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Research paper

A New Approach to Common Coupled Fixed Point of Caristi Type Contraction on a Metric Space Endowed with a Graph

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Abstract

In this paper we introduced a new notation G – fg – contraction of Caristi type and a new edge preserving property. With help of these we proved a some coupled fixed point results for four maps endowed with a graph in a complete metric space. Also we gave an application to integral equations.

Keywords*: Metric spaces with a graph, edge preserving property, coupled fixed point.*

1. Introduction

Banach Introduced the concept of fixed point theory in 1922([10]), it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is a modified $\mathcal{E}$ -variation principle of Ekeland([12]). In 1976, Caristi proved the following famous fixed point theorem.

Theorem 1.1

[11] Let (X, d) be complete metric space and $f: X \to R$ be lower semi continuous function and bounded below function. A mapping $T: X \to X$ is said to be Caristi type map on X dominated by f if T satisfies $d(x, Tx) \leq f(x) - f(Tx)$ for each $x \in X$. Then T has a fixed point.

It is well-known that the Caristi's fixed point theorem is one of the most valuable generalization of the Banach contraction principle. The concepts of fixed point theory and graph theory were combined by Espinola and Kirk ([4]). Jachymski([5]) came up with an interesting idea of using the language of graph theory in the study of fixed point results.

A graph is an ordered pair $G = (V, E)$, where V is a non empty set and the elements in V are called vertices or nodes and E is a binary relation on V. i.e., $E \subseteq (V \times \tau)$. The elements of E are called edges.

In this paper we concentrate on directed graphs.

Let G^{-1} be the conversion of the graph G. i.e., the graph obtained from G by reversing the direction of edges.

Simply, $E(G^{-1}) = \{(\gamma, \chi): (\chi, \gamma) \in E(G)\}$.

A directed graph G is called a oriented graph if $(\chi, \gamma) \in E(G)$, then $(\gamma, \chi) \notin E(G)$.

Bhaskar and Lakshmikantham ([13]) introduced the coupled fixed points concept, later several authors proved some coupled fixed point theorems in partial metric spaces (see [8, 9]).

In the same line, our aim is to extend some coupled coincidence and coupled common fixed point theorems for nonlinear contractions complete metric spaces endowed with a directed graph. Our results would bring about a more unified approach to the presentation of coupled coincidence and coupled common fixed point theorems for four maps. Also, as an application of our results, we aim to prove a theorem which can be used to test the existence of a solution for some particular integral equations.

Definition 1.2

[2] Let $F: X \times X \to X$. If $F(\chi, \gamma) = \chi$ and $F(\gamma, \chi) = \gamma$. for $(\chi, \gamma) \in X \times X$ then (χ, γ) is said to be a coupled fixed point of F.

Definition 1.3

[1] An element $(\chi, \gamma) \in X \times X$ is called

1. a coupled coincident point (CCoin) of mappings $F: X \times X \to X$ and $f: X \to X$ if $f\chi = F(\chi, \gamma)$ and $f\gamma = F(\gamma, \chi)$.
2. a common coupled fixed point(CCFP) of mappings $F: X \times X \to X$ and $f: X \to X$ if $\chi = f\chi = F(\chi, \gamma)$ and $\gamma = f\gamma = F(\gamma, \chi)$.

Definition 1.4

[7] Two functions $F: X \times X \to X$ and $f: X \to X$ are said to be commutative on a non empty set X if $f(F(\chi, \gamma)) = F(f\chi, f\gamma)$ and $f(F(\gamma, \chi)) = F(f\gamma, f\chi)$.

Definition 1.5

[5] A function $f: X \to X$ is said to be $G-$ continuous if

(a) $(\chi_{n_i}) \to p$ and $(\chi_{n_i}, \chi_{n_{i+1}}) \in E(G)$ implies $f(\chi_{n_i}) \to fp$, where $\chi, p \in X$
and $(n_i)_{i\in N}$ be a sequence of positive integers.

(b) $(\gamma_{n_i}) \to q$ and $(\gamma_{n_i}, \gamma_{n_{i+1}}) \in E(G^{-1})$ impels $f(\gamma_{n_i}) \to fq$, where $\gamma, q \in X$
and $(n_i)_{i\in N}$ be a sequence of positive integers.

Definition 1.6

[3] A function $S: X \times X \to X$ is said to be $G-$ continuous if $\{\chi_{n_i}\} \to p$, $\{\gamma_{n_i}\} \to q$ and $(\chi_{n_i}, \chi_{n_{i+1}}) \in E(G)$, $(\gamma_{n_i}, \gamma_{n_{i+1}}) \in E(G^{-1})$ implies $S(\chi_{n_i}, \chi_{n_{i+1}}) \to S(p, q)$ and $S(\gamma_{n_i}, \gamma_{n_{i+1}}) \to S(q, p)$ as $i \to \infty$, where $(\chi, \gamma), (p, q) \in X \times X$ and $(n_i)_{i\in N}$ be a sequence of positive integers.

Definition 1.7

[3] Let (X, d) be a complete metric space endowed with a directed graph G. Then the triplet (X, d, G) has property (A) if

(i) for any sequence $\{\chi_n\}_n \in N$ in X such that $\{\chi_n\} \to p$ and $(\chi_n, \chi_{n+1}) \in E(G)$ implies $(\chi_n, p) \in E(G)$

(ii) for any sequence $\{\gamma_n\}_n \in N$ in X such that $\{\gamma_n\} \to q$ and $(\gamma_n, \gamma_{n+1}) \in E(G^{-1})$ implies $(\gamma_n, q) \in E(G^{-1})$

Lemma 1.8 [6] Let $\lhd$ be a reflexive relation on a nonempty set M and $\phi: M \to R$ a function bounded from below, then $\chi \lhd \gamma$ and $\chi \neq \gamma$; then $\phi(\chi) > \phi(\gamma)$.

