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Research paper

The development of approximation theory and some proposed applications

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Abstract

In this survey article, we review the early history of approximation theorem that was introduced by Weierstrass in the late 18th century, together with its extension works. We also propose some applicable scenarios that best fit this theory. Such applications include the kinetics conditions related to manual and automatic vehicle transmission by using convex function, and the theory of calculating deviations of eye's layers (normal vision, hyperopia and myopia) in some patients by using monotone function.

Keywords: *Approximation Theory; Chebyshev's Polynomial; Weierstrass' Theorem; 2010 MSC: 42A10; 42A15; 41A52; 41Axx.*

1. Scope of the survey

In this survey, we present a historical overview of the approximation theory and its development over time. Then, we provide some open problems of approximation theory in mechanical engineering and ophthalmology, through examples in gear transmission and medical optics. These are pragmatic applications, given the centrality of vehicular mobility and human sight to improve the quality of life.

2. Review of the development of approximation theory

We begin by giving a broad perspective of approximation theory. It was created to provide value for the methods of infinite finitisation qualitatively and quantitatively. It was founded based on mathematical model provided by mapping, function, number and set, which contributed to understanding the problems of quadrature in approximation theory. The aim of approximation theory is to introduce and provide computation in theoretical mathematics. Such computational advantages is able to number operations including division, multiplication, inversion of a matrix, best polynomials and solution for equations [1]. In fact, if the word "approximation theory" is used, this term refer to more divisions in mathematical analysis, which was developed as a result of the works by Chebyshev (1853), Weierstrass (1885), Lebesgue (1898), Bernstein (1952, 1937), Nikol'skij (1945), Kolmogorov (1985) and followed by many of their followers such as [2], and [3]. The direction of approximation theory (e.g. functions by trigonometric, algebraic polynomials, splines, rational functions and estimates approximation on smooth functions), is not limited to technical sciences only. It is also relatable to practical human activities, such as the theory of mechanisms in steam engines. The applications highlighted by Chebyshev's works are notable examples of this practical application.
In the early work of approximation theory, the number Π that was developed by Euler (1747) was known as the period of approximation theory. By diverting their attention to mathematical analysis, Euler (1747), Bernoulli (1694), Kepler (1615), Newton (1665) and Lagrange (1898) developed methods to approximate numbers, operators and functions in the form of solutions for equations. However, Gauss (1986) popularising approximation theory by developing methods for optimal computation of integrals and solutions of equations. He also developed an approximation theory in the metric of the quadratic function. Later, Chebyshev (1853) developed methods for optimal approximations of abundant concrete functions [4].
In the era of modern technology however, many of his published works only gain historical interest. Nonetheless, it is important to note that the significance of Chebyshev's work had turned out to have wider pragmatic application. Such extension enabled a new direction in functional analysis, whereby research topics expanded towards algebraic polynomial and best approximation, to name just two area of interest. The complexity of functions approximation by means of polynomials are the core of the first stage of Chebyshev's work on approximation theory (see [5] and [6]). Gradually, this practical direction (better known as the second stage in approximation theory) describes the purely theoretical problem for the drawn properties of functions by the usage of their approximative characteristics. These include functions such as the Fourier and Taylor coefficients, Pade approximation and best polynomial approximation.
The third stage in the development of approximation theory focused on approximative possibilities of the algebraic and trigonometric polynomials and rational fractions. The works in this third phase include Kolmogorov [3] such that a new viewpoint on classical

approximation theory was promoted. For example, non-classical tools for approximations such as spline began to penetrate computational practice [7]. This resulted in the idea to use the rigor of approximation, the ∈-entropy.
Currently, we would like to believe that we stand on the edge of an emerging fourth stage in the development of approximation theory, characterised by attempts to analysis discrete mathematics in new perspective. Also, it is inevitable that the mix between connotations and theorems for accuracy and complexity should be developed.
The following subsections provides brief introduction to the developmental stages of approximation theory.

2.1. Approximation of pi (Π) [8]

Approximation methods of numbers, mappings and functions began in 17^{th} and 18^{th} centuries. However, prior to this, an ancient reference to approximation was documented in textbook by ancient Egyptians called Rhind papyrus (2000 – 1700 B.C.), stated that a circular area is equal to a square area by equaling the sides of the circle diameter, lessened by $\frac{1}{9}$ of its length. This gives a nearness value of $\Pi = \frac{256}{81} = 3.1604 \ldots$. This is the earliest reference to the problem of approximation of a number (in this case Π) and quadratures, i.e., the replacement of the area under a curve by the area of an equivalent square.

