Vertex Hyper-Zagreb Index of Graphs

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Abstract

In this paper, we obtain the vertex hyper-Zagreb index for some composite graphs. Using the results obtained here, the vertex hyper-Zagreb index for some important classes of graphs are obtained.

Keywords: Composite graphs; F-index; Hyper-Zagreb index; Topological index; Zagreb index.

1. Introduction

A chemical graph is a graph whose vertices denote atoms and edges denote bonds between those atoms of any underlying chemical structure. A topological index for a (chemical) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. Topological indices and graph invariants based on the distances between vertices of a graph or vertex degrees are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making their chemical applications. These indices may be used to derive quantitative structure-property or structure activity relationships (QSPR/QSAR) for more details see [Bollobas and Eids [3], Gutman and Trinajstic[13], Randic [17], Toodeschini and Consonni [20]].

For a (molecular) graph G, The first Zagreb index $M_1(G)$ is the equal to the sum of the squares of the degrees of the vertices, and the second Zagreb index $M_2(G)$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is,

$$
M_1(G) = \sum_{uv \in E(G)} d_u^2(u) = \sum_{uv \in E(G)} (d_u(u) + d_v(v))
$$

$$
M_2(G) = \sum_{uv \in E(G)} d_u d_v(v) .
$$

There are different types of Zagreb indices were studied by various authors. One of the important version of Zagreb indices, the vertex first and second Zagreb indices were proposed by Tavakoli et al.[10]. They are defined as

$$
\overline{M}_1^*(G) = \sum_{(u,v) \in V(G)} (d_u(u) + d_v(v))
$$

Shiradel et al. [9] proposed the another version of Zagreb index, namely, hyper-Zagreb index, which is defined as

$$
HM(G) = \sum_{uv \in E(G)} (d_u(u) + d_v(v))^2 .
$$

Similarly, the hyper-Zagreb co index is defined as

$$
\overline{HM}^*(G) = \sum_{uv \in E(G)} (d_u(u) + d_v(v))^2.
$$

In this connection, we introduce the new version of Zagreb index called vertex hyper-Zagreb index which is defined as

$$
\overline{HM}_1^*(G) = \sum_{(u,v) \in V(G)} (d_u(u) + d_v(v))^2.
$$

The F-index was introduced by Furtula and Gutman [4], and it is defined as

$$
F = F(G) = \sum_{uv \in V(G)} d_u^3(u)
$$

They established a few basic properties of the forgotten topological index and show that it can significantly enhance the physico-chemical applicability of the first Zagreb index.


In Fath-Tabar [3] and Shiradel et al. [9] the hyper and third Zagreb indices of some graph operations are obtained. The mathematical properties for the third and hyper-Zagreb coindices of graph operations containing the Cartesian product and composition are explained by Gao et al. [5]. Pattabiraman and Seenivasan [7] analyzed Bounds of vertex Zagreb indices of graphs for more details see [Gutman [8], Gutman [9], J Gutman ,Das [10], J Gutman ,Furtula ,

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In this paper, we obtain the vertex hyper-Zagreb index for some composite graphs. Using the results obtained here, the vertex hyper-Zagreb index for some important classes of graphs are obtained.

2. Composite Graphs

Some chemically interesting graphs can be obtained by different graph operations on general or particular graphs. Thus graph operations played an important role in chemical graph theory Dosić [3]. In this view, here we obtain the vertex hyper-Zagreb indices of different types of operations of graphs.

2.1 Join

The join $G+H$ of graphs $G$ and $H$ is obtained from the disjoint union of the graphs $G$ and $H$, where each vertex of $G$ is adjacent to each vertex of $H$.

