Fixed Point Theorems Under New Caristi Type Contraction in Bipolar Metric Space with Applications

B.Srinuvasa Rao³, G.N.V.Kishore⁴, S. Ramalingeswara Rao³

¹Department of Mathematics, K L E F, Vaddeswararam, Guntur, Andhra Pradesh, India - 522 502
², ³ Department of Engineering Mathematics, S. R. K. R. Engineering College, Bhimavaram, Andhra Pradesh, India-534 204
* Corresponding Author Email: kishore.apr2@gmail.com

Abstract

In this paper, the existence of fixed-point results in a complete bipolar metric spaces under new caristi type contraction is well established. Some attention gaining consequences are attained through our results. Finally, it presented an illustration which present applicability of the obtained results.

Keywords: Bipolar metric space; covariant map; fixed point; lower semi continuous function; new caristi type contraction.

1. Introduction and Preliminaries

Fixed point theory plays a vital role in applications of many branches of mathematics. Finding fixed points of generalized contraction mappings has become the focus of well-built research activity in fixed point theory. Recently, many investigators have published various papers on fixed point theory and applications in different ways. One of the recently popular topics in fixed point theory is to cast the existence of fixed points of contraction mappings in bipolar metric spaces which can be considered as generalizations of Banach contraction principle. In 2016, Mutlu and Gürdal [1] initiated the concepts of bipolar metric space and they also investigated some fixed point and coupled fixed point results on this space ([1], [2]).

Caristi’s fixed point theorem [3] is a renowned extension of Banach contraction principle [4]. The proof of caristi’s results has been generalized and extended in many directions see ([5] - [13]). In this paper, we shall establish some fixed point results for covariant mapping under different new caristi type contractive conditions. We have illustrated the validity of the hypotheses of our results. First we recall some basic definitions and results.

Definition 1.1: [1] Let U and V be a two non-empty sets. Suppose that d: U×V →[0,∞) be a mapping satisfying the below properties:

(i) If d (u, v) = 0, then u=v for all (u, v) ∈ U×V.
(ii) If v = u, then d (u, v) = 0, for all (u, v) ∈ U×V.
(iii) If d (u, v) ≤ d (u, w) + d (w, v), for all u, w ∈ U×V.
(iv) If d (u₁, v₂) ≤ d (u₁, v₁) + d (u₂, v₁) + d (u₂, v₂) for all u₁, u₂ ∈ U, and v₁, v₂ ∈ V.

Then the mapping d is termed as Bipolar-metric of the pair (U, V) and the triple (U, V, d) is termed as Bipolar-metric space.

Definition 1.2: [1] Assume (U₁, V₁) and (U₂, V₂) as two pairs of sets and a function as F: U₁ × V₁ → U₂ × V₂ is said to be a covariant map. If F (U₁) ⊆ U₂ and F (V₁) ⊆ V₂ and denote this with S: (A₁, B₁) ~ (A₂, B₂). The mapping S: U₁ × V₁ → U₂ × V₂ is said to be a contravariant map. If F (U₁) ⊆ V₂, and F (V₁) ⊆ U₂, and write F: (U₁, V₁) ~ (U₂, V₂). In particular, if d₁ and d₂ are bipolar metric on (U₁, V₁) and (U₂, V₂), respectively, we sometimes use the notation F: (U₁, V₁, d₁) ~ (U₂, V₂, d₂) and F: (U₁, V₁, d₁) ~ (U₂, V₂, d₂).

Definition 1.3: [1] Assume (U, V, d) as a bipolar metric space. A point v ∈ U × V is termed as a left point if v ∈ U, a right point if v ∈ V and a central point if both. Similarly, a sequence {vₙ} on the set U and a sequence {vₙ} on the set V are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence {vₙ} is considered convergent to a point v, if and only if {vₙ} is the left sequence, v is the right point and limₙ→∞ d(vₙ, v) = 0; or {vₙ} is a right sequence, v is a left point and limₙ→∞ d(v, vₙ) = 0. A bi-sequence ({{vₙ}}, {{vₙ}}) on (U, V, d) is a sequence on the set U × V. If the sequence {uₙ} and {vₙ} are convergent, then the bi-sequence ({{uₙ}}, {{vₙ}}) is said to be convergent. ({{uₙ}}, {{vₙ}}) is Cauchy sequence, if limₙ→∞ d(uₙ, vₙ) = 0. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent. A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent hence bi-convergent.