2.2. Approximation of functions [8]

The first function was deliberate by some methods. Letter by Newton to Oldenburg (1676) captures in detail the invention of getting logarithms from hyperbolic area. His enthusiasm was very strong, but a better description by Mercator in "Logarithmotekhnica" (1668) represented better by introducing series to better handle logarithm, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. Series were also used in the calculation of numbers, i.e., the series $\log 2 = \frac{1}{1*2} + \frac{1}{3*4} + \cdots$. Then, what we now call Taylor series was provided in the works of Newton (1665) and Taylor (1715). The trigonometric series occurred in the works of Euler in 1744 and was further improved in 1777 through Leibniz's and Euler's work on approximation of the series we now call splines.

2.3. Approximation of quadrature [8]

It is known that the term quadrature dates back to the ancient Greek civilization 2500 years ago. In modern times, Kepler (1615), Torricelli (1664) and Simpson (1743) used the simplest quadrature formulae. Method of approximating quadrature discussed within ways to provide variation of the function either by polynomial or spline.

2.4. Approximation of interpolation [8]

The renowned interpolation formula was developed by Lagrange (1795), in which the method of difference is used. Newton improved its description, and later general interpolation formula was finally developed by Cauchy.

2.5. Approximation of functionals and operators [8]

The most important component in approximation theory is a functional, and one of the simplest functional for the calculation of integrals is the quadrature. It is described as a foundation for the approximation method and involves the replacement of the functions by approximation and the representation of it to take near values like the functional or operator. Application of this method for numerical differentiation can be found in the solutions of differential equations.

3. Generalization of weierstrass' theorem [8]

The evolution of approximation theory comes from a long history of different approaches for the direct theorem and the inverse theorem. The initial and most conventional approach was the concept of Weierstrass' theorem (1885), proven that for any continuous function f on $[a, b]$, then approximation maybe accurately conducted by the algebraic polynomial [8]. To emphasize the basic role of the approximation theory from early history, Bernstein (1912), wrote the following result:
"The discovery of this theorem, remarkable in its generality, determined the future progress of the development of analysis".
Weierstrass' theorem on this method has difficulty in approximating the complex function. In addition, the generalization made by Stone to this theorem was is one of the foundations for many theorems in functional analysis such as Banach algebras theorem. Indeed, the theorem accurately formed the constructive theory of functions according to Bernstein's theorem (1912).
Now, we will adopt the following concepts, which play a fundamental in the next section.

Definition 3.1: *[9] If X is a vector space that has a topology τ, then we say that X is locally convex space if every point has a neighborhood base consisting of convex sets.*

Theorem 3.2: *The Stone – Weierstrass Theorem: [10] Let T be compact convex set in a locally convex space $X(T)$ and $\mathfrak{A} \subset X(T)$, then* $\forall\ \varepsilon > 0, t_1, t_2 \in T;\ t_1 \neq t_2, \exists\ f \in \mathfrak{A}$ and $|f(t_1) - f(t_2)| < \varepsilon$.

Stone presented this theorem on the subject of approximation, where exhausts some meaning of real functions by polynomials, but afterwards Weierstrass' name became widely known regarding this theorem [10].

Definition 3.3: *[11] The uniform norm $\|f\|_{[-1,1]} = \max_{x\in[-1,1]} |f(x)|$, equipped with the space of continuous functions f on $[-1,1]$, which denoted by $C[-1,1]$, and write $\|f\| := \|f\|_{[-1,1]}$.*

Definition 3.4: *[11] Let $E_n(f)$ and $E_n^{(2)}(f)$ be the best polynomial approximation, monotone and convex, respectively, of monotone and convex functions on $[-1,1]$, i.e.,*

$E_n(f) = \inf_{p_n \in \pi_n \cap \Delta^{(k)}} \|f - p_n\|$

and

$E_n^{(2)}(f) = \inf_{p_n \in \pi_n \cap \Delta^{(2)}} \|f - p_n\|,$

such that Δ^k *and* Δ^2 *are a class of* k*-monotone and convex functions on* $[-1,1]$*, respectively,* π_n *is the set of all algebraic polynomials of degree* $\leq n-1$*, and* p_n *algebraic polynomials of degree* $\leq n-1$.

Remark 3.6: *The set* C_ϕ^2 *is denote the space of twice continuous differentiable function on* $[-1,1]$.

