Travelling-Wave Similarity Solution for Gravity-Driven
Rivulet of a Non-Newtonian Fluid

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Abstract

Unsteady travelling-wave similarity solution describing the flow of a slender symmetric rivulet of non-Newtonian power-law fluid down an inclined plane is obtained. The flow is driven by gravity with strong surface-tension effect. The solution predicts that at any time \( t \) and position \( x \), the rivulet widens or narrows according to \( (x - ct)^{1/6} \), where \( c \) is velocity of a rivulet, and the film thickens or thins according to a free parameter \( F_0 \), independent of power-law index \( N \). The rivulet also has a quartic transverse profile which always has a global maximum at its symmetrical axis.

Keywords: Power-law fluid, Rivulet, Travelling-wave similarity solution, Thin film.

1. Introduction

Rivulet flows occur in a wide range of practical situations ranging from industrial situation such as coating processes to geophysical situation such as lava flow. There is therefore a considerable literature of both steady and unsteady flows of thin and slender rivulets. Following the approach of Smith1 for gravity-driven rivulet of a Newtonian fluid, Duffy and Moffat2 obtained a steady similarity solution for gravity-driven rivulet of a Newtonian fluid with strong surface-tension effect down a near-vertical plane. The similarity solution predicts that the width and the height of rivulet obtained by Smith1 is modified to \( x^{3/13} \) and \( x^{-1/13} \), where \( x \) is the distance down the plane. Wilson and Burgess3 obtained a steady similarity solution for driven rivulet of a non-Newtonian power-law fluid down an inclined plane. The similarity solution indicates that the width and the height of rivulet vary according to \( x^{(2N+1)/(5N+2)} \) and \( x^{-N/(5N+2)} \), where \( N \) is a power-law index. Wilson et. al4 obtained the steady similarity solutions for rivulets of a non-Newtonian power-law fluid driven by either gravity or constant surface shear stress down an inclined plane, for both weak and strong surface-tension effects. They found that, despite the rather different physical mechanisms driving the flow, the similarity solutions for gravity-driven and shear-stress-driven rivulets are qualitatively similar. Particularly, the solution for gravity-driven flow recovers the solutions of Wilson and Burgess3, while for shear-stress-driven flow, the width and the height of rivulet vary according to \( x^{-1/6} \), respectively, independent of power-law index \( N \).

The unsteady similarity solution for gravity-driven rivulet of a non-Newtonian power-law fluid on an inclined plane has been studied by Atiyat et. al5, both for converging sessile rivulet and diverging pendent rivulet. The solution predicts that the evolution of the width and the height of rivulets at any time \( t \) vary according to \( |x|^{(2N+1)/(8N+1)} \) and \( |x|^{-N/(8N+1)} \), respectively, while at any position \( x \) vary according to \( |t|^{-N/2(2N+1)} \) and \( |t|^{-N/(2N+1)} \), respectively, with cross-sectional profiles that are either single-humped or double-humped. More recently, Abas et. al6 obtained a different type similarity solution namely a travelling-wave similarity solution for the unsteady gravity-driven rivulet of a Newtonian fluid down an inclined plane, with strong surface-tension effect. In this study, the approach of Abas et. al6 is used to obtain travelling-wave similarity solution describing unsteady gravity-driven rivulet of a non-Newtonian power-law fluid down an inclined plane, with strong surface-tension effect.

2. Problem Formulation

Consider the unsteady flow of a thin slender rivulet of a non-Newtonian power-law fluid with constant density \( \rho \) and viscosity \( \mu = \mu_0\gamma^{-N-1} \), where \( \mu_0 \) is the consistency coefficient, \( \gamma \) is the shear rate and \( N > 0 \) is the power-law index on a plane inclined at an angle \( \alpha(0 < \alpha < \pi/2) \) to the horizontal subject to gravitational acceleration \( g \) with strong surface-tension effect \( \sigma \). The power-law fluid is characterized as a shear thinning when \( 0 < N < 1 \) and a shear thickening when \( N > 1 \); when \( N = 1 \), the special case of a Newtonian fluid with constant viscosity \( \mu_0 \) is recovered.

