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Research paper

A Comparative Study on Laplace Solutions of Ordinary Differential Equations between Analytical and Matlab Approach

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Abstract:

The aim of this paper is to give the Laplace solutions of ordinary differential equation through Matlab. Such a solution can be obtained effectively, through the strong knowledge on mathematical way of obtaining the solution of differential equation by applying Laplace transform technique and hence the paper contains both the approaches of analytical as well as by using the software Matlab of Lapalce solutions of differential equations. Also this paper provides the codlings to be used to execute the Laplace solutions graphically by which one can get the clear understanding of nature of the given problem and be able to interpret the results obtained.

***Keywords**: Matlab , Graphically, Laplace, Differential Equations*

1. Introduction

MATLAB is user friendly software which allows Matrix Manipulations, Plotting of Functions and data, Implementation of Algorithms and used for visualizing the effects by means of graphs. Also, an additional package, Simulink adds graphical multi-domain simulation and model-based design for dynamic and embedded systems.

Use of Transform techniques such as Laplace Transforms and Fourier transforms in any engineering field is customary. In this article, it has been discussed about the way of using matlab in finding solution by the method of Laplace and Fourier transform techniques.

1.1. Transform Techniques

Laplace transform is a powerful mathematical technique useful to the engineers and scientists, as it enables them to solve linear differential equations with given initial conditions by using algebraic methods..Particularly, it is used in signal processing. Laplace and Fourier transforms are the powerful tool to study a signal in frequency domain. For instance, if your signal is smooth over time, it means that, in the frequency domain, you're very likely to find only small frequencies. Similarly, the concept of filtering signal/data is based on a frequency domain interpretation. The only difference between a Laplace transform and a Fourier transform lies on the domain. A Laplace transform can be seen as a generalization of a Fourier transform. From a system point of view, a Fourier transform gives information on the steady state, while the Laplace gives information on the steady state as well as the transient state.

2. Solution of Differential Equation by Laplace Transform

2.1. Laplace Transform Definition:

If $f(t)$ is a function which is piecewise continuous function on every finite interval in the range $t \geq 0$ and is of exponential order, then the Laplace transform of $f(t)$ exists and it is defined by

We know that $L\left[f(t) \right] = \int_0^\infty e^{-st} f(t)\,dt$

2.2. Properties

Some of the properties of Laplace transform technique are stated here.

- ***Linear Property:***

The Laplace transformation is a linear transformation

i.e., $L[af_1(t)+bf_2(t)] = aL[f_1(t)]+bL[f_2(t)]$ where a & b are constants.

- ***First Shifting Property:***

If $L[f(t)] = F(s)$ then $L\left[e^{at} f(t)\right] = F(s-a)$

Example:

1. Find $L[\cosh t \cdot \sin 2t]$

Solution:

By First shifting theorem,

$$L[\cosh t.\sin 2t] = L\left[\left(\frac{e^t + e^{-t}}{2}\right)\sin 2t\right]$$

$$= \frac{1}{2}L\left[e^t \sin 2t\right] + \frac{1}{2}L\left[e^{-t}\sin 2t\right]$$

$$= \frac{1}{2}\frac{2}{(s-1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4}$$

$$= \frac{1}{(s-1)^2+4} + \frac{1}{(s+1)^2+4}$$

MATLAB CODE:

Following command can be used to attain the solution of the above problem.

```
>> syms t s
.>>f=cosh(t)*sin(2*t)
>> F=laplace(f,t,s)
% To get the solution same as (1)
>> simplify (F)
>>pretty(ans)
```

Otherwise one can directly write f(t) as follows

```
syms t s
F=laplace(cosh(t)*sin(2*t))
```

2. Find $L[t\sin 2t]$

Solution:

