The Controller Design Using Local Stability Analysis on a Nonlinear Inverted Pendulum

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Abstract

This paper is designed for the purpose of analytic design and numerical simulation of a controller for nonlinear mobile inverted pendulum. It needs to obtain a method to overcome the difficulties in the design problem of controller for nonlinear plant. The method was using local stability analysis of two fixed points in two-dimensional ordinary nonlinear differential equation. The result provides a stable response solution using a special pole placement design, and transient stability is simulated using simulink. The response behavior can be selected according to the desired poles. The results obtained at this work are different from the optimal control problem. Indeed, our results have been summarized in the design method, target response and simulation process.

Keywords: analytic design, special pole placement, simulation, stable response

1. Introduction

A nonlinear inverted pendulum model has dynamics with two movements, its angle and its wheel movements, as shown in Fig. 1. These dynamics[1–4] can be elaborated by two nonlinear second order differential equations shown with its arrangement as can be seen in equations (1) and (2).

\[
\begin{align*}
\dot{\theta}(t) &= \frac{W (M_p g L \sin \theta - x) - Q \cos \theta (M_p L \dot{\theta}^2 \sin \theta + \frac{1}{2} x)}{W P - Q^2 \cos \theta^2} \\
\dot{x}(t) &= \frac{(M_p L \dot{\theta}^2 \sin \theta + \frac{1}{2} x) P - (M_p g L \sin \theta - x) Q \cos \theta}{W P - Q^2 \cos \theta^2}
\end{align*}
\]

(1)

(2)

\(\theta\) is the position of pendulum; \(x\) is the position of the wheel.

The initial position of pendulum was \(\theta = \theta_0\) in the unit (rad), the initial position of the wheel with \(x = x_0 = 0\) in the unit (m). The parameters \(M_p\) and \(M_w\) are respectively by the mass of the wheel and the pendulum in the unit (kg), and \(g = 9.81\) (m/s^2) is the gravity acceleration. To do the simulation it needs the parameters: \(L = 0.6\) m, \(M_p = 1.6\) kg, \(M_w = 1.551\) kg, \(J_o = 0.005\) kg·m^2, \(J_p = 0.027\) kg·m^2, \(r = 0.2\) m [1].

The equations (1) and (2) can be modeled in the state space equation. It chooses four states, where, \(x_1 = \dot{\theta}, x_2 = \dot{x}, x_3 = x, x_4 = \dot{x}\). Based on equations (1) and (2) have been formed the state space equation (4) below[5]:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{W g \sin x_1 - Q^2 x_2^2 \sin x_1 \cos x_1 - \left(W^2 + Q^2 \cos x_1\right) x_2}{W P - Q^2 \cos x_1^2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{F Q x_2^2 \sin x_1 - Q^2 g \sin x_1 \cos x_1 + \left(W^2 + Q^2 \cos x_1\right) x_4}{W P - Q^2 \cos x_1^2}
\end{align*}
\]

(4)

The behavior of the open loop inverted pendulum has been simulated using matlab-simulink[6–8], as shown in Fig. 2.

Both equations use three parameters can be seen in equation (3).

\[
W = M_w + M_p + \frac{J_w}{r^2}, J_p + M_p L^2, Q = M_p L
\]

(3)
Let is assumed that there is a fixed point \((x^*, u^*)\) for which \(f(x^*, u^*) = f_1(x^*, u^*) = f_2(x^*, u^*) = 0\), occurs on the plant. The linear analysis involves to carry out a Taylor expansion of the nonlinear functions in \(f_1(x,u)\) and \(f_2(x,u)\) in the neighborhood of form \(f(x^*, u^*)\), as can be seen in the following below:

\[
f(x,u) = f(x^*, u^*) + \frac{\partial f_1}{\partial x}(x^*_x, u^*_x)(x-x^*) + \frac{\partial f_2}{\partial u}(x^*_u, u^*_u)(u-u^*) + \ldots
\]

(6)

Based on equation (6), it can be defined:

\[
X = x - x^*, \ U = \ u - u^*
\]

(7)

Then it can expand equation (6) become,

\[
\frac{ax}{dt} = AX + BU + \ldots \quad \text{and} \quad \frac{au}{dt} = CX + DU + \ldots
\]

(8)

The equation (8) needs two functions \(f_1\) and \(f_2\) related to equations (1) and (2) in the form of equation (6), then it found:

\[
A = \frac{\partial f_1}{\partial x}(x^*_x, u^*_x) \quad B = \frac{\partial f_1}{\partial u}(x^*_u, u^*_u) \quad C = \frac{\partial f_2}{\partial x}(x^*_x, u^*_u) \quad D = \frac{\partial f_2}{\partial u}(x^*_u, u^*_u)
\]

(9)