Cartesian coordinates \( Oxyz \) with the \( x \)-axis down the line of greatest slope and the \( z \)-axis normal to the substrate, with the substrate at \( z = 0 \) are adopted. The (unknown) free surface of the rivulet is denoted by \( z = h(x,y,t) \), where \( t \) is time. The rivulet is considered symmetric about \( y = 0 \) (i.e. for which \( h \) is even in \( y \)) with (unknown) semi-width \( a = a(x,t) \), so that \( h = 0 \) at the contact lines \( y = \pm a \). The geometry of the problem is sketched in Figure 1.

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Making the familiar lubrication approximation, the velocity \((u, v, w)\) and pressure \(p\) satisfy the governing equations
\[ u_x + v_y + w_z = 0, \tag{1} \]
\[ (\mu u_x)_x + p g \sin \alpha = 0, \tag{2} \]
\[ (\mu v_y)_y = p_y = 0, \tag{3} \]
\[ -p_z - p g \cos \alpha = 0, \tag{4} \]
subject to the boundary conditions of no slip and no penetration on the substrate \(z = 0\):
\[ u = v = w = 0 \tag{5} \]
and balances of normal and tangential stresses on the free surface \(z = h\):
\[ p = p_a - \sigma h_{yy}, \quad \mu u_x = \mu v_x = 0, \tag{6} \]
where \(p_a\) is atmospheric pressure, together with the kinematic condition on \(z = h\):
\[ h_x + u_x + v_y = 0, \tag{7} \]
where \(\bar{u} = (x, y, t)\) and \(\bar{v} = (x, y, t)\) are the local fluxes of the flow in the longitudinal (x-axis) and in the transverse (y-axis) direction, respectively, defined by
\[ \bar{u} = \int_0^h u dz, \quad \bar{v} = \int_0^h v dz \tag{8} \]
and the zero-mass-flux condition at the contact lines \(y = \pm \alpha(x, t)\):
\[ \bar{v} = \pm \alpha u \bar{u}. \tag{9} \]
Equations (1) – (4) can be solved to yield
\[ p = p_a - \rho g \cos \alpha (h - z) - \sigma h_{yy}, \tag{10} \]
\[ u = \frac{N}{N+1} \left( \frac{\rho g \sin \alpha}{\mu_0} \right)^\frac{1}{N} \frac{h^{N+1}}{h^{N+1} - (h - z)^{N+1}}, \tag{11} \]
\[ v = -\frac{N}{N+1} \left( \frac{\rho g \cos \alpha h - \sigma h_{yy}}{\mu_0} \right) \frac{h^{N+1}}{h^{N+1} - (h - z)^{N+1}} \tag{12} \]
and substitution of (11) and (12) into (8) gives
\[ \bar{u} = \frac{N}{2N+1} \left( \frac{\rho g \sin \alpha}{\mu_0} \right)^\frac{1}{N} \frac{h^{2N+1}}{h^{2N+1} - h^{N+1}}, \tag{13} \]
\[ \bar{v} = -\frac{N}{2N+1} \left( \frac{\rho g \cos \alpha h}{\mu_0} \right)^\frac{1}{N} \frac{h^{2N+1}}{h^{2N+1} - h^{N+1}}, \tag{14} \]
respectively. Therefore, the kinematic condition (7) yields the governing partial differential equation for \(h\):
\[ \frac{2N+1}{N} \left( \frac{\rho g \sin \alpha}{\mu_0} \right)^\frac{1}{N} \left( \frac{h^{N+1}}{h^{N+1} - h^{N+1}} \right)_y = \sigma \left( \frac{h^{N+1}}{h^{N+1} - h^{N+1}} \right)_y \tag{15} \]
with \(h\) satisfies the contact-line condition
\[ h = 0 \quad \text{at} \quad y = \pm \alpha, \quad h^{N+1} h_{yy} \rightarrow 0 \quad \text{as} \quad y \rightarrow \pm \alpha, \tag{16} \]