We know that $L\left[tf(t)\right] = -\frac{d}{ds}\left[f(t)\right]$

Here $f(t) = \sin(2t)$

$$F(s) = L\left[f(t)\right] = L[\sin 2t] = \frac{2}{s^2+4}$$

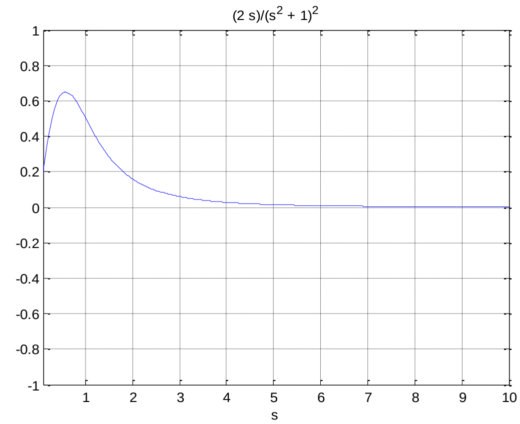
$$F(s) = \frac{2}{s^2+4}$$

$$\mathrm{L}(\mathrm{t}\sin 2\mathrm{t}) = \frac{-\mathrm{d}}{\mathrm{ds}}\left(\frac{2}{\mathrm{s}^2+4}\right)$$

$$= \frac{4\mathrm{s}}{\left(\mathrm{s}^2+4\right)^2}$$

we can easily plot the result too

```
syms t s
f=t*sin(t);
F=laplace(f,t,s);
simplify (F);
ezplot(F,[0  10])
xlim([0.1,10])
ylim([-1 1])
hold off
grid on
```



So the person readily got the solution by using simple command "laplace" in matlab.

Similarly one can easily use the command "ilaplace" for inverse laplace transform.

3. Transform Inverse Laplace

If the Laplace transform of a function $f(t)$ is $F(s)$ i.e., $L\left[f(t)\right] = F(s)$ then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is denoted by $f(t) = L^{-1}\left[F(s)\right]$

Example:

Find $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

Solution:

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2}\right] \quad \text{(using shifting property)}$$

$$= e^{-t}t$$

Matlab Code:

```
syms t s
F=ilaplace(1/(s+1)^2)
```

Here the simple matlab command is used to obtain the required solution.

Find $L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right] = L^{-1}\left[\frac{s-b+b}{(s-b)^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right] + L^{-1}\left[\frac{b}{(s-b)^2+a^2}\right]$$

$$= e^{bt}.\cos at + be^{bt}.\frac{\sin at}{a}$$

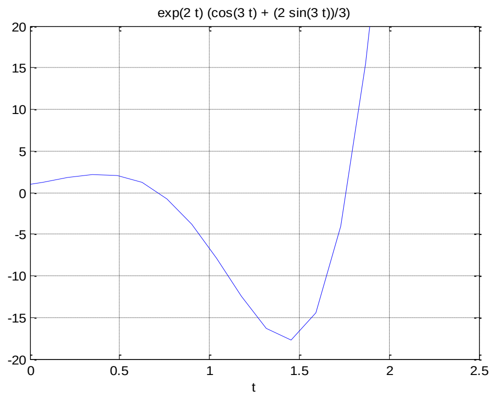
Particularly when we Put a=3,b=2 ,we get

$$L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right]$$

=exp(2t).((cos(3t)+
(2sin(3t))/3))

Analytical solution along with graphical solution can be studied using matlab code for the above as follows

```
syms t s
f=s/((s-2)^2+9);
F=ilaplace(f);
simplify (F)
ezplot(F,[-30 30])
xlim([0 2.5])
ylim([-20 20])
hold off
grid on
```



4. Applications of Laplace Transform:

4.1. Second Order Differential Equation

The main application of laplace transform technique is to find the solution of differential equations with the given initial conditions and to plot the graph of such solution to get clear vision of output of such equations.

Here the Laplace solution of second order differential equation with the known initial conditions is given in both direct method as well as the matlab coding for attaining the soltution both numerically and graphically.

Example:

Solve

$$(D^2+1)x = t\cos 2t,$$
$$x(t)=0, x'(t)=0, when\, t=0$$

Solution by Direct Laplace Method:

$$L[x'']+L[x]=L[t\cos 2t]$$

$$\Rightarrow s^2L[x]-sx[0]-x'[0]+L[x]=\frac{s^2-4}{(s^2+4)^2}$$

$$\Rightarrow (s^2+1)L[x]=\frac{s^2-4}{(s^2+4)^2}$$

$$\Rightarrow L[x]=\frac{s^2-4}{(s^2+1)(s^2+4)^2}$$

$$=\frac{-5}{9}\left(\frac{1}{s^2+1}\right)+\frac{5}{9}\left(\frac{1}{s^2+1}\right)+\frac{8}{3}\left(\frac{1}{(s^2+4)^2}\right)$$