In the first order of differential equation, the nonlinear form of equation (8) can be approximated by a linear equation (10).

\[
\frac{ax}{dt} = AX + BY \quad \text{and} \quad \frac{au}{dt} = CX + DY
\]

(10)

The poles of the linear approximation in equation (10) will be found with the formula below[5]:

\[
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

(11)

It found the poles \(\mu_1\) and \(\mu_2\) in equation (11), these are,

\[
|\mu - M| = \begin{bmatrix} \mu - A & -B \\ -C & \mu - D \end{bmatrix} = 0
\]

(12)

After taking determine of equation (12), both of the poles have equation as can be seen in the following below:

\[
\mu_1 = \frac{A + D}{2} \pm \sqrt{\left(\frac{A - D}{2}\right)^2 + 4BC}
\]

\[
\mu_2 = \frac{A + D}{2} \pm \sqrt{\left(\frac{A - D}{2}\right)^2 + 4BC}
\]

(13)

If the result of equation (13) has the real part with \(\mu_1 < 0\) and \(\mu_2 < 0\), the system is called stable and the responses of the system has steady-state value.

### 3. The Special Pole Placement

Based on the dynamics of nonlinear plant equation (1) and (2), it needs equation (9) and (10) to generate the linear form of nonlinear plant. Furthermore, it found the dynamics of nonlinear plant has changed into the linear equation, and the process of pole placement is explained with some steps in the following below. First, the four states have defined in equation (4), those are:

\[
x_1 = \theta, x_2 = \dot{\theta}, x_3 = x, x_4 = \dot{x}
\]

(14)

Second, refers to equation (6), it needs to make both equations (1) and (2) become \(f_1\) and \(f_2\). By using equations (9) and (10), it found all parameters for equation (11) as shown below:

\[
A_{11} = \frac{\partial f_1}{\partial x_1}(x_1, x_2, x_3, x_4, \delta), A_{12} = \frac{\partial f_1}{\partial x_2}(x_1, x_2, x_3, x_4, \delta),
\]

\[
A_{11} = \frac{\partial f_2}{\partial x_1}(x_1, x_2, x_3, x_4, \delta), A_{12} = \frac{\partial f_2}{\partial x_2}(x_1, x_2, x_3, x_4, \delta).
\]
Fourth, the next step was to perform the controller needed to stabilize the system. Define the desired closed loop system with four poles; those are \( \mu_1, \mu_2, \mu_3 \) and \( \mu_4 \). The characteristic equations to all poles are given in equation (20).

\[
L(\mu) = (\mu + \mu_1) (\mu + \mu_2) (\mu + \mu_3) (\mu + \mu_4)
\]  

(20)

If the controller in equation (17) is substituted in equation (19), the closed loop equation can be found as below:

\[
\hat{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ A_{41} & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ B_2 \\ 0 \\ B_4 \end{bmatrix} v - K_1 x_1 + K_2 x_2 - K_3 x_3 - K_4 x_4
\]  

(21)

Then the equation of closed loop system can be written down with equation (22) below:

\[
\hat{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21} - B_2 K_1 & -B_2 K_2 & -B_2 K_3 & -B_2 K_4 \\ 0 & 0 & 0 & 1 \\ A_{41} - B_4 K_1 & -B_4 K_2 & -B_4 K_3 & -B_4 K_4 \end{bmatrix} x + \begin{bmatrix} 0 \\ B_2 \\ 0 \\ B_4 \end{bmatrix} r
\]  

(22)

In general form, the equation (22) can be written with equation (23) below:

\[
\hat{x} = A_{CL} x + B_{CL} r
\]  

(23)

Where,

\[
A_{CL} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21} - B_2 K_1 & -B_2 K_2 & -B_2 K_3 & -B_2 K_4 \\ 0 & 0 & 0 & 1 \\ A_{41} - B_4 K_1 & -B_4 K_2 & -B_4 K_3 & -B_4 K_4 \end{bmatrix}
\]  

(24)

Finally the value for the special pole placement for state feedback should be found as \( K = [K_1, K_2, K_3, K_4] \) with the procedure below:

First, it needs to elaborate the equation (20) to find the characteristic equation below:

\[
L(\mu) = \mu^4 + m_3 \mu^3 + m_2 \mu^2 + m_1 \mu + m_0
\]  

(25)

Where,

\[
m_0 = \mu_1 + \mu_2 + \mu_3 + \mu_4
\]  

\[
m_1 = \mu_1 \mu_2 + \mu_1 \mu_3 + \mu_1 \mu_4 + \mu_2 \mu_3 + \mu_2 \mu_4 + \mu_3 \mu_4
\]  

\[
m_2 = \mu_1 \mu_2 \mu_3 + \mu_1 \mu_2 \mu_4 + \mu_1 \mu_3 \mu_4 + \mu_2 \mu_3 \mu_4
\]  

\[
m_3 = \mu_1 \mu_2 \mu_3 \mu_4
\]  

