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Research paper

Deficiency of finite difference methods for capturing shock waves and wave propagation over uneven bottom seabed

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Abstract

The implementation of finite difference method is used to solve shallow water equations under the extreme conditions. The cases such as dam break and wave propagation over uneven bottom seabed are selected to test the ordinary schemes of Lax-Friederichs and Lax-Wendroff numerical schemes. The test cases include the source term for wave propagation and exclude the source term for dam break. The main aim of this paper is to revisit the application of Lax-Friederichs and Lax-Wendroff numerical schemes at simulating dam break and wave propagation over uneven bottom seabed. For the case of the dam break, the two steps of Lax-Friederichs scheme produce non-oscillation numerical results, however, suffering from some of dissipation. Moreover, the two steps of Lax-Wendroff scheme suffers a very bad oscillation. It seems that these numerical schemes cannot solve the problem at discontinuities which leads to oscillation and dissipation. For wave propagation case, those numerical schemes produce inaccurate information of free surface and velocity due to the uneven seabed profile. Therefore, finite difference is unable to model shallow water equations under uneven bottom seabed with high accuracy compared to the analytical solution.

Keywords: *Numerical schemes, dam break, finite difference method, wave propagation, uneven bottom seabed*

1. Introduction

Coastal ocean is strongly affected by human interference and climate change issue as portrayed in the past [1]. Understanding of coastal change through numerical modelling is one of the most effective tools. Reconstructions of historic states, hindcasts and analysis of the dynamics in the last decades, short term forecasts of coastal ocean states, as well as for coastal climate projections and possible future scenarios are possible through numerical modelling implementation [2]. Initially, the implementations of numerical schemes were conducted at the shallow water model without considering the sediment transport and bed level models. Acquiring bed sediment size is crucial for sediment transport prediction [3]. Nevertheless, sediment transport study at coastal areas are differ significantly with the stream channel due to bed material size, directional flows and nature of water level fluctuations [4, 5]. Advanced computer technology has opened up the possibility of modelling the hydrodynamics and sediment transport to carry out the engineering appraisal for many coastal engineering works. One-dimensional models of sediment transport in stream have seen extensive development over the past decades. These models have been frequently used to simulate the cohesive sediment transport and morphological changes in river, tidal channels and the salt wedge formation in estuaries [6]. In [7] developed a one-dimensional sediment transport model, in which the standard shallow water equations were solved for two layers of water depth to provide the flow field. The main aim of this paper is to solve hydrodynamic equations using finite difference model of Lax-Wndroff scheme and Lax-Friederichs scheme. The ability of these schemes to model wave propagation will be compared with analytical results. In [8] successfully demonstrated the use of finite difference element to simulate three-dimensional shallow water flow. The robustness and correctness of Lax-Wendroff and Lax-Friederichs numerical schemes were put into test by looking at the scenario of dam break problem and wave propagation mechanics.

2. Methodology

There are a lot available finite difference schemes in the literature such as artificial dissipation (AD), localized artificial diffusivity (LAD) and weighted essentially non-oscillatory (WENO) schemes. Those are high order schemes as previously described by [9]. The shallow water equations can be solved using the ordinary finite difference methods such as Lax-Friederichs and Lax-Wendroff numerical schemes. The selection of those schemes are based on the stability of Lax-Friederichs and Lax-Wendroff numerical schemes to solve nonlinear problem [10]. The shallow water equations can be written in the conservation form as

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=S \;.... \qquad (1)$$

where

$$U=\begin{bmatrix}u\\uh\end{bmatrix},F=\begin{bmatrix}uh\\u^2+\frac{1}{2}gh^2\end{bmatrix},S=\begin{bmatrix}0\\S_1\end{bmatrix}$$

in which U is a conservation variable, F is a physical flux and S is the source term. All these terms are written in a vector form. The source term usually consists of the bed slope term, friction term and etc.