**Theorem 1.** Let $G$ be a graph with $n_1$ vertex and $m_1$ edges, $i=1,2$. Then

$$HM^*_1(G_1+G_2) = HM^*_1(G_1) + 4n_1M_1^t(G_1) + HM^*_1(G_2) + 4n_2M_1^t(G_2) + n_1n_2M_1(G_1,G_2) + n_1n_2M_2(G_1,G_2).$$

**Proof of theorem 1.** Set $V(G_1) = \{u_1,u_2,...,u_{n_1}\}$ and $V(G_2) = \{v_1,v_2,...,v_{n_2}\}$. By definition of the join of two graphs, one can see that,

$$d_{G_1\cup G_2}(x) = \begin{cases} 
  d_G(x) + n_2; & \text{if } x \in V(G_1), \\
  d_G(x) + n_1; & \text{if } x \in V(G_2).
\end{cases}$$

By the definition of vertex hyper Zagreb index,

$$HM^*_1(G_1+G_2) = \sum_{(u,v)\in E(G_1+G_2)}(d_{G_1\cup G_2}(u) + d_{G_1\cup G_2}(v))^2$$

$$= \sum_{(u,v)\in E(G_1)}(d_{G_1}(u) + d_{G_1}(v))^2 + \sum_{(u,v)\in E(G_2)}(d_{G_2}(u) + d_{G_2}(v))^2 + \sum_{(u,v)\in E(G_1,G_2)}(d_{G_1}(u) + d_{G_2}(v))^2 + \sum_{(u,v)\in E(G_1,G_2)}(d_{G_2}(u) + d_{G_1}(v))^2$$

$$= \sum_{(u,v)\in E(G_1)}(d_{G_1}(u) + d_{G_1}(v))^2 + 2n_1^2$$

$$+ \sum_{(u,v)\in E(G_2)}(d_{G_2}(u) + d_{G_2}(v))^2 + 2n_2^2$$

$$+ \sum_{(u,v)\in E(G_1) \cup E(G_2)}(d_{G_1}(u) + d_{G_2}(v) + n_2^2)$$

$$+ \sum_{(u,v)\in E(G_1) \cup E(G_2)}(d_{G_2}(u) + d_{G_1}(v) + n_1^2)$$

$$= \sum_{(u,v)\in E(G_1)}(d_{G_1}(u) + d_{G_1}(v))^2 + 4n_1^2 + 4n_2^2 + 4n_1d_{G_1}(u)d_{G_1}(v)$$

$$+ \sum_{(u,v)\in E(G_2)}(d_{G_2}(u) + d_{G_2}(v))^2 + 4n_2^2 + 4n_1d_{G_2}(u)d_{G_2}(v)$$

$$+ \sum_{(u,v)\in E(G_1) \cup E(G_2)}(d_{G_1}(u) + d_{G_2}(v))^2 + 4n_1^2 + 4n_2d_{G_1}(u)d_{G_2}(v)$$

$$= \sum_{(u,v)\in E(G_1)}(d_{G_1}(u) + d_{G_1}(v))^2 + 4n_1^2 + \frac{n_1(n_1-1)}{2} + 4n_2M_1^t(G_1)$$

$$+ n_2M_1(G_1) + n_1M_1(G_2) + (n_1+n_2)^2n_2 + 4m_1n_2(n_1+n_2) + 2m_1 + 2m_2 + 8m_1m_2.$$

Hence

$$HM^*_1(G_1+G_2) = \sum_{(u,v)\in E(G_1)}(d_{G_1}(u) + d_{G_1}(v))^2 + \sum_{(u,v)\in E(G_2)}(d_{G_2}(u) + d_{G_2}(v))^2$$

$$+ n_1M_1(G_1) + n_1M_1(G_2) + n_2M_2(G_1,G_2) + n_1n_2M_1(G_1,G_2) + n_1n_2M_2(G_1,G_2).$$

**Example 2.1.** The complete bipartite graph $K_{p,q}$ can be defined as $K_{p,q} = \overline{K_p} + K_q$. The vertex hyper-Zagreb index of $K_{p,q}$ is $p+q = (p+q)^2$.

