskii’s fixed point theorem to analyze the existence results for an impulsive neutral integro-differential equations with infinite delay and non-instantaneous impulses in Banach spaces. By deploying the fixed point theorem with semigroup theory, we developed the coveted outcomes.

Keywords: Neutral equations; Equations with impulses; Non-instantaneous impulse condition; Integro-differential equations; fixed point theorem.

1. Introduction

In this paper, we consider the impulsive neutral integro-differential equation with infinite delay of the model

\[ \frac{d}{dt}[w(t) - G(t, w(t))] = A w(t) + G_1(t, w(t)) + G_2(t, w(t)), \quad t \in (s_i, \alpha_{i+1}], i = 0,1,...,N, \quad I = [0,T) \]

(1)

\[ w(t) = \theta(t, \tau), \quad t \in (s_i, \alpha_{i+1}], i = 1,2,...,N, \]

(2)

\[ w_0 = \phi \in B_{h}, \]

(3)

where the operator A is the infinitesimal generator of an analytic semigroup \( \{T(t)\}_{t \geq 0} \) in a Banach space X having norm \( \| \cdot \| \) and \( \mathcal{M} \) is a positive constant to ensure that \( \|T(t)\| \leq \mathcal{M} \). G_1 : \( I \times B_{h} \rightarrow X \) is a continuous and for each s \( s \in I \), \( G_1(t, s) \) is given by

\[ G_1(t, s) = \frac{1}{\alpha} \int_{s}^{t} e_{\alpha}(t, s, w(t)) \, dw \]

where \( \alpha \) is a constant. The function \( e_{\alpha}(t, s, w(t)) \) is such that \( \| e_{\alpha}(t, s, w(t)) \| \leq \mathcal{M} \) for \( t, s \in I \), and \( e_{\alpha}(t, s, w(t)) \) is continuous and bounded on \( I \times I \). By deploying the fixed point theorem with semigroup theory, we developed the coveted outcomes.

2. Preliminaries

Let X be a Banach space provided with the norm \( \| \cdot \| \). Consider the analytic semigroup \( \{T(t)\}_{t \geq 0} \) of bounded linear operators in X. Let A : \( D(A) \rightarrow X \) be the infinitesimal generator of \( \{T(t)\}_{t \geq 0} \). Then it is possible to determine the fractional power \( A^\alpha \) for \( 0 < \alpha \leq 1 \), as a closed linear operator on its domain \( D(A^\alpha) \), being dense in X. If \( X^\alpha \) represents the space \( D(A^\alpha) \), endowed with the norm \( \| x \|_{X^\alpha} = \| A^\alpha x \| x \in D(A^\alpha) \).

Then the following properties are well known. With this discussion, we recall fundamental properties of fractional powers \( A^\alpha \) from Pazy [2]. Now consider the space

\[ B_{h}^\alpha = PC(\{s_i, T\} \times X) = \{ w : (s_i, T) \rightarrow X \text{ such that } w \in C(I, X) \text{ and } w(t^+) = 0 \in B_{h}, \quad k = 0,1,2,...,m \} \]

where \( B_{h}^\alpha \) is the restriction of w to \( I = (s_i, t_k) \), set \( \| x \|_{B_{h}^\alpha} = \sup \{ \| x(s) \| : s \in [0,T] \} \), \( x \in B_{h}^\alpha \).

3. Existence results

Definition 3.1: A function \( w : (s_i, T) \rightarrow X \) is called mild solution of model (1)-(3) if \( w_{w} = \phi \in B_{h}^\alpha \) on \( (s_i, 0) \) \( w(t^+) = h(t, w(t)) \) for t \( t \in (s, s_i] \) for each \( i = 0,1,...,N \), the constraint of \( w(.) \) to interval \( (0,T) - \{ t_1, t_2, t_3, \ldots, t_m \} \) is continuous and for each \( s \in (0,T) \) the function \( A^{\alpha}T(t)G_1(s, w_{s}) \int_{0}^{t} e_{\alpha}(t, s, w(t)) \, dt \) is integrable and sub-sequence impulsive integral equation is
The functions \( e_j : \mathcal{D} \times B_0 \times X \times X \rightarrow \mathbb{R} \) are continuous and there exist constants \( K_{e_j} > 0 \) to ensure that \( ||e_j(t, s, \varphi) - e_j(t, s, \psi)||_{B_0} \leq K_{e_j} \varphi \) for each \( (t, s) \) and all \( \varphi, \psi \in B_0 \).

(H6) The functions \( e_j : \mathcal{D} \times B_0 \times X \times X \rightarrow \mathbb{R} \) are continuous and there exist constants \( K_{e_j} > 0 \) to ensure that \( ||e_j(t, s, \varphi) - e_j(t, s, \psi)||_{B_0} \leq K_{e_j} \varphi \) for each \( (t, s) \) and all \( \varphi, \psi \in B_0 \).