2.1. One step Lax-Wendroff scheme

Lax-Wendroff scheme is a numerical method for the solution of hyperbolic partial differential equations, based on finite differences [11]. This method was applied in Lagrangian coordinates in one dimension [12]. The one step Lax-Wendroff scheme was applied to tidal wave propagation with uneven bottom test problem. This scheme provides truncation errors of $0(\Delta t^2, \Delta x^2)$, but preserves the dissipative nature of the scheme. The Lax-Wendroff equations start from a Taylor's series in t as

$$U(x, t+\Delta t) = U(x,t) + (\Delta t)\frac{\partial U}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 U}{\partial t^2} + O(\Delta t)^3 \ldots . (2)$$

or in general terms of indices

$$U_i^{n+1} = U_i^n + (\Delta t)\frac{\partial U}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 U}{\partial t^2} + O(\Delta t)^3 \ldots \quad (3)$$

Now, from equation (1), therefore

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right) - S\frac{\partial A}{\partial x} \ldots \quad (4)$$

where the Jacobian $A = \partial \mathrm{F}/\partial U$ is introduced. Substituting the Jacobian A into equation (3) gives

$$U_i^{n+1} = U_i^n - \Delta t\frac{\partial F}{\partial x} + \frac{(\Delta t)^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)..$$

$$+S\Delta t - \frac{S(\Delta t)^2}{2}\frac{\partial A}{\partial x} + o(\Delta t)^3 \ldots . \quad (5)$$

Using central differencing for the spatial derivates and discretised at point *i* (Figure 1), the one-step Lax-Wendroff is obtained as

$$\begin{aligned} U_i^{n+1} = U_i^n &- \frac{1}{2}\lambda(F_{i+1}^n - F_{i-1}^n) \\ &+ \frac{1}{2}\lambda^2\left[\frac{1}{2}(A_{i+1}^n + A_i^n)(F_{i+1}^n - F_i^n) - \frac{1}{2}(A_i^n + A_{i-1}^n)(F_i^n - F_{i-1}^n)\right] \\ &+ \frac{\lambda}{2}\left[S_{i+\frac{1}{2}}^n + S_{i-\frac{1}{2}}^n\right] \\ &- \frac{\lambda^2}{2}\left[\frac{(A_{i+1}^n - A_i^n)}{2}S_{i+\frac{1}{2}}^n - \frac{(A_i^n - A_{i-1}^n)}{2}S_{i-\frac{1}{2}}^n\right].. \quad (6)\end{aligned}$$

The equation (6) was used in hydrodynamic simulations.

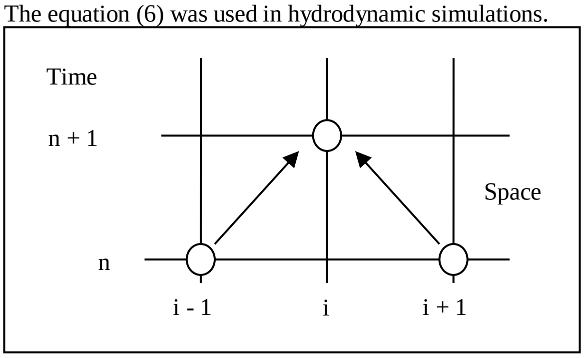

Fig. 1: Grid points for one step discretisation

2.2. Two step Lax-Wendroff scheme

For comparison purpose, the two step Lax Wendroff scheme was also applied to dam break problem. The accuracy of the Lax scheme is improved by interfacing with the leapfrog scheme and become known as two step Lax-Wendroff scheme. The two step Lax-Wendroff applies consists of

Predictor step

$$U_{i+1/2}^{n+1/2} = \frac{1}{2}[(U_{i+1}^n + U_i^n) - \lambda(F_{i+1}^n - F_i^n)] + S \ldots \ldots \quad (7)$$

Corrector step

$$U_i^{n+1} = U_i^n - \lambda\left(F_{i+\frac{1}{2}}^{n+\frac{1}{2}} - F_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right) + S\Delta t \ldots . \quad (8)$$