where the fluid occupies \(|y| \leq \alpha\). Once \(h\) is determined from (15), the complete solution for \(p, u, v\) is given by (10) – (12). Note that, in the special case of \(N = 1\), equation (15) reduces to the familiar equation describing the unsteady gravity-driven flow of a thin slender rivulet of Newtonian fluid studied by Abas et al.\(^8\). The draining down the plane driven by gravity is negligible in comparison with the flow down caused by surface tension; this is justified provided that
\[ \rho g \cos \alpha h \ll \sigma h_{yy}. \tag{17} \]
Consider the unsteady travelling-wave similarity solution of (15) in the form
\[ h = \eta \left( \frac{\ell (x - ct)}{\ell} \right)^\frac{1}{2}, \tag{18} \]
where the velocity of the rivulet \(c\) (up or down the substrate) and the dimensionless function \(F = F(\eta) (\geq 0)\) of the dimensionless similarity variable \(\eta\) are to be determined, and \(b > 0\) and \(\ell\) are constants, which, without loss of generality, can be written as \(\ell = 4\pi b S_p / p g \sin \alpha\), where \(S_p = \pm 1\). The rivulet lies in the region where \(\ell (x - ct) \geq 0\); along \(x = ct\), the fluid thickness \(h\) and its derivative \(h_{yy}\) are continuous (i.e. so that \(u, v, p\) are also continuous there), except at the apex of a rivulet, \(x = ct, \ y = 0\).
For simplicity in plotting results, the variables are scaled according to
\[ x = X x^*, \quad h = h^* h, \quad z = b z^*, \quad y = (\ell Y) y^*, \quad t = T t^*, \quad a = (\ell X) a^*, \tag{19} \]
\[ h_m = h h^*, \quad c = U c^*, \tag{20} \]
where \(X\) is a length scale in the \(x\)-direction which may be chosen arbitrarily and \(U\) is a velocity scale given by
\[ U = \frac{N}{2N+1} \left( \frac{\rho g \sin \alpha}{\mu_0} \right)^\frac{1}{N} \tag{21} \]
and hence (15) reduces to a fourth-order ordinary differential equation for \(F\), namely
\[ \left( F^{2N+1} \right)' - S b \left( F^{2N+1} - c F \right)' = 0, \tag{22} \]
where a prime denotes differentiation with respect to \(\eta\). For a symmetric rivulet, regular at \(y = 0\), appropriate boundary conditions are
\[ F = F_0, \quad F' = 0, \quad F'' = F_2, \quad F''' = 0 \tag{23} \]
where \(F_0 (\geq 0)\) and \(F_2\) are the free parameters. The position where \(F = 0\) is \(\eta = \eta_0\) (corresponding to the contact-line position \(y = \alpha\), so that
\[ F = 0 \quad \text{at} \quad \eta = \eta_0, \quad F^{2N+1} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \eta_0, \tag{24} \]
where the fluid now lies in \([\eta] \leq \eta_0\). The semi-width of the rivulet varies with \(x\) and \(t\) according to
\[ a = (x - ct)^{\frac{1}{2}} \eta_0. \tag{25} \]
In order to satisfy the assumption of thin and slender rivulet, the length scales in \(x, y\) and \(z\) directions (namely \(X, a\) and \(h_m\), respectively) must satisfy \(h_m \ll a \ll X\), which requires that
\[ \frac{\sigma X S_p}{b^2 p g \sin \alpha} \gg 1, \quad \frac{X^3 \rho g \sin \alpha S_p}{a b} \gg 1, \tag{26} \]
respectively, showing that \(X\) must be sufficiently large and that \(a\) cannot be close to 0.

