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Research paper

Gaussian Pell Numbers

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Abstract

Gaussian numbers means representation as Complex numbers. In this work, Gaussian Pell numbers are defined from recurrence relation of Pell numbers. Here the recurrence relation on Gaussian Pell number is represented in two dimensional approach. This provides an extension of Pell numbers into the complex plane.

Keywords: *Pell sequence, Gaussian integers, Recurrence relations, Gaussian Pell number.*

1. Introduction

“God invented the integers; all else the work of man” as quoted by Kronecker the research on spotting subsets of integers that follow recurrence relations is a delightful and inexhaustive one. It is actually a wide class of problem. Rather than as a science Mathematics is thought as a creative art by Mathematics.

The theory of numbers has always occupied a unique position in the world of mathematics. Noted mathematicians on one hand and numerous amateurs on the other hand share a similar interest and are attracted not towards any other theory but towards the theory of numbers.

Number theory, regarded ‘Queen of Mathematics’ provides all the weapons of mathematics and so both professional mathematician and upcoming research scholars are invariably attracted to number theory. Also, it is a fact that many important branches of mathematics has their origin in number theory. As the mathematician Sierpinski once said the progress of knowledge of numbers is advanced not only by what is known already but the realization of what yet to be known.

Leonardo Fibonacci, Mathematical innovator of the 13th Century was a solitary flame of mathematical genius during the middle ages. Fibonacci wrote in 1202, the Liber Abaci, in which he explained the Hindu Arabic Numerals and how they are used in Computation. Fibonacci is remembered particularly for the sequence of number $1,1,2,3,5,\ldots$ to which his name has been applied.

This sequence is the subject of continuing research especially by the association which publishes the Fibonacci Quarterly. Recursive definition of the n^{th} Fibonacci number F_n is

$F_n = F_{n-1} + F_{n-2}, n \geq 3$ and $F_1 = F_2 = 1.$

Using the Fibonacci recurrence relation and different initial conditions, many integral sequences can be constructed. A.F. Horadam [5] by introduced the concept of Complex Fibonacci numbers. C.J. Harman in [1] extended the complex Fibonacci numbers and established some recurrence relations concerning them.

As in Fibonacci sequence and Lucas sequence the sequence of Pell numbers is also expressed as by the recurrence relation

$P_{n+2} = 2P_{n+1} + P_n, P_0 = 0$ and $P_1 = 1$

The first few terms of the Pell sequence are $0,1,2,5,\ldots\ldots$.Here, in this paper the Gaussian Pell sequence is defined by a recurrence relation.

2. The gaussian pell number

The set of Gaussian Pell numbers is denoted by $GP(n,m)$ and is defined in analogy with the Pell recurrence relation

$$P_{n+2} = 2P_{n+1} + P_n, P_0 = 0 \text{ and } P_1 = 1 \qquad (1)$$

Then the two-dimensional recurrence relations, satisfied by $GP(n,m)$ will be

$$GP(n+2,m) = 2GP(n+1,m) + GP(n,m) \qquad (2)$$

$$GP(n,m+2) = 2GP(n,m+1) + GP(n,m) \qquad (3)$$

where

$$GP(0,0) = 0, GP(1,0) = 1, GP(0,1) = i, GP(1,1) = 1+i \qquad (4)$$

with $GP(n,m) = n + im$

The conditions are sufficient to specify unique value of $GP(n,m)$ at each point (n,m) in the plane .

when $m = 0$,

$GP(n+2,0) = 2GP(n+1,0) + GP(n,0)$

and hence

$$GP(n,0) = P_n \qquad (5)$$

$$GP(n,1) = 2P_n GP(1,1) + P_{n-1} GP(0,1)$$

By substitution,

$GP(n,1) = 2P_n(1+i) + iP_{n-1}$

$$= 2p_n + i\left(2p_n + p_{n-1}\right) \tag{6}$$

and so by (1)

$$GP(n,m) = P_m GP(n,1) + P_{m-1}\, GP(n,0)$$
$$= P_m\left(2P_n + i\left(2P_n + P_{n-1}\right)\right) + P_{m-1} GP(n,0)$$
$$= P_m\left(2P_n + iP_{n+1}\right) + P_{m-1}P_n$$
$$= 2P_m P_n + iP_m P_{n+1} + P_{m-1}P_n$$

$$GP(n,m) = P_n P_{m+1} + iP_m P_{n+1} \tag{7}$$

Here $GP(n,0) = P_n$, and $GP(0,m) = ip_m$.

3. Recurrence equations and identities

Combination of (2) and (3) gives

$$GP(n+2,m+2) = R_n\left[2GP(n+1,m+2) + GP(n,m+2)\right] +$$

$$I_m\left[2GP(n+2,m+1) + GP(n+2,m)\right] \tag{8}$$

which is an interesting two dimensional version of Pell recurrence relation and gives the growth-characteristic of the numbers in a unique fashion: Gaussian Pell number $GP(n,m)$ is defined as: $'n'$ is the sum of twice the real part of previous vertex and the real part vertex that precedes it along the real axis and $'m'$ is the sum of twice the imaginary part of the previous vertex and imaginary part of the vertex that precedes it along the imaginary axis on the Gaussian lattice.

$$GP(n+1,m+1) = P_{n+1}P_{m+2} + iP_{m+1}P_{n+2}$$
$$= P_{n+1}\left(2P_{m+1} + P_m\right) + i\left(2P_{n+1} + P_n\right)P_{m+1}$$
$$= 2P_{n+1}P_{m+1}(1+i) + P_m P_{n+1} + iP_n P_{m+1}$$