Throughout this paper, we assume that $\kappa: [0,\infty) \to [0,\infty)$ is an upper semi continuous function. Now we prove our main results.

2. Main Results

Definition 2.1

Suppose (X, d) be a metric space endowed with a directed graph G. Let us consider the mappings $S, T: X \times X \to X$ and $f, g: X \to X$ with defining the following sets

(I)

$(X \times X)_{Sf} = \{(\chi, \gamma) \in X \times X: (f\chi, S(\chi, \gamma)) \in E(G), (f\gamma, S(\gamma, \chi)) \in E(G^{-1})\}$ and (i) f is edge preserving. i.e., $(f\chi, f\xi) \in E(G), (f\gamma, f\tau) \in E(G^{-1})$

implies $(f(f\chi), f(f\xi)) \in E(G)$ and $(f(f\gamma), f(f\tau)) \in E(G^{-1})$.

(ii) S is f edge preserving. i.e., $(f\chi, f\xi) \in E(G), (f\gamma, f\tau) \in E(G^{-1})$ implies $(S(\chi, \gamma), S(\xi, \tau)) \in E(G)$ and $(S(\gamma, \chi), S(\tau, \xi)) \in E(G^{-1})$.

(II)

$(X \times X)_{Tg} = \{(\xi, \tau) \in X \times X: \big(g\xi, T(\xi, \tau)\big) \in E(G), \big(g\tau, T(\tau, \xi)\big) \in E(G^{-1})\}$ and

(i) g is edge preserving. i.e., $(g\chi, g\xi) \in E(G), (g\gamma, g\tau) \in E(G^{-1})$ implies $(g(g\chi), g(g\xi)) \in E(G)$ and $(g(g\gamma), g(g\tau)) \in E(G^{-1})$.

(ii) T is g edge preserving. i.e., $(g\chi, g\xi) \in E(G), (g\gamma, g\tau) \in E(G^{-1})$ implies $(T(\chi, \gamma), T(\xi, \tau)) \in E(G)$ and $(T(\gamma, \chi), T(\tau, \xi)) \in E(G^{-1})$.

(III) $(X \times X)_{ST}^{fg} = (X \times X)_{Sf} \cap (X \times X)_{Tg}$

ST are said to be $G - fg$ contraction if

(i) f, g are edge preserving respectively.

i.e., $(f\chi, g\xi) \in E(G), (f\gamma, g\tau) \in E(G^{-1})$

implies $(f(f\chi), g(g\xi)) \in E(G)$ and $(f(f\gamma), g(g\tau)) \in E(G^{-1})$.

(ii) S, T are fg −edge preserving.

i.e., $(f\chi, g\xi) \in E(G), (f\gamma, g\tau) \in E(G^{-1})$

implies $(S(\chi, \gamma), T(\xi, \tau)) \in E(G)$ and

$(S(\gamma, \chi), T(\tau, \xi)) \in E(G^{-1})$

(iii) for all $x, y, u, v \in X$ and for

$(f\chi, g\xi), (S(\chi, \gamma), T(\xi, \tau)) \in E(G)$

and $(f\gamma, g\tau), (S(\gamma, \chi), T(\tau, \xi)) \in E(G^{-1})$

Define

$$d\big(S(\chi, \gamma), T(\xi, \tau)\big) \le \max\Big\{\kappa\big(\psi(f\chi, g\xi)\big), \kappa\Big(\psi\big(S(\chi, \gamma), T(\xi, \tau)\big)\Big)\Big\} \big[\psi(f\chi, g\xi) - \psi\big(S(\chi, \gamma), T(\xi, \tau)\big)\big] - \max\Big\{\kappa\big(\phi(f\gamma, g\tau)\big), \kappa\Big(\phi\big(S(\gamma, \chi), T(\tau, \xi)\big)\Big)\Big\} [\phi(f\gamma, g\tau) - \psi(S(\gamma, \chi), T(\tau, \xi))]$$

where $\psi, \phi: X \times X \to [0,\infty)$ are lower semi continuous functions.

Theorem 2.2:

Let $S, T: X \times X \to X$ and $f, g: X \to X$. Suppose that S, T are fg-edge preserving and satisfies $G - fg$ contraction.

Let $S(X \times X) \subseteq f(X)$ and $T(X \times X) \subseteq g(X)$. Also let $\{\chi_{2n}\}, \{\gamma_{2n}\}, \{\xi_{2n}\}$ and $\{\tau_{2n}\}$ be sequences in the metric space (X, d) endowed with a directed graph G. Then the following statements are true.

(i) $(f\chi, g\xi) \in E(G)$ and $(f\gamma, g\tau) \in E(G^{-1})$ implies $(S(\chi_{2n}, \gamma_{2n}), T(\xi_{2n+1}, \tau_{2n+1})) \in E(G)$ and $(S(\gamma_{2n}, \chi_{2n}), T(\tau_{2n+1}, \xi_{2n+1})) \in E(G^{-1})$, $\forall n \in N$;

(ii) $(\chi, \gamma) \in (X \times X)_{ST}^{fg} \Rightarrow (\chi_{2n+1}, \gamma_{2n+1}) \in (X \times X)_{ST}^{fg}, \forall n \in N$;

(iii) $\{\Omega_{2n}\}$ and $\{\eta_{2n}\}$ are cauchy sequences and there exists $\chi^*, \gamma^* \in X$ such that $\Omega_{2n} \to \chi^*$ and $\eta_{2n} \to \gamma^*$.