Definition 3.7: *[12] The set* $\Delta^2(Y_s)$ *is the collection of all functions* $f \in C[-1,1]$ *that change convexity at the points of the set* Y_s*. The degree of best coconvex polynomial approximation of* f *is defined by*

$E_n^{(2)}(f, Y_s) = \inf_{p_n \in \pi_n \cap \Delta^2(Y_s)} \|f - p_n\|,$

where $Y_s = \{y_i\}_{i=1}^s$ *such that*$y_{s+1} = -1 < y_s < \cdots < y_1 < 1 = y_0$*, and are convex in* $[y_1, y_0]$.

Theorem 3.8: *[11] If* $f \in C_\phi^2 \cap \Delta^2$*, then*

$E_n^{(2)}(f) \leq c\left(n^{-2}\omega_{3,2}^{\phi}\left(f", \frac{1}{n}\right) + n^{-6}\|f"\|_{\left[\frac{-1}{2},\frac{1}{2}\right]}\right), n \geq N,$

where c *and* N *are absolute constants. Hence,*

$E_n^{(2)}(f) \leq cn^{-2}\omega_{3,2}^{\phi}\left(f", \frac{1}{2}\right), n \geq N(f).$

Theorem 3.9: *[13] Let* $f \in C \cap \Delta^2(Y_1)$*, that is, changes convexity once on* $[-1,1]$*. Then*

$E_n^{(2)}(f, Y_1) \leq c\omega_2^{\phi}\left(f, \frac{1}{n}\right), n \geq 1.$

Lemma 3.10: *For every* $\alpha > 0$, $\alpha \neq 2$*. Then, for a given nondecreasing* f *in* $C^r[-1,1]$, $r \geq 1$*, the following inequality is true:*

$E_n(f) \leq c\left(\frac{\sqrt{1-x^2}}{n}\right)^{\alpha}, x \in [-1,1]$*, where* $c = c(\alpha)$.

Proof: From ([14], Corollary 1.3) and ([15], Theorem 5), so the Lemma is proved.

4. Main results

Kopotun, Leviatan and Shevchuk suggested (2009, 2010, 2018) the search for reciprocal connection between the best polynomial approximation and its moduli of smoothness as among way to develop this field (see [15 - 17]. Indeed, convex polynomials have received wide attention over the last thirty years. Much progress was made in recent years by Leviatan (1988), Kopotun and Listopad (1994), DeVore et al. (1996), Leviatan and Shevchuk (2002), Kopotun et al. (2005, 2009, 2010) to name a few (see [18], [14], [19], [13], [11], [16] and [17]). More progress, then embarked in monotone function by Leviatan and Petrova (2017) and Kopotun et al. (2018), have made the road brighter with some interesting applications (see [20] and [15]).

Chebyshev in 1853 was able to develop methods for the approximation properties of functions. In the era of modern, industrial technology highlights are mostly given to approximation works such as Chebyshev that ability to combine the theory into mechanical engineering goals, image processing and artificial limbs, just to mention a few.

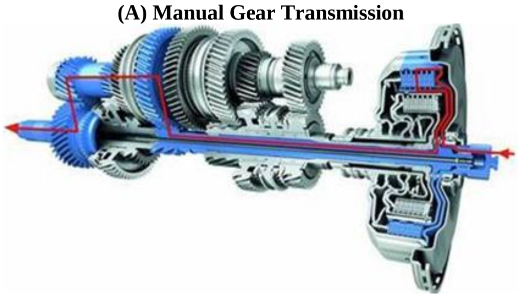
The initial Chebyshev's engagement in approximation theory focused on theory of mechanisms, in particular the application to steam engines in factories. His contribution revealed useful dimensions of parallelogram mechanisms reaching an approximately linear motion of steam engines.

As such kinematic design showed successes in adopting approximation theory in its fundamental, it is natural to assume extension work in this field to other mechanical systems. Thus, we propose the expansion of approximation theory application in the manual and automatic transmission for vehicles.

The world of gears transmission system originates from the definition of basic purpose of gears transmission and their torque between different columns and in accordance with available speed. Fixed transmission ratios usually occur the highest possible efficiency and least possible disturbance. The best convex polynomial of approximation is then derived from those requirements. Finally, we see that the manual and automatic transmission for vehicles are an appropriate application to the best convex approximation as described in these works (see [21 - 23]) and Figure 1, see [24]. We present selected open problems in the following:

Open Problem 1. Can we make use of approximation theory to determine the types of gear transmission systems (manual and automatic) needed by using a mathematical model? Can we determine the best convex polynomial approximation for the available speed of the vehicle? If not, what are the values of $E_n^{(2)}(f)$?