For the tidal wave propagation, the source term $S_{i+1/2}^n$ was discretized using centralized pointwise approach. It gives

$$S_{i+1/2}^n = \begin{bmatrix} 0 \\ -\frac{g}{2}(h_{i+1}^n - h_{i-1}^n)(B_{i+1}^n - B_{i-1}^n) \end{bmatrix}.. \quad (9)$$

2.3. Two step Lax-Friederichs scheme

The Lax–Friedrichs method is a basic method for the solution of hyperbolic partial differential equations (PDEs) [13]. The so-called two step scheme consists of a predictor step and corrector step. In the first step, a temporary value for the dependent variable was predicted, and then a corrected value was subsequently computed to provide the final value of dependent in the second step. The popularity of this scheme is due to its second-order accuracy and simplicity, although the behaviour around discontinuities is not completely satisfactory.

The Lax-Friederichs first step is obtained by forward differencing in time between time n and $(n+1/2)$ at the midpoint $(i+1/2)$, and forward differencing in space at point i to provide a staggered dual grid at time level $(n+1)$ (Figure 2). The formulation for equation 2 is then gives

$$\frac{U_{i+1/2}^{n+1/2} - U_{i+1/2}^n}{\Delta t/2} = -\frac{F_{i+1}^n - F_i^n}{\Delta x} + S..(10)$$

But $U_{i+1/2}^n = (U_{i+1}^n + U_i^n)/2$, so that the Lax-Friedrerichs first step is obtained as

Predictor step

$$U_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}[(U_{i+1}^n - U_i^n) - \lambda(F_{i+1}^n - F_i^n)] + S\frac{\Delta t}{2} \ldots \quad ..(11)$$

The scheme then uses the first step as a predictor to obtain the fluxes centred at time $(n+1)$, thus

Corrector step

$$U_i^{n+1} = \frac{1}{2}\left[\left(U_{i+1/2}^{n+1/2} + U_{i-1/2}^{n+1/2}\right) - \lambda\left(F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}\right)\right] + S\Delta t \ldots . \quad (12)$$

This scheme produces a second-order accurate scheme using half step. The equation 11 and 12 were used in hydrodynamic simulations.

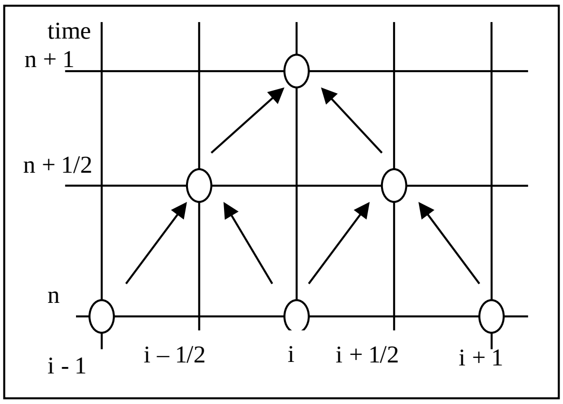

Fig. 2: Grid points for two step discretization

2.4. Verification of the finite difference schemes

The numerical schemes describe above were tested with dam break problem [14] and tidal wave propagation [15].

2.4.1. Dam break problem

For this test problem, the bed was taken as constant in depth and friction was ignored, thus the source term is zero. The test problem consists of 1D channel of length 1 m with walls at either end (Figure 3). The initial velocity is 0 and a barrier is present at which is removed at so the initial conditions are

$u(x,0) = 0$ and

$$h(x) = \begin{cases} h_L & \text{if } 0 \le x \le 0.5 \\ h_R & \text{if } 0.5 < x \le 1 \end{cases} \quad \text{............(13)}$$