Proof:

We have $S(X \times X) \subseteq f(X)$ and $T(X \times X) \subseteq g(X)$ so let us define the following sequences

$\Omega_{2n} = f\chi_{2n+1} = S(\chi_{2n}, \gamma_{2n}), \eta_{2n} = f\gamma_{2n+1} = S(\gamma_{2n}, \chi_{2n})$,

$\Omega_{2n+1} = g\chi_{2n+2} = T(\chi_{2n+1}, \gamma_{2n+1})$,

$\eta_{2n+1} = g\gamma_{2n+2} = T(\gamma_{2n+1}, \chi_{2n+1})$, for $n = 0,1,2, \cdots$

Now for $n = 0$ we have $(\chi_{2n}, \gamma_{2n}) = (\chi_0, \gamma_0)$, $(\chi_{2n+1}, \gamma_{2n+1}) = (\chi_1, \gamma_1)$

So let (i) Let $(f\chi, g\xi) \in E(G)$ and $(f\gamma, g\tau) \in E(G^{-1})$

By the $G - fg$ edge preserving property of ST we have

$(S(\chi, \gamma), T(\xi, \tau)) \in E(G)$ and $(S(\gamma, \chi), T(\tau, \xi)) \in E(G^{-1})$

$\Rightarrow (S(\chi_0, \gamma_0), T(\xi_1, \tau_1)) \in E(G)$ and

$(S(\gamma_0, \chi_0), T(\tau_1, \xi_1)) \in E(G^{-1})$

$\Rightarrow (f\chi_1, g\xi_2) \in E(G)$ and $(f\gamma_1, g\tau_2) \in E(G^{-1})$

Then by edge preserving property of S, T ,

$(S(\chi_1, \gamma_1), T(\xi_2, \tau_2)) \in E(G)$ and $(S(\gamma_1, \chi_1), T(\tau_2, \xi_2)) \in E(G^{-1})$

$\Rightarrow (f\xi_2, g\chi_3) \in E(G)$ and $(f\tau_2, g\gamma_3) \in E(G^{-1})$

by repeating the above process, we have

$(S(\chi_{2n}, \gamma_{2n}), T(\xi_{2n+1}, \tau_{2n+1})) \in E(G)$ and

$(S(\gamma_{2n+1}, \chi_{2n+1}), T(\tau_{2n+1}, \xi_{2n+1})) \in E(G^{-1})$.

(ii) Let $(\chi, \gamma) \in (X \times X)_{ST}^{fg}$

$\Rightarrow (\chi, \gamma) \in (X \times X)_{Sf} \cap (X \times X)_{Tg} \Rightarrow (f\chi, S(\chi, \gamma)) \in E(G)$, $(f\gamma, S(\gamma, \chi)) \in E(G^{-1})$ and

$(g\chi, S(\chi, \gamma)) \in E(G)$, $(f\gamma, S(\gamma, \chi)) \in E(G^{-1})$

$\Rightarrow (f\chi_0, S(\chi_0, \gamma_0)) \in E(G)$, $(f\gamma_0, S(\gamma_0, \chi_0)) \in E(G^{-1})$

and

$(g\chi_1, T(\chi_1, \gamma_1)) \in E(G)$, $(g\gamma_1, T(\gamma_1, \chi_1)) \in E(G^{-1})$

$\Rightarrow (f\chi_0, f\chi_1) \in E(G)$ and $(f\gamma_0, f\gamma_1) \in E(G^{-1})$ and

$(g\chi_1, g\chi_2) \in E(G)$ and $(g\gamma_1, g\gamma_2) \in E(G^{-1})$

Since S is f edge preserving and T is g edge prerving so we have,

$\Rightarrow (S(\chi_0, \gamma_0), S(\chi_1, \gamma_1)) \in E(G), (S(\gamma_0, \chi_0), S(\gamma_1, \chi_1)) \in E(G^{-1})$

and

$(T(\chi_1, \gamma_1), T(\chi_2, \gamma_2)) \in E(G), (T(\gamma_1, \chi_1), T(\gamma_2, \chi_2)) \in E(G^{-1})$ by using edge preserving property repeatedly, we get

$(S(\chi_{2n}, \gamma_{2n}), S(\chi_{2n+1}, \gamma_{2n+1})) \in E(G), (S(\gamma_{2n}, \chi_{2n}), S(\gamma_{2n+1}, \chi_{2n+1})) \in E(G^{-1})$ and

$(T(\chi_{2n+1}, \gamma_{2n+1}), T(\chi_{2n+1}, \gamma_{2n+1})) \in E(G), (T(\gamma_{2n+1}, \chi_{2n+1}), T(\gamma_{2n+1}, \chi_{2n+1})) \in E(G^{-1})$ $\forall n \in N$.

$\Rightarrow (f\chi_{2n+1}, S(\chi_{2n+1}, \gamma_{2n+1})) \in E(G)$

and $(f\gamma_{2n+1}, S(\gamma_{2n+1}, \chi_{2n+1})) \in E(G^{-1})$ and

$(g\chi_{2n+2}, T(\chi_{2n+2}, \gamma_{2n+2})) \in E(G)$

and $(g\gamma_{2n+2}, T(\gamma_{2n+2}, \chi_{2n+2})) \in E(G^{-1})$

$\Rightarrow (\chi_{2n+1}, \gamma_{2n+1}) \in (X \times X)_{ST}^{fg}$.

(iii) Define a relation $\lhd$ on X as follows: $S(\chi, \gamma) \lhd T(\xi, \tau) \Leftrightarrow S, T$ satisfies $G - fg$ contraction. Then clearly $\lhd$ is a reflexive relation on X.