(A) Manual Gear Transmission



(B)Automatic Gear Transmission

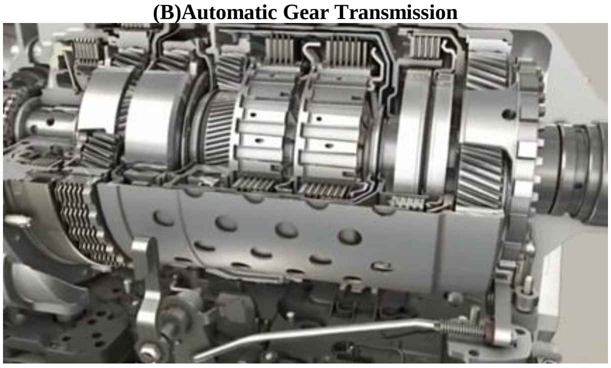


Fig. 1: How to Use (Manual and Automatic) Gear Transmission to Enter the First Gear (C1), the Second Gear (C2) … the Eight Gear (C8), by A New Estimate of Approximation.

Open Problem 2. Study the space of twice continuous differentiable function with the class of all convex functions on $[-1,1]$. In particular, verify whether a result as determine types of gear transmission systems? If not, what are the values of $E_n^{(2)}(f, Y_s)$?

Open Problem 3. Assume that p_n, $q_n \in \Delta^{(2)}$. If $\mathbb{D}_1$ and $\mathbb{D}_2$ are both disjoint and domains of convex polynomials of p_n and q_n respectively. Are p_n and q_n separated by hyperplane polynomial? That is, there exists a polynomial $\mathfrak{u}_n$ and a real number α such that $\mathfrak{u}_n(x) < \alpha < \mathfrak{u}_n(y)$, where $x \in \mathbb{D}_1$ and $y \in \mathbb{D}_2$.

Verify whether it is true that for all p_n and q_n are separated by a closed hyperplane?

Other problem to consider is in biological mechanics. A refractive surgery is the term used to describe surgical procedures that correct common vision problems (e.g. nearsightedness, farsightedness, astigmatism and presbyopia) to reduce dependence on prescription eyeglasses or contact lenses as described in Figure 2, see [25]. This kind of surgery is most popular in the United States. Despite this, many people don't have a good understanding of the anatomy of the eye, how vision works, and health problems that can affect the eye (see [26 - 28]).

Open Problem 4. Can we use k-monotone functions for solving of vision problems of the eye layers when using refractive surgery needed by using a mathematical model? If not, what are the values of $E_n(f)$?

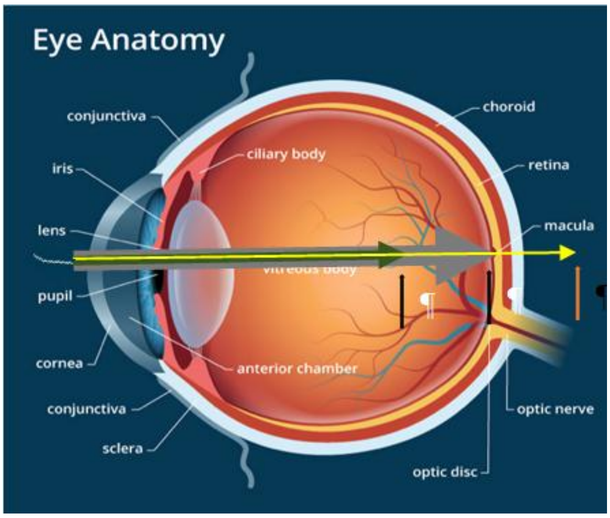


Fig. 2: Human Eye Anatomy with an Explanation for Phases Three Are Normal Vision, Hyperopia and Myopia.

5. Conclusion

The solution to these problems with this survey is a major assistant in PhD thesis of the first author (MALIK) who is still studying New Hybrid Separation and Approximation Theorems Based on Extended Domain of Convex and Coconvex Polynomial at School of Quantitative Sciences, Awang Had Salleh Graduate School of Arts and Sciences, Universiti Utara Malaysia under the supervision of Dr. Masnita Misiran and Prof. Zurni Omar.

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