(Using partial fractions)

$$x[t]=\frac{-5}{9}L^{-1}\left(\frac{1}{s^2+1}\right)+\frac{5}{9}L^{-1}\left(\frac{1}{s^2+1}\right)$$
$$+\frac{8}{3}L^{-1}\left(\frac{1}{(s^2+4)^2}\right) \quad ---------(1)$$

By using the results

$$L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t,$$

$$L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{\sin 2t}{2},$$

$$L^{-1}\left(\frac{1}{(s^2+4)^2}\right)=$$
$$\frac{1}{8}\left[\frac{\sin 2t}{4}+\frac{\sin 2t}{4}-t\cos 2t\right]$$

$$(u\sin g\; convlution theorem)$$

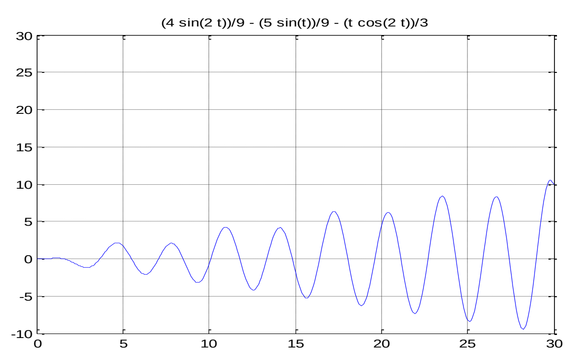
Substituting in (1), we get

$$x[t]=\frac{-5}{9}\sin t+\frac{4}{9}\sin 2t-\frac{1}{3}t\cos 2t$$

Matlab coding to obtain the solution as well as the graph of the above problem is given below.

Matlab Code

```
syms t s Y
  RHS=laplace(t*cos(2*t));
Y1=s*Y;
Y2= s*Y1;
S=solve(Y2+Y-RHS,Y);
Ans=ilaplace(S,s,t);
pretty(Ans)
ezplot(Ans,[0,30])
xlim([0,30])
ylim([-10 10])
hold off
grid on
```



4.2. Simultaneous Differential Equations

Laplace solution of simultaneous first order differential equations is given in both direct
method as well as the matlab coding. Both the solution and the corresponding graphs are obtained through matlab.

Example:
Solve

$\frac{dx}{dt}+y=\sin t\,;\frac{dy}{dt}+x=\cos t$ with $x=2$ and $y=0$ when $t=0$

Direct Method of Solution:
The given differential equation can be written as

$$x'(t)+y(t)=\sin t$$
$$y'(t)+x(t)=\cos t$$

Taking Laplace Transforms on both sides we get,

$$L\left[x'(t)\right]+L\left[y(t)\right]=L[\sin t]$$
$$L\left[y'(t)\right]+L\left[x(t)\right]=L[\cos t]$$
$$sL\left[x(t)\right]-x(0)+L\left[y(t)\right]=\frac{1}{s^2+1}$$
$$sL\left[y(t)\right]-y(0)+L\left[x(t)\right]=\frac{s}{s^2+1}$$

Putting $x(0)=2$, $y(0)=0$ and $L\left[x(t)\right]=\bar{x}$, $L\left[y(t)\right]=\bar{y}$ we get

$$s\bar{x}-2+\bar{y}=\frac{1}{s^2+1}$$
$$s\bar{y}+\bar{x}=\frac{s}{s^2+1}$$
$$s\bar{x}+\bar{y}=\frac{1}{s^2+1}+2 \qquad (1)$$
$$\bar{x}+s\bar{y}=\frac{s}{s^2+1} \qquad (2)$$