Let $(\chi, \gamma) \in (X \times X)_{ST}^{fg}$.Now from condition (ii) of Theorem 2.2, we have

$\Rightarrow (\Omega_{2n}, \Omega_{2n+1}) \in E(G)$ and $(\eta_{2n}, \eta_{2n+1}) \in E(G^{-1})$.

Now By $G - fg$ contraction and for $\Omega_{2n} \neq \Omega_{2n+1}$

$$\begin{aligned} 0 < \ & d(\Omega_{2n}, \Omega_{2n+1}) \\ = \ & d\big(S(\chi_{2n}, \gamma_{2n}), T(\chi_{2n+1}, \gamma_{2n+1})\big) \\ \le \ & \max\left\{\begin{matrix}\kappa(\psi(\Omega_{2n-1}, \Omega_{2n}), \\ \kappa(\psi(\Omega_{2n}, \Omega_{2n+1})\end{matrix}\right\} [\psi(\Omega_{2n-1}, \Omega_{2n}) - \psi(\Omega_{2n}, \Omega_{2n+1})] \\ & -\max\left\{\begin{matrix}\kappa(\phi(\eta_{2n-1}, \eta_{2n}), \\ \kappa(\phi(\eta_{2n}, \eta_{2n+1})\end{matrix}\right\} [\psi(\eta_{2n-1}, \eta_{2n}) - \psi(\eta_{2n}, \eta_{2n+1})] \\ \le \ & \max\left\{\begin{matrix}\kappa(\psi(\Omega_{2n-1}, \Omega_{2n}), \\ \kappa(\psi(\Omega_{2n}, \Omega_{2n+1})\end{matrix}\right\} [\psi(\Omega_{2n-1}, \Omega_{2n}) - \psi(\Omega_{2n}, \Omega_{2n+1})]. \end{aligned}$$

Similarly,

$d(\eta_{2n}, \eta_{2n+1})$

$\leq \max\left\{\begin{matrix}\kappa(\psi(\eta_{2n-1},\eta_{2n}),\\ \kappa(\psi(\eta_{2n},\eta_{2n+1})\end{matrix}\right\}[\psi(\eta_{2n-1},\eta_{2n})-\psi(\eta_{2n},\eta_{2n+1})].$

Since $\Omega_{2n} \lhd \Omega_{2n+1}, \eta_{2n} \lhd \eta_{2n+1}$ and $\Omega_{2n} \neq \Omega_{2n+1}, \eta_{2n} \neq \eta_{2n+1}$ for $n = 1,2,3\cdots$ so from Lemma 1.8 we have $\{\psi(\Omega_{2n},\Omega_{2n+1})\}$ and $\{\psi(\eta_{2n},\eta_{2n+1})\}$ are non increasing.
Let $\lim_{n\to\infty}\{\psi(\Omega_{2n},\Omega_{2n+1})\} = \lambda, \lim_{n\to\infty}\{\psi(\eta_{2n},\eta_{2n+1})\} = \mu$ for some $\lambda, \mu \geq 0$.
Since κ is upper semi continuous function so we have
$\lim_{n\to\infty}\sup\kappa(\psi(\Omega_{2n},\Omega_{2n+1})) = \kappa(\lambda)$
and
$\lim_{n\to\infty}\sup\kappa(\psi(\eta_{2n},\eta_{2n+1})) = \kappa(\mu)$.
So for any $m \in N$ with $n \geq n_0$ we have $\lim_{n\to\infty}\kappa(\psi(\Omega_{2n},\Omega_{2n+1})) = \kappa(\lambda) + 1$
and
$\lim_{n\to\infty}\kappa(\psi(\eta_{2n},\eta_{2n+1})) = \kappa(\mu) + 1$.
Therefore

$$d(\Omega_{2n},\Omega_{2n+1}) \quad \leq (\kappa(\lambda)+1)[\psi(\Omega_{2n-1},\Omega_{2n}) - \psi(\Omega_{2n},\Omega_{2n+1})]$$

Now for $m > n$, we have

$$\begin{aligned} d(\Omega_{2n},\Omega_{2m+1}) &= d(\Omega_{2n},\Omega_{2n+1}) + d(\Omega_{2n+1},\Omega_{2n+2}) + \cdots + d(\Omega_{2m},\Omega_{2m+1}) \\ &\leq (\kappa(\lambda)+1)[\psi(\Omega_{2n-1},\Omega_{2n}) - \psi(\Omega_{2n},\Omega_{2n+1})] \\ &\quad + (\kappa(\lambda)+1)[\psi(\Omega_{2n},\Omega_{2n+1}) - \psi(\Omega_{2n+1},\Omega_{2n+2})] \\ &\quad + \cdots \\ &\quad + (\kappa(\lambda)+1)[\psi(\Omega_{2m-1},\Omega_{2m}) - \psi(\Omega_{2m},\Omega_{2m+1})] \\ &\leq (\kappa(\lambda)+1)[\psi(\Omega_{2n-1},\Omega_{2n}) - \psi(\Omega_{2m},\Omega_{2m+1})]. \end{aligned}$$

as $m, n \to \infty$, $d(\Omega_{2n},\Omega_{2m+1}) \to 0$.
This shows that $\{\Omega_{2n}\}$ is a Cauchy sequence.
Similarly we can prove that $\{\eta_{2n}\}$ is a Cauchy sequence.
Since (X, d) is a complete so there exists $\chi^*, \gamma^* \in X$ such that $\Omega_{2n} \to \chi^*$ and $\eta_{2n} \to \gamma^*$.
Therefore $\lim_{n\to\infty}\Omega_{2n} = \chi^*$ and $\lim_{n\to\infty}\eta_{2n} = \gamma^*$

Theorem 2.3:

In addition to Theorem 2.2, assume that f, g are G continuous and (i) f commutes with S and g commutes with T [or] (ii) (X, d, G) has the property (A)
Then $CCoin(^{fg}_{ST}) \neq \phi$ iff $(X \times X)^{fg}_{ST} \neq \phi$.