Definition 1.4: [1] Let (U₁, V₁, d₁) and (U₂, V₂, d₂) be a bipolar metric spaces.

(i) A map F is called continuous, if it left continuous at each point u ∈ U₁ and right continuous at each point v ∈ V₂.
(ii) A contravariant map F: (U₁, V₁, d₁) ~ (U₂, V₂, d₂) is continuous if and only if it is continuous as a covariant map F: (U₁, V₁, d₁) ~ (U₂, V₂, d₂).

It can be seen from the definition (1.3) that a covariant or a contravariant map F: (U₁, V₁, d₁) ~ (U₂, V₂, d₂) is continuous if and only if (uₙ) → u on (U₁, V₁, d₁) implies F((uₙ)) → F(u) on (U₂, V₂, d₂).

2. Main Results

In this section, we give some fixed point theorems for covariant mapping satisfying various new caristi type contractive conditions in bipolar metric spaces.
Theorem 2.1: Let \((U, V, d)\) be a complete bipolar metric spaces, suppose that \(F: (U, V) \rightarrow (U, V)\) be a covariant mapping satisfies

\[
d(F(a), F(b)) \leq \psi(\psi(a) - \psi(b)) \psi(a) + \psi(b) - \psi(F(a)) - \psi(F(b)) \quad (1)
\]

for all \(a, b \in U\) and \(a, b \in V\), where \(\psi: \mathbb{R} \rightarrow \mathbb{R}^+\) are lower semi continuous functions and \(\psi: (-\infty, -\infty) \rightarrow (0, 1)\) be a continuous function and provided that \(F\) is continuous. Then the mapping \(F: U \rightarrow U\) has a unique fixed point.

Proof: Let \(a_0 \in U\) and \(b_0 \in V\), we construct the bisequence \((\langle a_n \rangle, \langle b_n \rangle)\) in \((U, V)\) as \(F a_n = a_{n+1}\) and \(F b_n = b_{n+1}\) for \(n = 0, 1, 2, \ldots\).

By using the (1), we have

\[
d(a_n, b_n) = d(Fa_{n-1}, Fb_{n-1})
\]

\[
\leq \psi(\psi(a_{n-1}) - \psi(a_n)) \psi(a_{n-1}) + \psi(b_n) - \psi(Fa_{n-1}) - \psi(Fb_{n-1})
\]

\[
\leq \psi(\psi(a_{n-1}) - \psi(a_n)) \psi(a_{n-1}) + \psi(b_n) - \psi(Fa_{n-1}) - \psi(Fb_{n-1})
\]

\[
\leq \alpha(a_{n-1}) - \alpha(a_n) + \beta(b_n) - \beta(b_{n-1})
\]

\[
\alpha(a_n) + \beta(b_n) \leq \alpha(a_{n-1}) - \alpha(a_n) + \beta(b_n) - \beta(b_{n-1})
\]

\[
< \alpha(a_{n-1}) + \beta(b_{n-1})
\]

(3)

and also

\[
d(a_n, b_n) = d(Fa_{n-1}, Fb_{n})
\]

\[
\leq \psi(\psi(a_{n-1}) - \psi(a_n)) \psi(a_{n-1}) + \psi(b_n) - \psi(Fa_{n-1}) - \psi(Fb_n)
\]

\[
\leq \psi(\psi(a_{n-1}) - \psi(a_n)) \psi(a_{n-1}) + \psi(b_n) - \psi(Fa_{n-1}) - \psi(Fb_n)
\]

\[
\leq \alpha(a_{n-1}) - \alpha(a_n) + \beta(b_n) - \beta(b_{n-1})
\]

\[
\alpha(a_n) + \beta(b_n) \leq \alpha(a_{n-1}) - \alpha(a_n) + \beta(b_n) - \beta(b_{n-1})
\]

\[
< \alpha(a_{n-1}) + \beta(b_{n-1})
\]