**Example 2.2.** Let $G$ be a graph with $n$ vertices and $m$ edges. Then

$$HM^*_1(G+K_p) = HM^*_1(G) + n_1M_1^t(G) + n_1M_1(G) + np(p-1)^2(2p+5n-2)$$

$$+ np(4p^2-2n(p+1) + (n+p)^2) + 2(n+p)(2m+n) + np(p-1) + 4m^2.$$
\[ + \sum_{u \in \text{V} (G), v \in \text{V} (G)} (d_{Gi}(u) + d_{Gi}(v))^2 \]
\[ = \overline{HM} [G_1] + \overline{HM} [G_2] + n_2 M_2(G_1) + n_1 M_1(G_2) + 4m_1m_2. \]

### 2.3 Cartesian Product

The Cartesian product of graphs, \( G \square H \), of the graphs \( G \) and \( H \) has the vertex set \( \text{V}(G \square H) = \text{V}(G) \times \text{V}(H) \) and \( (u, x) \times (v, y) \) is an edge of \( G \square H \) if \( u = v \) and \( x \in \text{E}(H) \) or \( x \in \text{E}(G) \) and \( (x, y) \in \text{E}(H) \). To each vertex \( u \in \text{V}(G) \), there is an isomorphic copy of \( H \) in \( G \square H \) and to each vertex \( v \in \text{V}(H) \), there is an isomorphic copy of \( G \) in \( G \square H \).

**Theorem 3.** Let \( G \) be a graph with \( n \) vertex and \( m \) edges, \( i = 1, 2 \). Then
\[ \overline{HM} [G] = \frac{1}{2} [\text{M}(G) + n_1 \text{HM}(G) + M_1(G)] [2n^2 - n_1 - 2n + 9]. \]

**Theorem 4.** Let \( G \) be a graph with \( n \) vertex and \( m \) edges, \( i = 1, 2 \). Then
\[ \overline{HM} [G] = \frac{1}{2} [\text{M}(G) + n_1 \text{HM}(G) + M_1(G)] [2n^2 - n_1 - 2n + 9]. \]

### 2.5 Corona Product

For given graphs \( G \) and \( H \), we define their corona product \( G \circ H \) as the graph obtained by taking \( [V (G)] \) copies of \( H \) and joining each vertex of the \( i \)-th copy with vertex \( v \in \text{V}(G) \). Obviously, \( [V (G \circ H)] = [V (G)] \times [V (H)] \). By using Theorem 4.1 of [5] and Theorem 3 respectively, we have the degree of a vertex \( x \in \text{V}(G) \) is given by
\[ \overline{d}_{G_1 \circ G_2}(x) = \begin{cases} d_{G_1}(x) + 1, & \text{if } x \in \text{E}(G_1) \\ d_{G_2}(x) + 1, & \text{if } x \in \text{E}(G_2), i = 1, 2, \ldots, n. \end{cases} \]

where \( G_{2i} \) is the \( i \)-th copy of \( G_2 \).

The following theorem is presented in [11].

**Theorem 6.** Let \( G \) be a graph with \( n \) vertex and \( m \) edges, \( i = 1, 2 \). Then
\[ \overline{HM} [G_1 \circ G_2] = [n_1 \overline{HM}(G_2) + n_2 M_2(G_1)] + \left( n_1 + n_2 - 3m - 1 \right) M_1(G_1) + \left( n_1 + n_2 - 4m \right) M_2(G_1) + n_1 n_2 (n_1 - 1) - 2m_1 + 2m_2 + 2n_1 - 2n_2 + 4m_1 + 4m_2.
\]

The proof of the following theorem follows by applying Theorem 2.5 of [2] and Theorem 6, respectively.

**Theorem 7.** Let \( G \) be a graph with \( n \) vertex and \( m \) edges, \( i = 1, 2 \). Then
\[ \overline{HM} [G_1 \circ G_2] = [n_1 \overline{HM}(G_2) + n_2 M_2(G_1)] + \left( n_1 + n_2 - 3m - 1 \right) M_1(G_1) + \left( n_1 + n_2 - 4m \right) M_2(G_1) + n_1 n_2 (n_1 - 1) - 2m_1 + 2m_2 + 2n_1 - 2n_2 + 4m_1 + 4m_2.
\]

For a given graph \( G \), its t-fold bristled graph Brst(G) is obtained by attaching \( t \) vertices of degree 1 to each vertex of \( G \). This graph can be represented as the corona product of \( G \) and complement of a complete graph on \( t \) vertices. The t-fold bristled graph of a given graph is also known as its t-thorny graph.