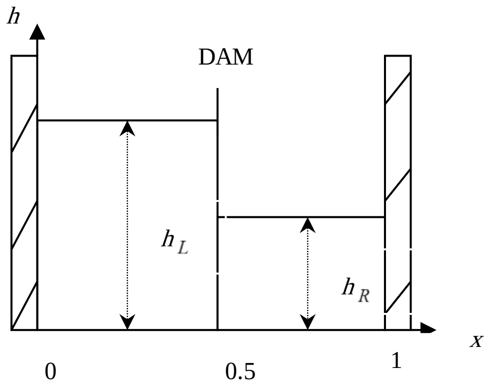

Fig. 3: Initial conditions for the dam break problem

In [14] derived an analytical solution of the dam break problem which can be used to check the accuracy of the numerical schemes. The exact solution is

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0.5 - t\sqrt{gh_L} \\ \frac{1}{3}(2(x + t\sqrt{gh_L}) - 1) & \text{if } 0.5 - t\sqrt{gh_L} \le x \le (u_2 - c_2)t + 0.5 \\ u_2 & \text{if } (u_2 - c_2)t + 0.5 < x \le St + 0.5 \\ 0 & \text{if } x > St + 0.5 \end{cases}$$

......................14

and

$$h(x,t) = \begin{cases} h_L & \text{if } x < 0.5 - t\sqrt{gh_L} \\ \frac{1}{9g}\left(2\sqrt{gh_L} - \frac{1}{2t}(2x-1)\right)^2 & \text{if } 0.5 - t\sqrt{gh_L} \le x \le (u_2 - c_2)t + 0.5 \\ \frac{h_R}{2}\left(\sqrt{1 + \frac{8S^2}{gh_R}} - 1\right) & \text{if } (u_2 - c_2)t + 0.5 < x \le St + 0.5 \\ h_R & \text{if } x > St + 0.5 \end{cases}$$

................15

where

$$u_2 = S - \frac{gh_R}{4S}\left(1 + \sqrt{1 + \frac{8S^2}{gh_R}}\right)$$

and

$$c_2 = \sqrt{\frac{ghR}{2}\left(\sqrt{1 + \frac{8S^2}{gh_R}} - 1\right)}$$

The bore speed, S is the positive root of

$$u_2 + 2c_2 - 2\sqrt{gh_L} = 0 \text{.........16}$$

The exact solution with $h_L = 1\,m$ and $h_R = 0.5\,m$ is illustrated in Figure 4, where the bore speed is approximately $S = 2.958$. The exact solution is only valid until either the bore or the rarefraction wave hits the wall.

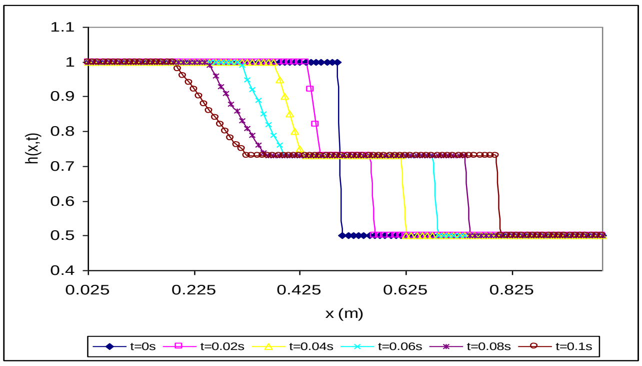

Fig. 4: Exact solution of the dam break test problem for t = 0 to 0.1 s.

2.4.2. Tidal wave propagation with uneven bottom test problem

In [15] discussed a tidal wave propagation test problem that consists of the initial and boundary conditions as (Figure 5).

$$h(x,0) = 60.5 - B(x),$$
$$u(x,0) = 0,$$
$$h(0,t) = \begin{cases} 64.5 + 4\sin\left[\pi\left(\frac{t}{10800} - \frac{1}{2}\right)\right] & \text{, if } t \le 43{,}200s \\ 60.5 & \text{if } t > 43200s \end{cases}$$
$$u(L,t) = 0$$