Solving (1) and (2) simultaneously we get

$$\bar{x}=\frac{\begin{vmatrix}\frac{1}{s^2+1}+2 & 1\\ \frac{s}{s^2+1} & s\end{vmatrix}}{\begin{vmatrix} s & 1\\ 1 & s\end{vmatrix}}$$
$$=\frac{s\left(\frac{1}{s^2+1}+2\right)-\frac{s}{s^2+1}}{s^2-1}$$
$$=\frac{s\left[2\left(s^2+1\right)+1\right]-s}{\left(s^2-1\right)\left(s^2+1\right)}$$
$$=\frac{2s}{s^2-1}$$
$$L\left[x(t)\right]=\frac{2s}{s^2-1}$$
$$x(t)=2L^{-1}\left(\frac{s}{s^2-1}\right)$$
$$x(t)=2\cosh t$$
$$\bar{y}=\frac{\begin{vmatrix} s & \frac{1}{s^2+1}+2\\ 1 & \frac{s}{s^2+1}\end{vmatrix}}{\begin{vmatrix} s & 1\\ 1 & s\end{vmatrix}}$$

L[y(t)]= $L\left[y(t)\right]=\frac{s^2+3}{\left(s^2+1\right)\left(1-s^2\right)}$

$$y(t)=L^{-1}\left[\frac{s^2+3}{\left(s^2+1\right)\left(1-s^2\right)}\right]$$
$$\frac{s^2+3}{\left(s^2+1\right)(1+s)(1-s)}=\frac{A}{1+s}+\frac{B}{1-s}+\frac{Cs+D}{s^2+1}$$
$$s^2+3=A(1-s)\left(s^2+1\right)+B(1+s)\left(s^2+1\right)$$
$$+(Cs+D)(1+s)(1-s)$$

Putting $s=1, 4B=4 \Rightarrow B=1$
Putting $s=-1, 4A=4 \Rightarrow A=1$
Putting $s=0, A+B+D=3 \Rightarrow D=1$
Equating coefficient of s^3, we get $-A+B-C=0 \Rightarrow C=0$

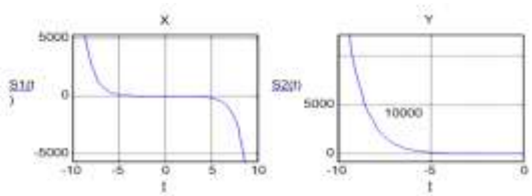
$$\frac{s^2+3}{\left(s^2+1\right)(1+s)(1-s)}=\frac{1}{1+s}+\frac{1}{1-s}+\frac{1}{s^2+1}$$
$$L^{-1}\left[\frac{s^2+3}{\left(s^2+1\right)(1+s)(1-s)}\right]=L^{-1}\left[\frac{1}{1+s}+\frac{1}{1-s}+\frac{1}{s^2+1}\right]$$
$$=L^{-1}\left[\frac{1}{1+s}-\frac{1}{s-1}+\frac{1}{s^2+1}\right]$$
$$=e^{-t}-e^{t}+\sin t$$

Therefore $x(t)=2\cosh t$

$$y(t)=e^{-t}-e^{t}+\sin t$$

Matlab Code:

```
syms t s x(t) y(t)
dx(t)=diff(x(t),t);
dy(t)=diff(y(t),t);
eq1(t)=dx(t)+y(t)-sin(t);
eq2(t)=dy(t)+x(t)-cos(t);
eqn1L1=laplace(eq1,t,s)-2;
eqn2L2=laplace(eq2,t,s);
L1(t)=laplace(eq1,t,s);
L2(t) = laplace(eq2,t,s);
syms LF LS
X1 = subs(L1(t),{x(0),y(0)},{2,0});
Y1 = subs(L2(t),{x(0),y(0)},{2,0});
X1=subs(X1,{laplace(x(t),t,s),laplace(y(t),t,s)},{LF,LS});
Y1=subs(Y1,{laplace(x(t),t,s),laplace(y(t),t,s)},{LF,LS});
X1=collect(X1,LF);
Y1=collect(Y1,LS);
[LF,LS]=solve(Y1,LS,X1,LF);
S1=ilaplace(LF,s,t);
S2=ilaplace(LS,s,t);
subplot(2,2,1); ezplot(S1,[-10,10]);
title('X'); ylabel('S1(t)'); grid
subplot(2,2,2); ezplot(S2,[-10,0]);
title('Y'); ylabel('S2(t)'); grid
```



5. Conclusion

MATLAB is software which has special and powerful features to solve ODE and system of linear differential equations and also to execute the concern graphs to have a visual impact of such equations. In this paper, Some of the matlab coding are given to obtain the Laplace Solutions and it can be extended to Fourier Transform also.

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