(5)

From (3) and (5) which shows that the bisequence \((\langle a_n \rangle, \langle b_n \rangle)\) is non increasing bi-sequences of non-negative real numbers. So they must converges to point \(\lambda, \mu \geq 0\) respectively. Suppose \(\lambda \lambda > 0\) or \(\lambda < 0\). Letting \(n \rightarrow \infty\) in equation (3) and (5), we get contradiction. Therefore,

\[
\lim_{n \rightarrow \infty} \alpha(a_n) = \lim_{n \rightarrow \infty} \beta(b_n) = 0
\]

(6)

Now from (4), we have

\[
\sum_{m=1}^{\infty} d(a_n, b_{n+1}) \leq d(a_1, b_2) + d(a_2, b_3) + \cdots + d(a_m, b_{m+1})
\]

\[
< \alpha(a_0) + \beta(b_1) + \beta(b_2) + \alpha(a_1) + \alpha(a_2) + \beta(b_3) + \cdots
\]

\[
< \alpha(a_m) + \beta(b_{m+1})
\]

This shows that \(\sum_{m=1}^{\infty} d(a_n, b_{n+1})\) is a bi-convergent series. Similarly, we can prove that \(\sum_{m=1}^{\infty} d(a_n, b_{m+1})\) is a bi-convergent series. Hence convergent. Using the property (iv) in definition (1.1), for each \(n, m \in N\) with \(n < m\) and from (2) and (4), then we have

\[
d(a_m, b_m) \leq d(a_m, b_{m+1}) + d(a_{m+1}, b_{m+1}) + \cdots
\]

\[
< \alpha(a_{m-1}) + \alpha(a_m) + \beta(b_{m+1}) + \beta(b_{m+2}) + \cdots
\]

\[
< \alpha(a_{m-1}) + \alpha(a_m) + \beta(b_{m+1}) + \beta(b_{m+2}) + \cdots
\]

\[
\rightarrow 0\ as\ n, m \rightarrow \infty.
\]

Similarly, \(d(a_m, b_m) \rightarrow 0\ as\ m \rightarrow \infty\). This shown that \((a_n, b_n)\) is Cauchy bi-sequence in \((U, V)\). Therefore,

\[
\lim_{n \rightarrow \infty} a_{n+1} = k = \lim_{n \rightarrow \infty} b_{n+1}
\]

(7)

We prove that \(k\) is fixed point of \(F\). Since \(\alpha, \beta\) are lower semi continuous functions, so

\[
\lim a_n = \alpha(k), \lim b_n = \beta(k)\ from\ (6),\ we\ get\ \alpha(k) = 0 = \beta(k).\ since\ \lim_{n \rightarrow \infty} F a_n = F k. From\ (1)\ and\ (iv)\ in\ definition\ (1.1),\ we\ have
\]

\[
d(Fk, x) \leq d(Fk, b_{n+2}) + d(a_{n+2}, b_{n+2}) + d(a_{n+2}, k)
\]

\[
\leq d(Fk, Fb_{n+2}) + d(b_{n+2}, b_{n+2}) + d(a_{n+2}, k)
\]

\[
\leq \psi(\psi(k) - \psi(a_{n+2})) \psi(a_{n+2}) + \psi(b_{n+2}) - \psi(Fk) - \psi(Fb_{n+2})
\]

\[
\leq \psi(\psi(k) - \psi(a_{n+2})) \psi(a_{n+2}) + \psi(b_{n+2}) - \psi(Fk) - \psi(Fb_{n+2})
\]

\[
\leq \alpha(a_{n+2}) - \alpha(a_{n+2}) + \beta(b_{n+2}) - \beta(b_{n+2})
\]

\[
< \alpha(k) - \alpha(Fk) + \beta(b_{n+2}) - \beta(b_{n+2}) + d(a_{n+2}, k)
\]

\[
+ d(a_{n+2}, k) \rightarrow 0\ as\ k \rightarrow \infty.
\]