...........(17)

and bathymetry

$$B(x) = \frac{40x}{L} + 10\left(1 - 10\sin\left[\pi\left(\frac{4x}{L} + \frac{1}{2}\right)\right]\right) \text{.....(18)}$$

where L = 648,000 m is the channel length. Under these conditions, the tidal wave of 4 m amplitude entering the region at the upstream boundary (Figure 6). The tidal wave reaches a full height of 8 m at t = 21,600s and since the wave propagates at approximately with $\sqrt{gh}$ then at t = 10,800 s, the tidal wave could has

only reached as far as x = 216,000 m for L = 648,000 m. In this test problem, the bottom stress was not taken into account.
Figure 7 and Figure 8 show the surface elevation (h) and velocity (u) after t = 10,800 s using one step Lax-Wendroff respectively.

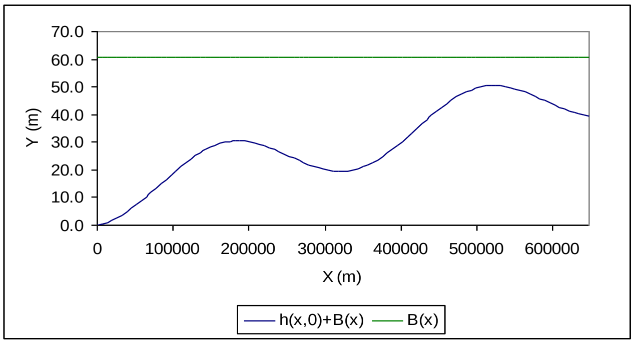

Fig. 5: Initial condition of the tidal wave propagation test problem

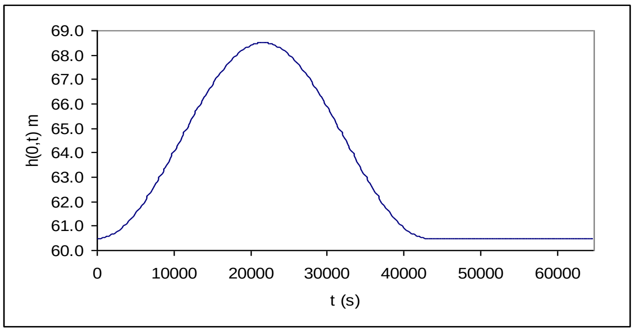

Fig. 6: Upstream boundary condition of the tidal wave propagation test problem.

3. Results and Discussion

The Lax-Friederichs and Lax-Wendroff numerical schemes have been tested using the shallow water equations. The test cases include the source term for wave propagation and without the source term for dam break. For the case of dam break, the two step of Lax-Friederichs scheme produces non oscillation numerical results, but suffer with some of bad dissipation output. On contrary Lax-Wendroff scheme produces a very bad oscillation. It seems that both schemes cannot solve the problem at discontinuities which finally leads to extreme oscillation and dissipation. These output are well sgree with the finding of [16]. They indicate that these methods have some drawbacks albeit the two step Lax-Wendroff scheme is second order numerical scheme, but bad oscillatory is quite evident. Same observation goes to the two step Lax-Friederichs scheme which is non-oscillatory but also excessively diffusive.

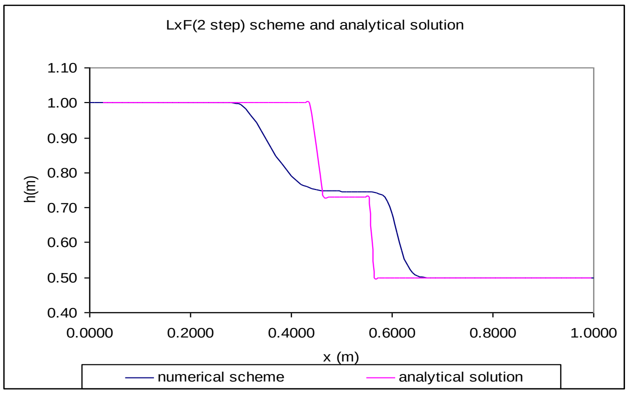

Fig. 7: Two step Lax Friederichs Numerical scheme and analytical solution of the dam break test problem for t = 0.02 s

