A Study on Fuzzy TT-Ideals in Ternary Γ-Semiring

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Abstract

The concepts of fuzzy theory in T-semiring in terms of fuzzy TT-ideals in TT-semirings are introduced and also we made a study on some properties of fuzzy TT-ideals in TT-semiring.

Keywords: TT-Semiring, TT-ideal, fuzzy TT-ideal, TT-subsemiring, regular TT-semiring.

1. Introduction


2. Preliminaries

Note 2.8: For preliminaries refer the references [2], [3].

3. Fuzzy TT-ideals:

Def 3.1: A fuzzy set ξ of a TT-semiring is said to be a fuzzy TT-subsemiring of Q if

(i) ξ(u + v) ≥ min{ξ(u), ξ(v)}

(ii) ξ((au)hv) ≥ min{ξ(a), ξ(v), ξ(w)} ∀ u, v, w ∈ M

Ex: 3.2: Let M be the set of rational numbers and Γ is the set of natural numbers. Define a mapping from M×M×M to M by usual addition and ternary multiplication defined by aabβc = usual product of a, b, c; for a, b, c ∈ M, a, b, c ∈ Γ. Then M is a TT-semiring. Define μ:M → [0, 1] as μ(x) = {1 if x ∈ Q

0, otherwise

Then μ is a fuzzy TT-semiring of M.

Example 3.4: Consider the set Z = {0, 1, -1, 2, -2, …} and Γ be the set of even natural numbers. Then with respect to usual addition Z and multiplication is infinite TT-semiring. Clearly ZZ is a proper Γ-subsemiring of Z. Define μ:Z → [0, 1] by

ξ(x) = \begin{cases} 0.9 & \text{if } x \in 2Z \\ 0.8 & \text{if } x \in 2Z+1 \\ \end{cases} 

It is easy to verify that ξ is a fuzzy T Γ-sub semi ring of the TT-semi ring Z.

Def 3.5: A fuzzy T Γ -sub semi ring μ of a TT-semiring M is called improper if μ is constant on the T Γ-semi ring T, otherwise μ is termed as proper.

The 3.6 : Let M be a T-smarandah and F(M) be the set of all non-empty fuzzy subsets of TT-semiring M. If e, f, g, h ∈ F(M), then

(i) e ∩ (f ∩ g) = (e ∩ f) ∩ g

(ii) (f ∩ g) ∩ h = f ∩ (g ∩ h)

(iii) g ∩ (e ∩ f) ∩ h = (g ∩ f) ∩ (h ∪ (f ∩ g))

The 3.7: Suppose Q be a T-semiring and φ a fuzzy sub set of M. Then (i) φ(d) = min{φ(d), φ(m), φ(k)} ∀ d, m, k ∈ Q and φ ∈ Γ and z(d) = min{φ(d), φ(m), φ(k)} are equivalent ∀ d, m, k ∈ Q and φ ∈ Γ.

Proof: Assume that φ(d) = min{φ(d), φ(m), φ(k)} ∀ d, m, k ∈ Q and φ ∈ Γ. We may assume that φ(d) = φ(m) ≡ φ(k). Then φ(d) = φ(m) and φ(d) = φ(k), so z(d) = 1 - φ(d) = 1 - φ(k) = z(k) = z(φ). Therefore, z(d) = min{φ(d), φ(m), φ(k)} = min{z(d), z(m), z(k)} ∀ d, m, k ∈ Q and φ ∈ Γ.

Conversely, suppose that φ(d) = min{φ(d), φ(m), φ(k)} ∀ d, m, k ∈ Q and φ ∈ Γ. We may assume that φ(d) = φ(m) ≡ φ(k). Then z(d) = min{φ(d), φ(m), φ(k)} = min{z(d), z(m), z(k)} = z(φ) = z(k) = z(d) = min{φ(d), φ(m), φ(k)} = min{z(d), z(m), z(k)} = z(k). Therefore, φ(d) = min{φ(d), φ(m), φ(k)} ∀ d, m, k ∈ Q and φ ∈ Γ.
Th 3.8: Let $R$ be a $T_p$-semi ring and $K \subseteq R$, $K \neq \emptyset$. Then $K$ is a $T_p$-sub semi ring of $R$ iff the fuzzy subset $\pi_K$ is a fuzzy $T_p$-sub-semiring of $R$.

Proof: Obviously, $\pi_K$ is a fuzzy subset of $R$. Let $u, v, w \in R$ and $\gamma, \delta \in \Gamma$. Then $\pi_K(u) = \pi_K(v) = 1$. Since $u \notin K$ or $v \notin K$ or $w \notin K$, we have $\pi_K(u) = 0$ or $\pi_K(v) = 0$ or $\pi_K(w) = 0$, then $\pi_K(u+v) \geq 1 = \pi_K(u) + \pi_K(v)$ and $\pi_K(uw) \geq 1 = \pi_K(u) \pi_K(w)$.