Therefore, \(d(Fk, x) = 0\ implies \(Fk = k\). Now we prove the uniqueness, we begin by take \(v\ be another fixed point of covariant map \(F\). Then \(Fv = v\ implies\ \forall\ v \in U \lor V\ and\ we\ have

\[
d(k, v) = d(Fk, Fv) \leq \psi(\psi(k) - \psi(a_{n+2})) \psi(a_{n+2}) + \psi(b_{n+2}) - \psi(Fk) - \psi(Fb_{n+2})
\]

\[
\leq \psi(\psi(k) - \psi(a_{n+2})) \psi(a_{n+2}) + \psi(b_{n+2}) - \psi(Fk) - \psi(Fb_{n+2})
\]

\[
\leq \psi(\psi(k) - \psi(a_{n+2})) \psi(a_{n+2}) + \psi(b_{n+2}) - \psi(Fk) - \psi(Fb_{n+2})
\]

\[
< \alpha(k) - \alpha(Fk) + \beta(b_{n+2}) - \beta(b_{n+2}) + d(a_{n+2}, k)
\]

\[
+ d(a_{n+2}, k) \rightarrow 0\ as\ k \rightarrow \infty.
\]

Therefore, \(d(k, v) = 0\ implies \(k = v\). Hence \(k\ is unique fixed point of covariant mapping \(F\).

Example 2.2: Let \(U = \{U_i(R)|i \in R\}\) is upper triangular matrices over \(R\) and
\[ V = \{ L_m(R) / L_m(R) \} \text{ is lower triangular matrices over } R. \]

Define \( d(u_m, v_m) \) for \( u_m, v_m \in L_m(R) \). Then obviously, \((U, V, d)\) is a complete bipolar metric space.

Define \( d(P, Q) = \sum_{i,j} |(P_{ij} - Q_{ij})|^2 \) for all \( P = (p_{ij})_{m \times n}, Q = (q_{ij})_{m \times n} \in L_m(R) \). Then obviously, \((U, V, d)\) is a complete bipolar metric space.

Now, from (9), we have
\[
\sum_{n=1}^{\infty} d(a_n, b_{n+1}) \leq \sum_{n=1}^{\infty} \sum_{i,j} |(a_{n,j} - b_{n+1,j})|^2 + \sum_{n=1}^{\infty} \sum_{i,j} |(a_{n,j} - b_{n+1,j})|^2
\]

Now, \( \psi : \mathbb{R} \to [0, \infty) \) is lower semi continuous function and \( \sum_{n=1}^{\infty} \sum_{i,j} |(a_{n,j} - b_{n+1,j})|^2 \) is a convergent series. Hence \( \sum_{n=1}^{\infty} |(a_{n,j} - b_{n+1,j})|^2 \) is Cauchy bi-sequence in \((U, V, d)\). Therefore,
\[
\lim_{n \to \infty} \psi(a_n, b_n) = 0. \]

Similarly, \( \psi(a_n, b_n) \to 0 \) as \( n \to \infty \). This shows that \((a_n, b_n)\) is Cauchy bi-sequence in \((U, V, d)\). Therefore,
\[
\lim_{n \to \infty} a_n + b_n = \kappa \quad \text{ or } \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n
\]

We prove that \( \kappa \) is fixed point of \( F \). Since \( F \) is continuous function, So \( \lim F a_n = F \kappa \). Since \( \psi \) is lower semi continuous function and
\[
\lim_{n \to \infty} \psi(a_n, b_n) = \psi(\kappa, \kappa). \]

Uniqueness follows from the Theorem 2.1.