Proof:

Case(i): Let f commutes with S and g commutes with T.
Suppose $CCoin(^{fg}_{ST}) \neq \phi$,
Then there exists $(\xi,\tau) \in CCoin(^{fg}_{ST})$
$\Rightarrow (\xi,\tau) \in CCoin(Sf) \cap CCoin(Tg)$
i.e., $f\xi = S(\xi,\tau)$, $f\tau = S(\tau,\xi)$ and $g\xi = T(\xi,\tau)$, $g\tau = T(\tau,\xi)$.
So $(f\xi, S(\xi,\tau)) = (f\xi, f\xi) \in E(G)$ and $(f\tau, S(\tau,\xi)) = (f\tau, f\tau) \in E(G^{-1})$
and $(g\xi, T(\xi,\tau)) = (g\xi, g\xi) \in E(G)$ and
$(g\tau, T(\tau,\xi)) = (g\tau, g\tau) \in E(G^{-1})$
$\Rightarrow (\xi,\tau) \in (X \times X)^{fg}_{ST}$
$\Rightarrow (X \times X)^{fg}_{ST} \neq \phi$.
Conversly, suppose that $(X \times X)^{fg}_{ST} \neq \phi$.
Then there exists some $(\chi_0,\gamma_0) \in (X \times X)^{fg}_{ST}$
so we have $(f\chi_0, S(\chi_0,\gamma_0)) \in E(G)$ and $(f\gamma_0, S(\gamma_0,\chi_0)) \in E(G^{-1})$
and $(g\chi_1, T(\chi_1,\gamma_1)) \in E(G)$ and $(g\gamma_1, T(\gamma_1,\chi_1)) \in E(G^{-1})$
Then by Theorem 2.2, condition (ii) and (iii), there exists a sequence $\{n_i\}_{i\in N}$ of positive integers such that
$\lim_{n\to\infty}S(\chi_{2n_i},\gamma_{2n_i}) \to \chi^*$, $\lim_{n\to\infty}S(\gamma_{2n_i},\chi_{2n_i}) \to \gamma^*$ and
$\lim_{n\to\infty}T(\chi_{2n_{i+1}},\gamma_{2n_{i+1}}) \to \chi^*$, $\lim_{n\to\infty}T(\gamma_{2n_{i+1}},\chi_{2n_{i+1}}) \to \gamma^*$
Since f is G continuous, by the definition we have
$\lim_{n\to\infty}f(S(\chi_{2n_i},\gamma_{2n_i})) \to f\chi*$.
Since (S, f) are commute so we have
$f(S(\chi_{2n_i},\gamma_{2n_i})) = S(f\chi_{2n_i}, f\gamma_{2n_i})$
and $f(S(\gamma_{2n_i},\chi_{2n_i})) = S(f\gamma_{2n_i}, fx_{2n_i})$.
Now

$$\lim_{n\to\infty}f(S(\chi_{2n_i},\gamma_{2n_i})) = S\lim_{n\to\infty}(f\chi_{2n_i}, f\gamma_{2n_i})$$
$$\Rightarrow f\chi^* = S(\chi^*,\gamma^*)$$

Similarly

$$\lim_{n\to\infty}f(S(\gamma_{2n_i},\chi_{2n_i})) = S\lim_{n\to\infty}(f\gamma_{2n_i}, fx_{2n_i})$$
$$\Rightarrow f\gamma^* = S(\gamma^*,\chi^*)$$

This shows that $(\chi^*,\gamma^*) \in CCoin(Sf)$.
Since g is G continuos, (T, g) are commute, by the above process we get $(\chi^*,\gamma^*) \in CCoin(Tg)$.
Therefore $CCoin(^{fg}_{ST}) \neq \phi$.
Case(ii): Let (X, d, G) satisfies property (A).
Since, as $i \to \infty$, $\Omega_{2n_{i+1}} = T(\chi_{2n_{i+1}},\gamma_{2n_{i+1}}) \to \chi^*$ and
$\eta_{2n_{i+1}} = T(\gamma_{2n_{i+1}},\chi_{2n_{i+1}}) \to \gamma^*$ as $i \to \infty$.
Also $(S(\chi_{2n_i},\gamma_{2n_i}), T(\chi_{2n_{i+1}},\gamma_{2n_{i+1}})) \in E(G)$ and
$(S(\gamma_{2n_i},\chi_{2n_i}), T(\gamma_{2ni+1},\chi_{2ni+1})) \in E(G^{-1})$
so by property (A) $(S(\chi_{2n_i},\gamma_{2n_i}),\chi^*) \in E(G)$ and $(S(\gamma_{2n_i},\chi_{2n_i}),\gamma^*) \in E(G^{-1})$.
Now as $i \to \infty$