Conversely, suppose that $u, v, w \in K$ and $\gamma, \delta \in \Gamma$. Then $\pi_K(u) = \pi_K(v) = \pi_K(w) = 1$. Since $\pi_K$ is a fuzzy $T_p$-sub semiring of $R$, we have $\pi_K(u+v) \geq 1 = \pi_K(u) + \pi_K(v) = \pi_K(w)$ and $\pi_K(uw) \geq 1 = \pi_K(u) \pi_K(w)$. Therefore the fuzzy subset $\pi_K$ is a fuzzy $T_p$-sub semi ring of $R$.

Def 3.9: A non-empty fuzzy sub-set $\varphi$ of a $T_p$-semi ring $Q$ is called a fuzzy $L(L_a, R)$ $T_p$-ideal or simply fuzzy left $T_p$-ideal of $Q$ if

(i) $\varphi(u+v) \geq \pi(\varphi(u), \varphi(v))$
(ii) $\varphi(uv) \geq \pi(\varphi(u), \varphi(v)) \varphi(uv) \geq \varphi(u) \varphi(v)$ $\varphi(uv) \geq \varphi(u)$ $\forall u, v, w \in Q, \forall \gamma, \delta \in \Gamma$.

Ex 3.10: Let $Q = [0, s, l, u]$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha, \beta' \in \Gamma$ is defined below:

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Clearly $Q$ is a $T_p$-semi ring. Define a fuzzy subset $\pi: Q \rightarrow [0, 1]$ by $\pi(x) = \begin{cases} 1 & \text{if } x=0, \alpha \\ 0 & \text{otherwise} \end{cases}$. Clearly, $\pi$ is a fuzzy $L$-$T_p$-ideal of $Q$.

Def 3.11: A non-empty fuzzy sub-set $\sigma$ of a $T_p$-semi ring $Q$ is called a fuzzy $L(L_a, R)$ $T_p$-ideal of $Q$ if

(i) $\sigma(f + g) \geq \min\{\pi(f), \pi(g)\}$
(ii) $\sigma(fgh) \geq \pi(f) \pi(g) \pi(h)$ for any $f, g, h \in Q$ and $\gamma, \delta \in \Gamma$.

Ex 3.12: Let the set $\mathcal{W}$ of -ve integers with 0 be $P$ and the set of negative even integers with 0 be $B$. Then $P$ is a $T_p$-semi ring if $u, v, w \in P$ as well as $u+v, uv, uw$ as well as $u, v, w \in Q$ as well as $u, v, w \in \mathcal{W}$.

Let $\eta$ be a fuzzy sub-set of $Q$, defined as follows:

$$\eta(m) = \begin{cases} 1, & \text{if } m = 0 \\ 0.1, & \text{if } m = -1, -2 \\ 0.2, & \text{if } m < -2 \end{cases}$$

Then $\eta$ is a fuzzy $T_p$-ideal of $Q$.

Ex 3.13: Let $M = \{0, u, v, w\}$ and $\Gamma = \{\alpha, \beta\}$ be the non-empty set of binary operations defined below:
Th 3.16: Let $P(\emptyset)$ be a subset of a $T$-semiring $Q$ and $\pi$ be the characteristic function of $I$, then $I$ is a $L(La, R)$ $T$-ideal of $Q$ iff $\pi$ is a fuzzy $L(La, R)$ $T$-ideal of $Q$.

**Proof:** Let $P$ be a $L$-$T$-ideal of a $T$-semi ring $Q$. Let $x, y \in Q$ and $\gamma \in \Gamma$. Let $x + y \in I$ and $xy \in I$ if $x \in I$. It follows that $X_{\gamma}(x + y) = 1$ and $X_{\gamma}(xy) \in I = X_{\gamma}(x) \oplus X_{\gamma}(y)$. If $y \notin I$, then $X_{\gamma}(x) = 0$. In this case $X_{\gamma}(xy) = 0 = X_{\gamma}(x)$. Therefore $X_{\gamma}$ is a fuzzy $L$-$T$-ideal of $Q$.

Conversely, suppose that $X_{\gamma}$ be a fuzzy left $T$-ideal of $S$. Let $x, y \in I$, if $x + y \in I$, then $X_{\gamma}(x) = X_{\gamma}(y) = 1$ and $X_{\gamma}(x + y) \geq \min \{ X_{\gamma}(x), X_{\gamma}(y) \} = 1$ if $x + y \in I$. Thus $x + y \in I$. Therefore $X_{\gamma}$ is a fuzzy $L$-$T$-ideal of $Q$.