3. Applications

3.1. Application to the Existence of Solutions of Integral Equations

**Theorem 3.1:** Let us consider the integral equation
\[
\int_{\kappa} f(\kappa, \nu) d\nu = \psi(\kappa, \kappa).
\]

Where \( f(\kappa, \nu) \) is a continuous function and \( \psi(\kappa, \kappa) \) is lower semi continuous. This shows that \( \psi(a_n, b_n) \) is Cauchy bi-sequence in \((U, V, d)\). Therefore,
\[
\lim_{n \to \infty} a_n + b_n = \kappa \quad \text{ or } \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n
\]

We prove that \( \kappa \) is fixed point of \( F \). Since \( F \) is continuous function, So \( \lim F a_n = F \kappa \). Since \( \psi \) is lower semi continuous function and
\[
\lim_{n \to \infty} \psi(a_n, b_n) = \psi(\kappa, \kappa). \]

Uniqueness follows from the Theorem 2.1.
Proof: Let \( U = L^w(E_1) \) and \( V = L^w(E_2) \) be two normed linear spaces, where \( E_1 \) and \( E_2 \) are two Lebesgue measurable sets with \( m(E_1 \cup E_2) < \infty \). Consider \( d: U \times V \to [0, \infty] \) be defined by \( d(f, g) = \|f - g\|_w \) for all \( (f, g) \in U \times V \). Then \((U, d)\) is complete in \( L^w \) with respect to the \( L^w \) norm. Define \( \psi: U \times V \to \mathbb{R} \) by \( \psi(f, g) = \|f - g\|_w \). Then \((U, \psi)\) is a bipolar metric space. Define \( \nu = U \times V \). Define \( \psi(U \times V) = [0, \infty) \) by \( \psi(f, g) = \|f - g\|_w \). Then \((U \times V, \psi, \nu)\) is a bipolar metric space. Define \( \alpha(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \) for all \( t \in \mathbb{R} \). Notice that \( \alpha \) is a non-decreasing function and \( 0 \leq \alpha(t) \leq 1 \) for all \( t \in \mathbb{R} \).

Thus \( \alpha \) satisfies all the conditions of Theorem 2.3. Then \( \alpha \) has a unique fixed point in \( U \times V \).

3.2. Application to Homotopy Theory

Theorem 3.2: Let \((U, V, d)\) be a complete bipolar metric space, \((A, B)\) be an open subset of \((U, V)\) and \((\bar{A}, \bar{B})\) be a closed subset of \((U, V)\) such that \((A, B) \subseteq (\bar{A}, \bar{B}) \). Suppose \( H: (\bar{A}, \bar{B}) \times [0, 1] \to \mathbb{R} \) be a function satisfying the following conditions:

(i) \( x \neq H(x, k) \) for each \( x \in \bar{A} \cup \bar{B} \) and \( k \in [0, 1] \).

(ii) \( d(H(x, k), H(y, \xi)) \leq \psi(a(x, k))a(x, k) + \psi(b(y, \xi))b(y, \xi) \) for all \( x \in \bar{A}, \ y \in \bar{B} \) and \( k, \xi \in [0, 1] \), where \( a, b: U \times V \to [0, \infty) \) are lower semi continuous functions with \( \psi(t) = \|t\|_w \) for all \( t \in \mathbb{R} \).

(iii) \( \exists M \geq 0 \) such that \( d(H(x, k), H(y, \xi)) \leq M|k - \xi| \) for every \( x \in \bar{A}, \ y \in \bar{B} \) and \( k, \xi \in [0, 1] \). Then \( H(., .) \) has a fixed point \( \bar{x} \) if \( H(., .) \) has a fixed point.

Proof: Let the set \( X = \{ x \in [0, 1] : x = H(x, k) \text{ for some } x \in A \} \).

Since \( H(., .) \) has a fixed point in \( A \cup B \). So \( 0 \notin X \cup Y \). Thus \( X \cup Y \) is non-empty.