**Example 2.10.** The vertex hyper-Zagreb index of the t-fold bristled graph of \( C_n \) is given by
\[ \overline{HM}_1 (C_n \circ K_t) = n^t + 10n^{t-1} - 2nt + 17 + 10nt - 10n^t + 8n^2. \]

### 2.6 Composition

The composition \( G[H] \) of graphs \( G \) and \( H \) with disjoint vertex sets \( V(G) \) and \( V(H) \) and edge sets \( E(G) \) and \( E(H) \) is the graph with vertex set \( V(G) \times V(H) \) and \( (u, v) \) is adjacent with \( (u, v) \) whenever \( u \) is adjacent with \( u_2 \) or \( u_1 \) is adjacent with \( v_2 \). From the structure of \( G[H] \), the degree of a vertex \( u \in V(G) \) is given by
\[ \overline{d}_{G[H]} (u, v) = [V(H) \overline{d}(u) + \overline{d}_v(v)] \]

The following theorem is presented in Veylaki et al. [11].

**Theorem 8.** Let \( G \) be a graph with \( n \) vertex and \( m \) edges, \( i = 1, 2 \). Then
\[ \overline{HM} [G] = \frac{1}{2} \left[ \text{M}(G) + n_1 \text{HM}(G) + M_1(G) \right] [2n^2 - n_1 - 2n + 9]. \]

The proof of the above theorem follows by applying Theorems 2.3 and 2.4 of [8, respectively.]
\[ -n_i M_2(G_2) - n_2^3 F(G_2) - n_i F(G_2) \\
+ 4m n_i^2 (m_2 n_2^3 + 2m_2 n_1) + 4m_2 n_1^2 \\
+ 8m_2 m_1 (n_2 n_1 - 2) n_2 - m_3. \]

The proof of the following theorem follows by applying Theorem 2.3 of [2] and Theorem 8, respectively.

**Theorem 9.** Let \( G \) be a graph with \( n \) vertex and \( m \) edges, \( i=1,2 \).

Then \( \overline{HM}^* (G(G_2)) = n_1^2 HM(G_1) + n_2^2 HM(G_2) \\
+ n_3^2 M_2(G_1)(n_2 - n_1) + M_2(G_2) (2n_2 - n_1) \\
- n_2^3 M_1(G_2)(n_2 - n_1) + n_2^2 F_3(G_2) - n_3 F(G_2) \\
+ 4m n_2^2 (m_2 n_2^3 + m_2 n_1) + 4m_2 n_1^2 \\
+ 8m_2 m_1 (n_2 n_1 - 2). \)

Using Theorem 9, we have the following examples.

**Example 2.11.** The vertex hyper-Zagreb indices of \( P_d[P_m] \) and \( C_d[C_m] \) are given by

(i) \( \overline{HM}^* (P_d[P_m]) = 4m^2(n^2 - n - 1) \\
+ m^2(12n^2 - 22n) - 4m^3(n^2 - n - 1) + 20m. \)

(ii) \( \overline{HM}^* (C_d[C_m]) = 4m^2(n^2 - n + 1) \\
+ 4m^3(5n^2 - 4n) - 8m^3(n^2 + n - 16) \\
- 4m^3(2n - 1) + 4m(3n - 4). \)

**Example 2.12.** The open fence graph is defined as \( P_d[P_3] \). Similarly, the closed fence graph is defined as \( C_d[P_3] \).

The vertex hyper-Zagreb indices of the open and closed fence graphs are given by

(i) \( \overline{HM}^* (P_d[P_3]) = 228n^2 + 132n - 168. \)

(ii) \( \overline{HM}^* (C_d[P_3]) = 168n^2 + 17n. \)

**References**