3. Results and Conclusion

Since a closed-form solution of (22) is not available, so it must be solved numerically for \(F\) subject to the boundary conditions (23) and (24), where \(c\) and \(\eta_0\) are parameters to be determined. There are four cases to consider, namely case 1: \(S_2 = 1, \ c > 0, \ 0 < N < 1\), case 2: \(S_2 = -1, \ c < 0, \ 0 < N < 1\), case 3: \(S_2 = -1, \ c > 0, \ N > 1\) and case 4: \(S_2 = 1, \ c < 0, N > 1\); however, it turns out that the system (22) – (24) has solutions only in case 1, case 3 and case
4. Therefore, from now on only these three cases shall be considered. 

Equation (22) was solved numerically for \( F \) subject to (23) for a given value of \( F_0 \) and \( F_2 < 0 \) by using a shooting technique via Mathematica 9.0 software, the value of \( c \) and \( \eta_0 \) being determined as the point where \( F = 0 \). It was found that there are solutions when \( 0 < c < c_{\text{max}} \) for case 1, \( c \geq c_{\text{min}} \) for case 3 and \( c < 0 \) for case 4, where the value of \( c_{\text{min}} \) and \( c_{\text{max}} \) vary according to \( F_0 \) and \( F_2 \). 

In case 1, the relation between \( F_0 \) and \( \eta_0 \) is monotonic; for any value of \( F_2 \) there is a corresponding unique solution of \( \eta_0 \), but for any value of \( \eta_0 \) there is no solution when \( \eta_0 < \eta_{\text{bc}} \) while there is a unique solution of \( F_0 \) when \( \eta_0 \geq \eta_{\text{bc}} \). In case 3 and case 4, the relation between \( F_0 \) and \( \eta_0 \) is also monotonic; with a unique solution occurs for any choice of \( F_0 > 0 \) and \( \eta_0 \).

Also, it is found that \( F \) satisfies

\[
F = F_0 + \frac{F_2}{2} \eta^2 + \frac{F_2 S_c}{360 F_0^N} \left[ \frac{N+1}{N} \right] \left( 2N + 1 \right) F_0^{N+1} N \eta^{N+1} + O(\eta^N)
\]

near \( \eta \to 0 \) and

\[
F - \left( \frac{S_c}{3 N(N-1)(N+2)} \right)^N \left[ \frac{N+1}{N} \right] \left( 2N + 1 \right) F_0^{N+1} N \eta^{N+1}
\]

if either \( S_c > 0 \), \( 0 < N < 1 \) or \( S_c < 0 \), \( N > 1 \) at the leading order, as \( \eta \to \eta_0 \). Therefore, so far, the family of solutions of (22) parameterized by \( F_0 \) and \( F_2 \) are obtained, with \( c \) and \( \eta_0 \) determined in terms of \( F_0 \) and \( F_2 \). Figure 2 shows a plot of \( \eta_0 \) as a function of \( F_0 \), together with \( F_{\text{bc}} \) and \( \eta_{\text{bc}} \) (shown as dots), while Figure 3 shows three-dimensional plots of the free-surface profile \( z = h \) at different times \( t \) which demonstrate that the rivulets become narrower as time elapses while maintaining their cross-sectional shapes and their thickness.

Fig. 2: Plot of \( \eta_0 \) as a function of \( F_0 \) for \( F_2 = -1 \), \( c = 1 \) and \( N = 1/5, 2/5, 3/5 \) and \( 4/5 \), together with \( F_{\text{bc}} \) (shown as dots).

Fig. 3: Three-dimensional plots for \( F_0 = 2 \), \( F_2 = -1 \), \( c = 1 \) at times (a) \( t = 1 \) and (b) \( t = 5 \) with \( N = 1/2 \).

The travelling-wave similarity solutions describing the unsteady gravity-driven flow of a thin slender rivulet of a non-Newtonian power-law fluid down an inclined plane are obtained. The velocity and pressure are given by (10)–(12) in terms of free surface profile \( h \), where \( h \) is given by (21). There were four cases to consider (labelled as case 1, case 2, case 3 and case 4), but there is no solution found in case 2. Numerical analysis showed that the rivulet has a quartic shape which always has a maximum thickness at \( y = 0 \). This work also generalized the work of Abas et. al\(^b \) when \( N = 1 \).

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