In the similar manner one can prove remaining two parts.

Th 3.17: Suppose $X$ be a $L(La, R)$ $T$-ideal of a $T$-semi ring $V$ and $\gamma \leq \theta \neq 0$ be any two elements in $[0, 1]$, then the fuzzy subset $\pi$ of $X$, defined by

$$\pi(x) = \begin{cases} 0 & \text{if } \theta x \in I \\ \gamma & \text{otherwise} \end{cases}$$

is a fuzzy $L(La, R)$ $T$-ideal of $X$.

**Proof:** Let $X$ be a $L$-$T$-ideal of a $T$-semi ring $V$ and $x, y \in [0, 1]$. If $x, y \in I$, then $x \in I$. Suppose $x \in I$ and $y \in X \leq \theta \neq 0$ and $y \in I$. Then $y \in I$. Therefore $x \in I$ and $y \in I$. Since $X$ is a fuzzy $T$-ideal of $V$, the fuzzy subset $\pi(x)$ is a fuzzy $L$-$T$-ideal of $X$.

Th 3.18: Let $V$ be a $T$-semiring and $\pi$ be a non-empty fuzzy subset of $V$. Then $\pi$ is a fuzzy $L(La, R)$ $T$-ideal of $V$ if and only if $\pi$'s are $L(La, R)$ $T$-ideal of $V$ for all $t \in \text{Im}(\pi)$ where $\pi = \{ x \in V : \mu(x) \geq t \}$.

**Proof:** Let $\pi$ be a fuzzy $L$-$T$-ideal of $V$. Let $t \in \text{Im}(\pi)$, then $\exists a \in V, \exists \pi(a) = t$ and so $\pi \geq \theta$. Suppose $\pi \notin \theta$. Let $a \in V$ and $\pi(a) = t$. Therefore $\pi \geq \theta$. Now let $a \in V$, then $\pi(a) = t$. So $\pi(a) = t$. Now $\pi(a) = t$. Therefore $\pi \geq \theta$. Similarly, $\pi$ is a fuzzy $L$-$T$-ideal of $V$.

Conversely, suppose that $\pi$ is a $L$-$T$-ideal of $V$ for all $t \in \text{Im}(\pi)$. Again let $a, q \in V, s \in V$ and $x, y \in \Gamma$, then $\pi(a) = \pi(q) = t$. Since $\pi$ is a $L$-$T$-ideal of $V$, the fuzzy subset $\pi$ is a fuzzy $L$-$T$-ideal of $V$.

Conversely, suppose that $\pi$ be a fuzzy $L$-$T$-ideal of $V$ for all $t \in \text{Im}(\pi)$. Again let $a, q \in V, s \in V$ and $x, y \in \Gamma$, then $\pi(a) = \pi(q) = t$. Since $\pi$ is a $L$-$T$-ideal of $V$, the fuzzy subset $\pi$ is a fuzzy $L$-$T$-ideal of $V$.

Def 3.19: Let $\pi$ be a $T$-semi ring and $\pi$ be a fuzzy $L(La, R)$ $T$-ideal of a $T$-semi ring $V$. Then the $T$-ideal $\pi$'s are know as fuzzy $L(La, R)$ $T$-ideal of $V$.

Th 3.20: Let $\pi$ be a fuzzy $L(La, R)$ $T$-ideal of a $T$-semiring $V$ and $t_1 > t_2$. Then $\pi_{\geq t_1} \subseteq \pi_{\geq t_2}$. Equality occurs iff there is no $x \in V \not\exists t_1 \leq \pi(x) < t_2$.

**Proof:** The 1st section of the theorem follows easily. Now let $\pi$ be a fuzzy $L$-$T$-ideal of $V \not\exists \pi = \pi_{\geq t_1}$. If possible $t \in V \not\exists t \leq \pi(x) < t_2$. Then $x \not\in \pi_{\geq t_1}$ but $x \in \pi_{\geq t_2}$, it is contradiction. So $\pi_{\geq t_1} \subseteq \pi_{\geq t_2}$.

Conversely, let $\pi$ be a fuzzy $L$-$T$-ideal of $V$ that does not exist $x \in V$ with $t_1 \leq \pi(x) < t_2$. Then $t_1 \leq \pi(x) < t_2$. Then $x \not\in \pi_{\geq t_1}$ but $x \in \pi_{\geq t_2}$, it is contradiction. So $\pi_{\geq t_1} \subseteq \pi_{\geq t_2}$.