(H7) The following inequalities holds: Let \( M_3 \{ K_{G_0} (D_1^2 q + c_n) + c_n \} + K_{G_0} \} + (M_3 (1 + M_1) + M_1 - M_1^2 \beta \)) ||K_{G_0} + K_{G_0} (T) \psi \|| \leq \|D_1^2 q + p_1 M_1 T \mu_0 \psi \| (1 + T) K_{G_0} + M_3 M_1 T (K_{G_0} + K_{G_0} (T K_{G_0} )) \).
\( \xi_2 = T(t-s) G_2(s, u_0 + v_{*0}) e_2(s, \tau, u_0 + v_{*0}) dt, \)

\( \xi_4 = T(t-s) G_2(s, u_0 + v_{*0}) e_2(s, \tau, u_0 + v_{*0}) dt \)

\( \xi_5 = G_0(s, u_0 + v_{*0}) e_2(s, \tau, u_0 + v_{*0}) dt. \)

From this, it is understood that the operator \( \Phi \) has a fixed point if and only if \( \Phi \) has a fixed point. Give us a chance to demonstrate that \( \Phi \) has a fixed point. Now, we enter the main proof of the theorem. To apply Krassnoselskii’s fixed point theorem, we introduce the decomposition

\[ \Phi = \sum_{n=1}^{N} \Phi_n + \sum_{n=0}^{N} \Phi_n^* \]

\( (\Phi_1^* u)(t) = \begin{cases} 
-T(t)(G_1(0,0,0,0) + \int_{s_1}^{t} \xi_2 ds + \int_{s_1}^{t} \xi_4 ds, & \text{for } t \in [0,t_1], \\
0, & \text{for } t \in [t_r, t_{r+1}], \\
h(t, u_1 + v_r), & \text{for } t \in (t_s, t_r + 1], i = 1, 2, ..., N, \\
T(t-s_0)[h(u_0 + v_{*0} - \xi_3 + \xi_1 + \int_{s_1}^{t} \xi_4 ds, & \text{for } t \in [s_1, t_{s+1}], i = 1, 2, ..., N, \\
\int_{s_1}^{t} \xi_4 ds, & \text{for } t \in [t_r,s_r], i = 1, 2, ..., N. 
\end{cases} \]

(\( \Phi_1^* u)(t) = \begin{cases} 
\int_{s_1}^{t} \xi_4 ds, & \text{for } t \in [0,t_1], \\
0, & \text{for } t \in [t_r, t_{r+1}], \\
h(t, u_1 + v_r), & \text{for } t \in (t_s, t_r + 1], i = 1, 2, ..., N, \\
T(t-s_0)[h(u_0 + v_{*0} - \xi_3 + \xi_1 + \int_{s_1}^{t} \xi_4 ds, & \text{for } t \in [s_1, t_{s+1}], i = 1, 2, ..., N, \\
\int_{s_1}^{t} \xi_4 ds, & \text{for } t \in [t_r,s_r], i = 1, 2, ..., N. 
\end{cases} \]

Step 2: Next we will show that \( \Phi_1^* u(t) \) is compact and \( \Phi_1^* u(t) \) is continuous. We split the proof into three parts.

Let the sequence \( u_n \) such that \( u_n \to u \) in \( B_{r_0} \). Then for all \( t \in [0, T] \), by the definition of \( \Phi_2^* u(t) \) and \( \Phi_2^* u(t) \) and by assumptions H(3) and H(6).

For better readability, we divide our results into four steps.

Step 1: First we show that \( \Phi_1^* u(t) + \Phi_2^* u(t) \in B_{r_0} \)

(i) \( ||\Phi_1^* u(t) + \Phi_2^* u(t)|| \leq M_1 M_0 [K_0, \| \| b_n + K_0, \| \| b_n + K_0 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_1 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2] \leq q, t \in [0,t_1], \)

(ii) \( \Phi_1^* u(t) + \Phi_2^* u(t) \leq K_0 (D_t^2 q + c_n) + K_0 \leq q, \)

(iii) \( \Phi_1^* u(t) + \Phi_2^* u(t) \leq M_1 [K_0 (D_t^2 q + c_n) + K_0] + M_0 (1 + M_1 + M_0 - \frac{e^r}{r}) [K_0 + G_0 T K e_0] D_t^2 q + p_1 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2 \leq q, \)

(iv) \( \Phi_1^* u(t) + \Phi_2^* u(t) \leq M_1 [K_0 (D_t^2 q + c_n) + K_0] + M_0 (1 + M_1 + M_0 - \frac{e^r}{r}) [K_0 + G_0 T K e_0] D_t^2 q + p_1 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2 \leq q, \)

Step 2: Next we will show that \( \Phi_1^* u(t) + \Phi_2^* u(t) \) is a contraction.