$$\begin{aligned} &d(f\chi^*, S(\chi^*,\gamma^*)) \\ &= d(f\chi^*, f(S(\chi_{2n_i},\gamma_{2n_i}))) + d(f(S(\chi_{2n_i},\gamma_{2n_i})), S(\chi^*,\gamma^*)) \\ &= d\left(f\left(T(\chi_{2n_{i+1}},\gamma_{2n_{i+1}})\right), f\left(S(\chi_{2n_i},\gamma_{2n_i})\right)\right) \\ &\qquad + d\left(f\left(S(\chi_{2n_i},\gamma_{2n_i})\right), S(\chi^*,\gamma^*)\right) \\ &\leq d\left(T(\chi_{2n_{i+1}},\gamma_{2n_{i+1}}), S(\chi_{2n_i},\gamma_{2n_i})\right) \\ &\qquad + d\left(S(f\chi_{2n_i}, f\gamma_{2n_i}), S(\chi^*,\gamma^*)\right) \\ &\leq d\left(S(\chi_{2n_i},\gamma_{2n_i}), T(\chi_{2n_{i+1}},\gamma_{2n_{i+1}})\right) \\ &\qquad + d\left(S(f\chi_{2n_i}, f\gamma_{2n_i}), S(\chi^*,\gamma^*)\right) \\ &\leq (\kappa(\lambda)+1)\left[\psi(\Omega_{2n_{i-1}},\Omega_{2n_i}) - \psi(\Omega_{2n_i},\Omega_{2n_{i+1}})\right] \\ &\qquad + d\left(S(f\chi_{2n_i}, f\gamma_{2n_i}), S(\chi^*,\gamma^*)\right) \\ &\leq (\kappa(\lambda)+1)[\lambda - \lambda] + d\left(S(\chi^*,\gamma^*), S(\chi^*,\gamma^*)\right) \\ &= 0. \end{aligned}$$

Therefore $f\chi^* = S(\chi^*,\gamma^*)$.
Similarly we can prove that $f\gamma^* = S(\gamma^*,\chi^*)$.
In the same way we can prove that $g\chi^* = T(\chi^*,\gamma^*)$ and $g\gamma^* = T(\gamma^*,\chi^*)$.
This shows that S, T, f, g have a CCoin point.

Theorem 2.4:

Suppose that hypothesis of Theorem2.3 holds. Besides, let for every $(a^*,b^*), (c^*,d^*) \in (X \times X)$, there exists $(\xi,\tau) \in (X \times X)$ such that
$(S(a^*,b^*), T(\xi,\tau)) \in E(G)$, $(S(b^*,a^*), T(\tau,\xi)) \in E(G^{-1})$ and
$(S(c^*,d^*), T(\xi,\tau)) \in E(G)$, $(S(d^*,c^*), T(\tau,\xi)) \in E(G^{-1})$
Also
$(S(\xi,\tau), T(a^*,b^*)) \in E(G)$, $(S(\tau,\xi), T(b^*,a^*)) \in E(G^{-1})$ and
$(S(\xi,\tau), T(c^*,d^*)) \in E(G)$, $(S(\tau,\xi), T(d^*,c^*)) \in E(G^{-1})$.
Then S, T, f and g have a unique CCFP.

Proof:

Let (a^*,b^*),(c^*,d^*) are CCoin points S, T, f and g.
Then

$fa^* = S(a^*,b^*)$, $fb^* = S(b^*,a^*)$, $ga^* = T(a^*,b^*)$, $gb^* = T(b^*,a^*)$
$fc^* = S(c^*,d^*)$, $fd^* = S(d^*,c^*)$, $gc^* = T(c^*,d^*)$, $gd^* = T(d^*,c^*)$

Set $T(\xi_{2n+1},\tau_{2n+1}) = g\xi_{2n+2}$, $\xi = \xi_1$ and
$T(\tau_{2n+1},\xi_{2n+1}) = g\tau_{2n+2}$, $\tau = \tau_1$.
Then by the assumption
$(S(a^*,b^*), T(\xi_1,\tau_1)) \in E(G)$, $(S(a^*,a^*), T(\tau_1,\xi_1)) \in E(G^{-1})$
$(S(c^*,d^*), T(\xi_1,\tau_1)) \in E(G)$, $(S(d^*,c^*), T(\tau_1,\xi_1)) \in E(G^{-1})$.
This implies that, $(fa^*, g\xi_2) \in E(G)$, $(fb^*, g\tau_2) \in E(G^{-1})$ and
$(fc^*, g\xi_2) \in E(G)$, $(fd^*, g\tau_2) \in E(G^{-1})$.

Now by using the edge preserving property repeatedly for $n \geq 1$, we have
$(fa^*, g\xi_{2n+2}) \in E(G)$, $(fb^*, g\tau_{2n+2}) \in E(G^{-1})$ and
$(fc^*, g\xi_{2n+2}) \in E(G)$, $(fd^*, g\tau_{2n+2}) \in E(G^{-1})$.
Now

$$\begin{aligned} & d(fa^*, fc^*) \\ & \leq d(fa^*, g\xi_{2n+1}) + d(g\xi_{2n+1}, fc^*) \\ & = d\big(S(a^*, b^*), T(\xi_{2n+1}, \tau_{2n+1})\big) + d\big(T(\xi_{2n+1}, \tau_{2n+1}), S(c^*, d^*)\big) \\ & = d\big(S(a^*, b^*), T(\xi_{2n+1}, \tau_{2n+1})\big) + d\big(S(c^*, d^*), T(\xi_{2n+1}, \tau_{2n+1})\big) \\ & \leq \max\{\kappa(\psi(fa^*, g\xi_{2n+1}), \kappa(\psi(fa^*, g\xi_{2n+1})\}[\psi(fa^*, g\xi_{2n+1}) - \psi(fa^*, g\xi_{2n+1})] \\ & \quad + \max\{\kappa(\psi(fc^*, g\xi_{2n+1}), \kappa(fc^*, g\xi_{2n+1})\}[\psi(fc^*, g\xi_{2n+1}) - \psi(fc^*, g\xi_{2n+1})] \\ & = 0. \end{aligned}$$