Now we show that \( X \cup Y \) is both closed and open in \([0, 1] \). Hence by the connectedness \( X \cup Y \) is connected.

Let \((x_n, y_n) \to (x, y) \in [0, 1] \) as \( n \to \infty \). We must show that \( x, y \in X \cup Y \).

Since \( (x_n, y_n) \in X \cup Y \) for \( n = 1, 2, \ldots \), there exists \( x_n \in X \) and \( y_n \in Y \) such that \( H(x_n, \xi) = x_n \) and \( H(\xi, y_n) = y_n \).

Consider:

\[
\alpha(x_n) + \beta(y_n) = \psi(\alpha(x_n - 1))\alpha(x_n - 1) + \psi(\beta(y_n))\beta(y_n) \\
< \alpha(x_n) + \beta(y_n).
\]

And

\[
d(x_n, y_n) = d(H(x_n, \xi), H(y_n, \eta)) \\
\leq \psi(\alpha(x_n - 1))\alpha(x_n - 1) + \psi(\beta(y_n))\beta(y_n) \\
< \alpha(x_n) + \beta(y_n).
\]

Also we have:

\[
d(x_n, y_n) = d(H(x_n, \xi), H(y_n, \eta)) \\
\leq \psi(\alpha(x_n - 1))\alpha(x_n - 1) + \psi(\beta(y_n))\beta(y_n) \\
< \alpha(x_n) + \beta(y_n).
\]

From (16) and (18), which shows the monotonicity of \( (\{x_n\}, \{y_n\}) \) is non-increasing. Since \( (\{x_n\}, \{y_n\}) \) is non-negative and convergent to \( x, y \), we get contradiction. Therefore

\[
\lim_{n \to \infty} \alpha(x_n) = \lim_{n \to \infty} \beta(y_n) = 0
\]

Now from (15), we have:

\[
\sum_{n=1}^m d(x_n, y_{n+1}) \leq d(x_1, y_2) + d(x_2, y_3) + \cdots + d(x_m, y_{m+1}) \\
< \alpha(x_0) + \beta(y_1) + \alpha(x_1) + \beta(y_1) + \cdots \\
< \alpha(x_0) + \beta(y_1) + \cdots
\]

This shows that \( \sum_{n=1}^m d(x_n, y_{n+1}) \) is a bi-convergent series. Similarly, we can prove that \( \sum_{n=1}^m d(x_n, y_{n+1}) \) is a bi-convergent series.

Hence convergent. Using the property (iv) in definition (1.1), for each \( m, n \in \mathbb{N} \), we have:

\[
d(x_n, y_m) \leq d(x_n, y_{n+1}) + d(x_{n+1}, y_{n+1}) + \cdots + d(x_{m-1}, y_{m-1}) + d(x_m, y_m) \\
< \alpha(x_0) + \beta(y_1) + \cdots
\]

Similarly, \( d(x_n, y_m) \) is Cauchy bi-sequence in \( (A, B) \). By the completeness, there exist \( \eta \in A \) and \( \mu \in B \) such that

\[
\lim_{n \to \infty} x_n = \eta \text{ and } \lim_{n \to \infty} y_n = \mu
\]
\[
-\beta(H(y_n, \xi_\alpha)) + M|\kappa_n - \xi_\alpha| + d(x_{n+1}, \mu) \\
< \alpha(\eta) - \alpha(H(\eta, \kappa)) + \beta(y_n) + \beta(y_{n+1}) + \\
+ M|\kappa_n - \xi_\alpha| + d(x_{n+1}, \mu) \xrightarrow{\text{as } n \to \infty} 0
\]

It follows that \(d(H(\eta, \kappa), \mu) = 0\) implies \(H(\eta, \kappa) = \mu\). Similarly, we have \(H(\mu, \xi) = \eta\). On the other hand, from (20), we have

\[
d(\eta, \mu) = d(lim_{n \to \infty} y_n, lim_{n \to \infty} x_n) = lim_{n \to \infty} d(x_n, y_n) = 0.
\]

Thus, \(\eta = \mu\). Therefore, \(\kappa = \xi \in X \cap Y\). Clearly, \(X \cap Y\) is closed in \([0, 1]\).