The next form equation of the closed loop system in the equation (24) was found with equation below:

\[
[| \mu I - A_{CL} | = 0]
\]  

(26)

The determinant of equation (26) gave the characteristic equation was shown in equation (27) below:

\[
L(\mu) = \mu^4 + [B_4 K_4 + B_3 K_3] \mu^3 + [B_4 K_3 + B_2 K_2 + A_{21}] \mu^2 + [A_{41} B_4 - A_{21} B_3 + B_2] K_1 \mu + [A_{41} B_4 - A_{21} B_3] K_4
\]  

(27)

To find the special feedback \( K = [K_1, K_2, K_3, K_4] \), the equation (27) must be equal to equation (25). If both equations are equal, it found these four equations below:

\[
B_4 K_4 + B_3 K_3 = m_3
\]  

\[
B_4 K_3 + B_2 K_2 + A_{21} = m_2
\]  

\[
[A_{41} B_4 - A_{21} B_3] K_1 = m_1
\]  

\[
[A_{41} B_4 - A_{21} B_3] K_4 = m_0
\]  

(28)

By using the analytical process in equation (28), finally the equation to obtain the value of \( K \) by the four equations was obtained below:
\[ K_3 = \frac{m_0}{A_{41} B_2 - A_{21} B_4} \]
\[ K_4 = \frac{m_1}{A_{41} B_2 - A_{21} B_4} \]
\[ K_2 = \frac{m_3 - B_4 K_4}{B_2} \]
\[ K_1 = \frac{m_2 + A_{44} B_4 K_2}{B_2} \]  
\[ (29) \]

4. Simulation

Using the parameters of the open loop system in equations (18) and (19) it found the state-space equation below:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
6.2943 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
\end{bmatrix} F 
\]  
\[ (30) \]

It needs to choose four the desired poles refer to the equation (25) with the value in the following below:

\[
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{bmatrix} =
\begin{bmatrix}
-3 \\
-3 \\
-3 \\
-3 \\
\end{bmatrix} 
\]

By using equations (29) and (25) it found the value of state feedback in the following below:

\[
K = [-170.2331 - 68.5773 - 99.3821 - 94.4130]  
\]  
\[ (31) \]

The state feedback can be written as in the following below:

\[
F(t) = v(t) - [k_1 \ k_2 \ k_3 \ k_4] [x_1 \ x_2 \ x_3 \ x_4]^T 
\]  
\[ (32) \]

By inserting \( F(t) \) from the equation (32) to equation (30), the closed loop system has the result in the following below:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
6.2943 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} -
\begin{bmatrix}
0 \\
1 \\
0 \\
1 \\
\end{bmatrix} [k_1 \ k_2 \ k_3 \ k_4] 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
\end{bmatrix} v(t) 
\]  
\[ (33) \]

Finally, with the input \( v(t) = 0 \), the equation (33) can be applied to the nonlinear system in equations (1) and (2) as in the following below:

\[
\ddot{\theta}(t) = \frac{W_M g L \sin \theta - Q_M g L \dot{\theta}^2 \sin \theta \cos \theta - \left( W + Q \cos \theta \right) [ -k_1 \dot{\theta} - k_2 \dot{\theta} - k_3 \dot{x} - k_4 x]}{W_P - Q^2 \cos \theta^2} 
\]  
\[ (34) \]

\[
\dot{x}(t) = \frac{P_M \dot{\theta}^2 \sin \theta - Q_M g L \sin \theta \cos \theta + \left( \frac{P}{Q} \cos \theta \right) [ -k_1 \dot{\theta} - k_2 \dot{\theta} - k_3 \dot{x} - k_4 x]}{W_P - Q^2 \cos \theta^2} 
\]  
\[ (35) \]

To build the simulation using simulink diagram for the system with equations (34) and (35), it performed four sub-systems, as can be seen in Fig. 5.
There are two figures that captured by scope in the simulation, those are $x_1, x_2$ as can be seen in Fig. 10 and $x_3, x_4$ as can be seen in Fig. 11.

Based on the results in Fig. 10 and 11, all the responses of the nonlinear plant will run toward stable with oscillation damped until the responses toward zero, and all the responses are eligible.

5. Conclusion

Based on the result, it can be described that responses of linearization of the nonlinear plant has been adequate. Then the controller has been produced for the nonlinear plant by using the local stability analysis. The process needed the special feedback pole placement method like equation (21) to (32) that should be applied at simulink process. Lastly, the controller has been simulated by using simulink techniques. All the behavior of the responses was stable revering to the poles given. The results of these responses on the Fig. 10 and 11 have been taken after choosing the best poles by the trial and error process.

References