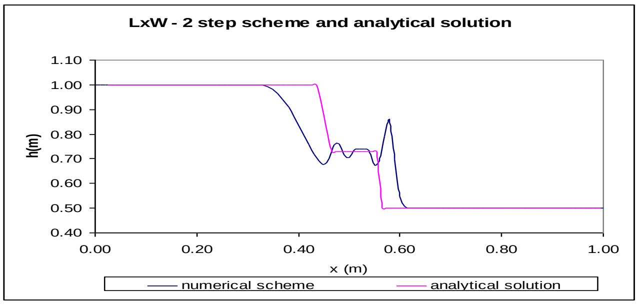

Fig. 8: Two step Lax Wendroff and analytical solution of the dam break test problem for $t = 0.02s$.

For tidal propagation case, the significant source term exists due to the uneven bathymetries. Figure 9 and Figure 10 show the surface elevation (h) and velocity (u) after t = 10,800 s using one step Lax-Wendroff respectively. The one step of Lax-Wendroff numerical scheme gives wrong information on the free surface and velocity profiles. Suppose the water elevation will not enter the distance at an approximate 216,000 m from the seaward but the result shows there is oscillation beyond this point. In [17] suggested that a few approaches can be adopted such flux or slope limiter associate with the scheme that can improves the oscillation problem. Although some improvement can be obtained, but yet the oscillation issues cannot be solved completely.

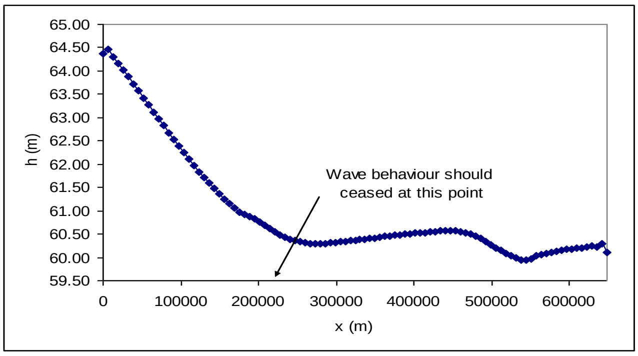

Fig. 9: Numerical result of one step Lax-Wendroff for tidal propagation at t = 10800s

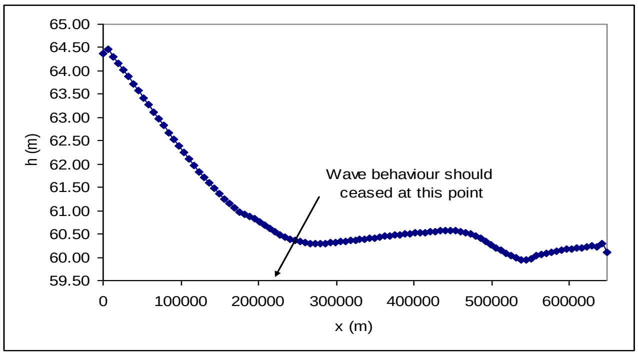

Fig. 10: Numerical result of one step Lax-Wendroff for tidal propagation at t = 10800s

4. Conclusion

The Lax-Friederichs and Lax-Wendroff numerical schemes have been tested using the shallow water equations. The test cases include the source term for wave propagation and exclude the source term for dam break. For the case of dam break, the two step of Lax-Friederichs scheme produces non oscillation numerical results, however, suffering from a dissipation. Unlike the Lax-Friederichs scheme, two step of Lax-Wendroff produces a very bad oscillation. It seems that these numerical schemes cannot solve the problem at discontinuities which finally leads to very bad oscillation and dissipation. The future trends for coastal modelling should include higher-order numerical schemes and integration into growing complexity using finite volume schemes.

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