By (7), we have

$$GP(n+1,m+1) = 2P_{n+1}P_{m+1}(1+i) + GP(m,n) \tag{9}$$

Repeatedly applying (9), we get

$$GP(n+1,m+2) = 2P_{n+1}P_{m+2}(1+i) + P_{n+1}P_{m+1} + iP_n P_{m+2}$$
$$GP(n+2,m+2) = 2(1+i)P_{n+2}P_{m+2} + P_{m+1}P_{n+2} + iP_{n+1}\, P_{m+2}$$
$$= 2(1+i)P_{n+2}P_{m+2} + P_{m+1}\left(2P_{n+1} + P_n\right) + iP_{n+1}\left(2P_{m+1} + P_m\right)$$
$$= 2(1+i)P_{n+2}P_{m+2} + 2P_{n+1}P_{m+1} + P_n P_{m+1} + 2iP_{n+1}P_{m+1} + iP_{n+1}P_m$$
$$= 2(1+i)P_{n+2}P_{m+2} + 2P_{n+1}P_{m+1}(1+i) + P_n P_{m+1} + iP_{n+1}P_m$$
$$GP(n+2,m+2) = 2(1+i)\left[P_{n+2}P_{m+2} + P_{n+1}P_{m+1}\right] + GP(n,m) \tag{10}$$

Repeatedly applying (9) and (10), we get

$$GP(n+3,m+3) = 2(1+i)\left[P_{n+3}P_{m+3} + P_{n+2}P_{m+2}\right] + GP(n+1,m+1)$$
$$GP(n+4,m+4) = 2(1+i)\left[P_{n+4}P_{m+4} + P_{n+3}P_{m+3}\right] + GP(n+2,m+2)$$
$$GP(n+2k,m+2k) = 2(1+i)\sum_{j=1}^{2k} P_{n+j}P_{m+j} + GP(n,m) \tag{11}$$

$$GP(n+2k+1,m+2k+1) = 2(1+i)\sum_{j=1}^{2k+1} P_{n+j}P_{m+j} + GP(m,n) \tag{12}$$

From (11),

$$2(1+i)\sum_{j=1}^{2k} P_{n+j}P_{m+j} = GP(n+2k,m+2k) - GP(n,m)$$

and so by (7)

$$\sum_{j=1}^{2k} P_{n+j}P_{m+j} = \frac{1}{4}(1-i)\left[P_{n+2k}P_{m+2k+1} - P_n P_{m+1} + iP_{n+2k+1}P_{m+2k} - iP_{n+1}P_m\right]$$

Equating real and imaginary parts,

$$P_{n+2k}P_{m+2k+1} - P_{n+2k+1}P_{m+2k} + P_{n+1}P_m - P_n P_{m+1} = 0 \tag{13}$$

and

$$\sum_{j=1}^{2k} P_{n+j}P_{m+j} = \frac{1}{4}\left[P_{n+2k}P_{m+2k+1} - P_n P_{m+1} + P_{n+2k+1}P_{m+2k} - P_{n+1}P_m\right] \tag{14}$$

Substitution of $P_{n+2k}p_{m+2k+1}$ from (13) into (14) gives

$$2\sum_{j=1}^{2k} P_{n+j}P_{m+j} = P_{n+2k+1}P_{m+2k} - P_{n+1}P_m \tag{15}$$

Similarly,

$$2\sum_{j=1}^{2k+1} P_{n+j}P_{m+j} = P_{n+2k+2}P_{m+2k+1} - P_n P_{m+1} \tag{16}$$

Identities (15) and (16) unify and generalize certain identities and provides some examples as special cases.

For example, $n = m = 0$ yields the well known identity:

$$2\left(P_1^2 + P_2^2 + \ldots + P_N^2\right) = P_N P_{N+1}$$

From (15), in this case $m = 0, n = 1$ gives

$$2\left(P_1P_2 + P_2P_3 + \ldots + P_{2k}P_{2k+1}\right) = P_{2k}P_{2k+2}$$

From (16), in this case $n = 0, m = 1$ gives

$$2\left(P_1P_2 + P_2P_3 + \ldots + P_{2k+1}P_{2k+2}\right) = P_{2k+2}^2$$

By the choice of various parameters we can get many identities. For example, equation (15) with $m = 0, n = 2$ gives

$$2\left(P_1P_3 + P_2P_4 + \ldots + P_{2k}P_{2k+2}\right) = P_{2k}P_{2k+3}$$

From (16), in this case $m = 2, n = 0$ gives

$$2\left(P_1P_3 + P_2P_4 + \ldots + P_{2k+1}P_{2k+3}\right) = P_{2k+2}P_{2k+3}$$

Taking $m = 2k+1$, in (13) we get

$$P_{n+2k+1}P_{m+2k+2} - P_{n+2k+2}\, P_{m+2k+1} = P_{n+1}P_m - P_n P_{m+1} \tag{17}$$

and together (13) and (17) constitute a generalization of some well-known classical identities. For example if $n = 1$, $m = 0$ they give

$$P_{N-1}P_{N+1} - P_N^2 = (-1)^N, N \geq 1.$$

with $n = 1, m = -2$ equations (13) and (17) yield the identity

$$P_{N-1}P_{N+1} - P_{N-2}P_{N+2} = 5(-1)^N.$$

4. Conclusion

Fibonacci and Lucas numbers have always been an enthusiastic material to study for both amateur and established mathematicians. Also, Gaussian approach is adding feather to it. Combination of these two stimulates the researchers to extend the study to other sequence of numbers which may be given by recurrence relations. So, I have tried the single case of Pellian numbers. Further study will be made in future for other sequence of numbers.

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