Therefore $fa^* = fc^*$ Similarly, we get $fb^* = fd^*$, $ga^* = gc^*$ and $gb^* = gd^*$.
Now for an arbitrary $(m, n) \in X \times X$,
Let $fa^* = fc^* = m$ and $fb^* = fd^* = n$ and $ga^* = gc^* = m$ and $gb^* = gd^* = n$. Since (S, f) are commute so
$f(fa^*) = f(S(a^*, b^*)) = S(fa^*, fb^*)$ and
$f(fb^*) = f(S(b^*, a^*)) = S(fb^*, fa^*)$.
Therefore $fm = S(m, n)$ and $fn = S(n, m)$.
Since (T, g) are commute so
$g(ga^*) = g(T(a^*, b^*)) = T(ga^*, gb^*)$
and
$g(gb^*) = g(T(b^*, a^*)) = T(gb^*, ga^*)$.
Therefore $gm = T(m, n)$ and $gn = T(n, m)$.
This shows that (m, n) is a CCoin point of S, T, f, g.
Now it remains to prove that (m, n) is fixed point of S, T, f, g.
Let (a^*, b^*) and (m, n) are CCoin points of S, T, f, g.
Then $fa^* = fm$ and $fb^* = fn$, also $ga^* = gm$ and $gb^* = gn$.
Therefore $m = fm$, $n = fn$ and $m = gm$, $n = gn$.
This shows that (m, n) is a CCFP of S, T, f and g.
Finally to prove tha uniqueness of (m, n).
Let (s, t) be another CCFP.
Then $s = fs = S(s, t)$, $t = ft = S(t, s)$ and $s = gs = T(s, t)$, $t = gt = T(t, s)$
Since $fs = fm = m$, $ft = fn = n$ and $gs = gm = m$, $gt = gm = n$
Thus $s = m, t = n$.
Hence the theorem is proved.

It is easy to prove our main results for two maps.

Corollary 1:
Suppose (X, d) be a complete metric space endowed with a directed graph G. Let us consider the mappings $T: X \times X \to X$ and $f: X \to X$ with defining the following sets
$(X \times X)_{Tf} = \{(\chi, \gamma) \in X \times X: (f\chi, T(\chi, \gamma)) \in E(G), (f\gamma, T(\gamma, \chi)) \in E(G^{-1})\}$Then Theorem 2.2, 2.3, 2.3 holds for the two maps T, f with the $G - f_1$contarction

$$\begin{aligned} & d(T(\chi, \gamma), T(\xi, \tau)) \\ & \leq \max\Big\{\kappa\big(\psi(f\chi, f\xi)\big), \kappa\Big(\psi\big(T(\chi, \gamma), T(\xi, \tau)\big)\Big)\Big\}\big[\psi(f\chi, f\xi) - \psi\big(T(\chi, \gamma), T(\xi, \tau)\big)\big] \\ & \quad - \max\{\kappa(\phi(f\gamma, f\tau)), \kappa(\phi(T(\gamma, \chi), T(\tau, \xi)))\}[\phi(f\gamma, f\tau) - \psi(T(\gamma, \chi), T(\tau, \xi))] \end{aligned}$$

where $\psi, \phi: X \times X \to [0, \infty)$ are lower semi continuous functions.

3. Application to Integral Equations

In this section, to discuss the application of our main results we establish an existence theorem in a metric space with graph for the solution of the integral equations.
Let us consider the following integral equations:

$$\chi(t) = \int_0^t f(t, \chi(s), \gamma(s))ds, \gamma(t) = \int_0^t f(t, \gamma(s), \chi(s))ds. \quad (3.1)$$

for all $t \in I = [0,1]$ and $f: I \times R \times R \to R$. Let $X = C(I, R)$ and define $d: X \times X \to R$ as $d(\chi, \gamma) = |\chi - \gamma|$ for $\chi, \gamma \in X$.
Define $\psi: X \times X \to [0, \infty)$ by $\psi(a, b) = |a - b|$, and
$\phi: X \times X \to [0, \infty)$ by $\phi(a, b) = \frac{|a-b|}{2}$ and define
$\kappa: [0, +\infty) \to [0, +\infty)$ as

$$\kappa(t) = \begin{cases} \frac{3}{2} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \end{cases}$$

Further define graph G on X using partial ordering relation.
i.e., $\chi, \gamma \in X, \chi \leq \gamma \Leftrightarrow \chi(t) \leq \gamma(t)$ for any $t \in I$.So we have,
$E(G) = \{(\chi, \gamma) \in X \times X: \chi \leq \gamma\}$, $E(G^{-1}) = \{(\chi, \gamma) \in X \times X: \gamma \leq \chi\}$.
Also, $\Delta(X \times X) \subseteq E(G)$ and (X, d, G) has property (A).
It is routine verification to check that (X, d) is a complete metric space with a directed graph G.