Since \((\kappa_0, \xi_0) \in (X, Y)\), then there exists a sequence \((x_0, y_0)\) with \(x_0 = H(x_0, \kappa_0)\) and \(y_0 = H(y_0, \xi_0)\). Since \(A \cup B\) is open, there exist \(r > 0\) such that \(B_2(r, x_0) \subseteq A \cup B\) and \(B_2(r, y_0) \subseteq A \cup B\). Choose \(k \in (\xi_0, \epsilon, \kappa_0 + \epsilon)\) such that \(|k - \xi_0| \leq \frac{1}{M^\frac{1}{2}}\) and \(\xi \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)\) such that \(|\xi - \xi_0| \leq \frac{1}{M^\frac{1}{2}}\) also \(|k_0 - \xi_0| \leq \frac{1}{M^\frac{1}{2}}\).

Then \(y \in \overline{B_2}(r, y_0) = \{y, y_0 \in B / d(x_0, y) \leq r + d(x_0, y_0)\}\) and \(x \in \overline{B_2}(r, x_0) = \{x, x_0 \in A / d(x, y) \leq r + d(x, y_0)\}\), also

\[
d(H(x, k), y_0) = d(H(x, \kappa), H(y_0, \xi_0)) \\
\leq d(H(x, \kappa), H(\xi_0)) + d(H(\xi_0), H(y_0, \xi_0)) \\
\leq 2M|k - \xi_0| + \rho(\alpha(x_0), \alpha(\xi_0) - \alpha(H(x, \kappa))) \\
+ \psi(\beta(y_0), \beta(y) - \beta(H(y, \xi_0))) \\
< 2M^\frac{1}{2} + \alpha(x_0) - \alpha(H(x, \kappa)) + \beta(y) - \beta(H(y, \xi_0))
\]

Letting \(n \to \infty\), we get

\[
d(H(x, k), y_0) < \alpha(x_0) - \alpha(H(x, \kappa)) + \beta(y) - \beta(H(y, \xi_0))
\]

\[
\leq d(x_0, y_0) \leq r + d(x_0, y_0)
\]

Similarly, we can prove \(d(x_0, y_0) \leq d(x, y_0) \leq r + d(x_0, y_0)\).

On the other hand

\[
d(x_0, y_0) = d(H(x_0, \kappa), H(y_0, \xi_0)) \\
\leq M|\kappa_0 - \xi_0| \xrightarrow{\text{as } n \to \infty} 0
\]

So \(d(x_0, y_0) = 0\) implies \(x_0 = y_0\). Thus \(k = \xi\) and

\(H(\cdot, \kappa) : \overline{B_2}(r, x_0) \to \overline{B_2}(r, y_0)\). Thus we conclude that \(H(\cdot, \kappa)\) has a fixed point in \(A \cup \overline{B}\). But this must be in \(A \cup B\). Therefore, all conditions of Theorem 3.2 are satisfied. Hence \(H(\cdot, \kappa)\) has a fixed point in \(A \cup \overline{B}\). But this must be in \(A \cup B\). Therefore, \(k = \xi \in X\cap Y\) for \(k \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon) \subseteq X \cap Y\).

Clearly, \(X \cap Y\) is open in \([0, 1]\).

To prove the reverse, we can use the similar process.

4. Conclusion

In the present research, we have continued to investigate postulates of bipolar metric spaces. We have presented fixed point results by using new Caristi type contractive conditions defined on bipolar metric spaces, suitable examples that supports our main results. Also, applications to integral equations as well as Homotopy theory are provided.

Acknowledgement

The authors are very thanks to the reviewers and editors for valuable comments, remarks and suggestions which improved the paper in good form.

References