From the definition of \( \Phi_1^* u(t) \) and \( \Phi_2^* u(t) \) and the assumption of (H3),

\( \| (\Phi_1^* u(t) - (\Phi_2^* u(t)) \| \leq D_1 [M_0 (1 + M_1 + M_0 - \frac{e^r}{r}) [K_0 + G_0 T K e_0] D_t^2 q + p_1 + M_0 M_1 T [K_0 + G_0 T K e_0] D_t^2 q + p_2] \leq \epsilon |A| \to 0 \text { as } \epsilon \to 0. \)

which are relatively compact sets arbitrarily close to the set \( V_\epsilon(t) \), \( t > 0 \) as a result \( V_\epsilon(t) \) is relatively compact in \( X \). From the above steps, it follows by the Krasnoselskii’s fixed point theorem, we get that \( \Phi \) has at least one fixed point \( u(t) \in B_{r_0} \). With these, a fixed point of the operator \( \Phi \) is the mild solution u of the problem (1) - (3). This finishes the verification of the hypothesis.
3. Applications

To epitomize our hypothetical results, now, we consider the following INIDE with infinite delay of the structure

\[ \frac{\partial}{\partial t}u(t, x) \left\{ \int_{-\infty}^{t} a_1(t, x, s = t)P_1(u(s, x)) \, ds + \int_{-\infty}^{t} \int_{-\infty}^{t} K_2(s, \tau) \left[ \int_{-\infty}^{\tau} a_2(t, x, s = t)Q_1(u(s, x)) \, ds + \int_{0}^{t} \int_{0}^{t} K_3(s, \tau)Q_2(u(t, x)) \, dr d\tau \right] \right\} , \quad x \in [0, \pi], t \in [0, b], t \neq t_k, \]

(8)

\[ u(t, 0) = u(t, \pi) = 0, \quad t \geq 0 \]

(9)

\[ u(t, x) = \varphi(t, x), t \in (-\infty, 0], x \in [0, \pi], \]

(10)

\[ u_0(t, x) = \int_{-\infty}^{t} \eta(t_1 - x)u(s, \lambda) \, ds, (t, x) \in (t_k, s_k) \times [0, \pi], \]

(11)

The prefixed real numbers are 0 < t_1 < t_2 < \cdots < t_n < b and \( \varphi \in B_b \). Let \( X = L^2[0, \pi] \) whose norm is \( \| \cdot \|_{L^2} \) and determine the operator \( A : D(A) \subset X \rightarrow X \) by \( Av = w^* \) with the domain \( D(A) = \{ w \in X : w, w' \text{ are absolutely continuous}, w' \in X, w(0) = w(\pi) = 0 \} \). Then \( Av = \sum_{n=1}^{m} \sqrt{n} \langle w, w_n \rangle w_n, w \in D(A) \), in which \( w_n(s) = \frac{\sqrt{n}}{\sqrt{\pi}} \sin(ns), n = 1, 2, \ldots \)

is the orthogonal set of eigenvectors of \( A \). \( T(t)w = \sum_{n=1}^{m} e^{-nt} \langle w, w_n \rangle w_n \), for all \( w \in X \), and every \( t > 0 \). For each \( w \in X, (A)^* = \sum_{n=1}^{m} (n) \langle w, w_n \rangle w_n \) and \( \| (A)^* \| = 1 \).

The continuous functions \( Q_k, k = 1, 2 \) are defined for each \( (0, x) \in (-\infty, 0] \times [0, \pi], \) and

\[ 0 \leq Q_k(u(\theta)(x)) \leq \Pi \left( \int_{0}^{s} e^{2s} \| u(s, \cdot) \|_{L^2} \, ds \right) \]

and the continuous non decreasing function is defined as \( \Pi : [0, \infty) \rightarrow (0, \infty) \) and we can take \( \Omega_G(r) = \Pi(r) \) in (H3).

Presently we can see that

\[ \| G_2(t, \varphi, H_2\varphi) \|_{L^1} = \left( \int_{0}^{\pi} \left( \int_{-\infty}^{t} a_2(t, x, \theta)Q_1(\varphi(\theta)) (x) \, d\theta + (H_2\varphi(\theta)) (x) \right)^2 \, dx \right)^{1/2} \]

\[ \leq \left( \int_{0}^{\pi} (m_1(t, x))^2 \, dx \right)^{1/2} + \left( \int_{0}^{\pi} (m_2(t))^2 \, dx \right)^{1/2} \Pi(\| \varphi \|_{h}) \]

\[ \leq \left( m_1(t) + \sqrt{n} m_2(t) \right) \Pi(\| \varphi \|_{h}) \leq m(t)(\| \varphi \|_{h}) \]

Since \( \Pi : [0, \infty) \rightarrow (0, \infty) \) is a continuous and non decreasing function, we can take \( m(t) = \left( m_1(t) + \sqrt{n} m_2(t) \right) \). Along these lines the condition (H3) holds. Hence by theorem 3.1, we comprehend that the system (8)-(11) has a unique mild solution on \( I \).

References