Theorem 3.1:
Suppose for the integral equation 3.1,
(i) $f \in C(I \times R \times R)$ is continuous.
(ii) for all $t \in I$ and $\chi, \gamma, \xi, \tau \in R$ with $\chi \leq \xi, \tau \leq \gamma$ and
$f(t, \chi, \gamma) \leq f(t, \gamma, \tau)$ and

$$\left|\int_0^t f(t, \chi(s), \gamma(s))ds - \int_0^t f(t, \xi(s), \tau(s))ds\right| = \frac{1}{2}|\chi - \xi| - |\gamma - \tau|$$

(iii) there exists $(\chi_0, \gamma_0) \in X \times X$ such that for all $t \in I$,
$\chi(t) \leq \int_0^t f(t, \chi_0(s), \gamma_0(s))ds$ and $\int_0^t f(t, \gamma_0(s), \chi_0(s))ds \leq \gamma(t)$.
Then there exists atleast one solution of 3.1 in $X = C(I, R)$

Proof:
Define $T: X \times X \to X, g: X \to X$ by
$T(\chi, \gamma)(t) = \int_0^t f(t, \chi(s), \gamma(s))ds$ and $g(\chi)(t) = \chi(t)$.
Then 3.1 is equivalent to $g(\chi) = T(\chi, \gamma)$, $g(\gamma) = T(\gamma, \chi)$
This shows that the solution of 3.1 is a coupled coincidence point of the mapping T, g , provided we verify the conditions of Corollary 1.
Since graph G is defined on X and $g: X \to X$ so g is edge preserving.
Now using g edge preserving property, suppose that $\chi, \gamma, \xi, \tau \in R$ such that $g(\chi) \leq g(\xi), g(\tau) \leq g(\gamma)$.
Then for each $t \in I$

$$\begin{aligned} T(\chi, \gamma)(t) &= \int_0^t f\big(t, \chi(s), \gamma(s)\big)ds \\ &= \int_0^t f\big(t, g(\chi)(s), g(\gamma)(s)\big)ds \\ &\leq \int_0^t f(t, g(\xi)(s), g(\tau)(s))ds \\ &= \int_0^t f\big(t, \xi(s), \tau(s)\big)ds \\ &= T(\xi, \tau)(t) \end{aligned}$$

$\Rightarrow T(\chi, \gamma)(t) \leq T(\xi, \tau)(t)$
$\Rightarrow (T(\chi, \gamma)(t), T(\xi, \tau)(t)) \in E(G)$.
Similarly, $T(\tau, \xi)(t) \leq T(\gamma, \chi)(t) \Rightarrow (T(\gamma, \chi)(t), T(\tau, \xi)(t)) \in E(G^{-1})$ Thus T is edge preserving.
Now consider $G - f_1$ contraction,

$$\begin{aligned} L.H.S &= d\big(T(\chi, \gamma)(t), T(\xi, \tau)(t)\big) \\ &= |T(\chi, \gamma) - T(\xi, \tau)| \\ &= \left|\int_0^t f(t, \chi(s), \gamma(s))ds - \int_0^t f(t, \xi(s), \tau(s))ds\right| \\ &= \frac{1}{2}|\chi - \xi| - |\gamma - \tau| \end{aligned}$$

$$\leq \frac{3}{2}\left[|\chi-\xi|+\frac{5}{4}|\gamma-\tau|\right]$$
$$=\frac{3}{2}\left[|\chi-\xi|-\frac{1}{2}|\chi-\xi|+|\gamma-\tau|+\frac{1}{2}|\gamma-\tau|-\frac{1}{4}|\gamma-\tau|+\frac{1}{2}|\chi-\xi|\right]$$
$$=\frac{3}{2}[|g\chi-gu|-|T(\chi,\gamma)-T(\xi,\tau)|]-\frac{3}{2}\left[\frac{|g\gamma-g\tau|}{2}-\frac{|T(\gamma,\chi)-T(\tau,\xi)|}{2}\right]$$
$$=\max\left\{\kappa(\psi(g\chi,g\xi)),\kappa\left(\psi(T(\chi,\gamma),T(\xi,\tau))\right)\right\}[\psi(g\chi,g\xi)-\psi(T(\chi,\gamma),T(\xi,\tau))]$$
$$-\max\{\kappa(\phi(g\gamma,g\tau)),\kappa(\phi(T(\gamma,\chi),T(\tau,\xi)))\}[\phi(g\gamma,g\tau)-\psi(T(\gamma,\chi),T(\tau,\xi))]$$

Therefore L.H.S $\leq$ R.H.S

Thus T is $G-(f)_1$ contraction.

Now, condition (iii) of hypothesis implies that there exists $(\chi_0,\gamma_0)\in X\times X$ such that

$\chi_0(t)\leq\int_0^t f(t,\chi_0(s),\gamma_0(s))ds$ and $\int_0^t f(t,\gamma_0(s),\chi_0(s))ds\leq\gamma_0(t)$.

But $\chi_0(t)=g(\chi_0)(t)$ and $\gamma_0(t)=g(\gamma_0)(t)$ so we have

$g(\chi_0)\leq T(\chi_0,\gamma_0)$ and $T(\gamma_0,\chi_0)\leq g(\gamma_0)$

$\Rightarrow (g(\chi_0),T(\chi_0,\gamma_0))\in E(G)$ and $(g(\gamma_0),T(\gamma_0,\chi_0))\in E(G^{-1})$

$\Rightarrow (\chi\times X)_T g\neq 0$

Also, T, g are commutative and $T(X\times X)\subseteq g(X)$. Thus T, g satisfies the condition of Corollary 1. Hence by Corollary 1 there exists a point $(\chi^*,\gamma^*)\in X\times X$ such that $g\chi^*=T(\chi^*,\gamma^*)$ and $g\gamma^*=T(\gamma^*,\chi^*)$.

Hence by the definition of g we have, $\chi^*=g\chi^*=T(\chi^*,\gamma^*)$ and $\gamma^*=g\gamma^*=T(\gamma^*,\chi^*)$. Therefore (χ^*,γ^*) is a solution of the equation 3.1.

4. Conclusion

In this paper we defined two different graphs sets with some properties and intersection of those two graphs as new set and finally proved that all the four maps in the graph have a unique common coupled fixed point.

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