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Research paper

On A Subclass of Harmonic Univalent Functions Associated with the Differential Operator

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Abstract

In this paper, a new subclass of harmonic univalent functions in the unit disk $U = \{z \in C : |z| < 1\}$ is introduced using a differential operator. Also the coefficient estimates, convolution conditions, extreme points and convex combinations are obtained.

Keywords: *Harmonic function; Univalent functions; Differential operator.*

1. Introduction

Let D be a simply connected domain in the complex plane C and let $f = u + iv$ be harmonic in D. We can write $f = h + \overline{g}$, where h and g are analytic in D. Here h and g are called as the analytic part and co-analytic part of f respectively. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|h'(z)| > |g'(z)|$ in D [1].

In 1984, the class S_H was introduced by Clunie and Sheil-Small [1] and their study gave insight of few coefficient bounds. Since then, several papers were published related to S_H and its subclasses. In fact, by investigating new subclasses, Sheil-Small [18], Jahangiri [15], Silverman and Silvia [9] and Ahuja [2] presented a unified and systematic study of harmonic univalent functions.

Let S_H denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense- preserving in the unit disk $U = \{z \in C : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. We may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

The differential operator $D^n_{\eta,\mu}(\lambda, w)\ (n \in N_0)$ was initiated by Bucur et al. [10].

For $f = h + \overline{g}$, given by (1.1), Sahsene Altinkaya and Sibel Yalcin [5] defined the operator $D^n_{\eta,\mu}(\lambda, w)$ as

$$D^n_{\eta,\mu}(\lambda, w)\ f(z) = D^n_{\eta,\mu}(\lambda, w)\ h(z) + (-1)^n \overline{D^n_{\eta,\mu}(\lambda, w)\ g(z)} \tag{1.2}$$

where $D^n_{\eta,\mu}(\lambda, w)\ h(z) = z + \sum_{k=2}^{\infty} [(k-1)(\mu w^{\lambda} - \eta) + k]^n a_k z^k$

and $D^n_{\eta,\mu}(\lambda, w)\ g(z) = \sum_{k=1}^{\infty} [(k+1)(\mu w^{\lambda} - \eta) + k]^n b_k z^k$,

where

$n, \eta \in N_0$, $\mu, \lambda, w \geq 0$, $0 \leq \eta \leq \mu w^{\lambda}$, $D^n_{\eta,\mu}(\lambda, w)\ f(0) = 0$.

Recently S. Porwal and Shivam Kumar [11] and A.L. Pathak et al. [12] have defined and studied a subclass of harmonic univalent functions using differential operator. They have obtained the coefficient and distortion bounds, extreme points, convolution and convex combinations.

Motivated by aforementioned work, in this paper we define a subclass $B_H(n, \eta, \alpha, \rho, \lambda, w)$ of harmonic univalent functions and study the coefficient bounds, convolution, extreme points and convex combinations.

Definition 1.1: We define a new subclass $B_H(n, \eta, \alpha, \rho, \lambda, w)$ of functions of the kind (1.1) that satisfy the condition

$$\mathrm{Re}[(1+\rho e^{ir}) \frac{D^{n+1}_{\eta,\mu}(\lambda, w) f(z)}{D^n_{\eta,\mu}(\lambda, w) f(z)} - \rho e^{ir}] > \alpha,\ 0 \leq \alpha < 1, r \in R, \rho \geq 0, n, \eta \in N_0 \tag{1.3}$$

where $D^n_{\eta,\mu}(\lambda, w)\ f(z)$ is defined by (1.2).

Let $\overline{B_H}(n, \eta, \alpha, \rho, \lambda, w)$ denote the subclasses of $B_H(n, \eta, \alpha, \rho, \lambda, w)$ which contains harmonic functions $f_n = h + \overline{g_n}$ such that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k \tag{1.4}$$

2. Coefficient Bounds

We start with a sufficient coefficient condition for functions in $B_H(n, \eta, \alpha, \rho, \lambda, w)$.

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Theorem 2.1: Let $f = h + \overline{g}$ be given by (1.1). If

$$\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|$$

$$+\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|$$

$$\le 1-\alpha \tag{2.1}$$

where $n,\eta \in N_0$, $\rho \ge 0, 0 \le \alpha < 1$, $\mu,\lambda,w \ge 0, 0 \le \eta \le \mu w^{\lambda}$, then f is harmonic univalent, orientation-preserving in U and $f \in B_H(n,\eta,\alpha,\rho,\lambda,w)$.

Proof: If $z_1 \ne z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|}$$

$$\ge 1-\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|}{1-\alpha}$$

$$1-\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

≥ 0, which demonstrates univalence.

Note that f is sense-preserving in U. This is because

$$|h'(z)| \ge 1-\sum_{k=2}^{\infty}k|a_k|z^{k-1}$$

$$>1-\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

$$>\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k||z|^{k-1}}{1-\alpha}$$

$$\ge \sum_{k=1}^{\infty}k|b_k|z^{k-1}$$

$$\ge |g'(z)|.$$

Using the fact that $\mathrm{Re}(w) > \alpha$ if and only if $|1-\alpha+w| \ge |1+\alpha-w|$.

$$w=\frac{(1+\rho e^{ir})D^{n+1}_{\eta,\mu}(\lambda,w)f(z)-\rho e^{ir}D^{n}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}=\frac{A(z)}{B(z)}$$

It suffices to show that

$$|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)| \le 0. \tag{2.2}$$

Consider

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$\ge 2(1-\alpha)|z|+$$

$$2\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}$$

$$[(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)-(\alpha+\rho)]|a_k||z|^{k}-$$

$$2\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}$$

$$[(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)+(\alpha+\rho)]|b_k||\overline{z}|^{k}$$

$$>2(1-\alpha)|z|\left\{1-\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}\right.\right.$$

$$\left.[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k||z|^{k-1}\right)$$

$$-\frac{1}{1-\alpha}\left(\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$

$$\left.\left.[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k||z|^{k-1}\right)\right\}$$

This is non-negative by (2.1), and so the proof is complete.

The harmonic function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{(1-\alpha)x_k z^{k}}{[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]}$$

$$+\sum_{k=1}^{\infty}\frac{(1-\alpha)\overline{y_k z^{k}}}{[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]} \tag{2.3}$$

where $n,\eta \in N_0$, $\rho \ge 0, 0 \le \alpha < 1$, $\mu,\lambda,w \ge 0, 0 \le \eta \le \mu w^{\lambda}$ with $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$,

shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $B_H(n,\eta,\alpha,\rho,\lambda,w)$ because

$$\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$

$$\left.[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|\right)+$$

$$\frac{1}{1-\alpha}\left(\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$

$$\left.[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|\right)$$

$$=\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1\ . \tag{2.4}$$

In the following theorem, it is shown that the condition (2.1) is also necessary for the functions $f_n = h + \overline{g_n}$, to be in the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ where h and g_n are of the form (1.4).

Theorem 2.2: Let $f_n = h + \overline{g}_n$ be given by (1.4).

Then $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$, if and only if

$$\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$
$$\left.[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|\right)$$
$$+\frac{1}{1-\alpha}\left(\sum_{k=2}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}\right.$$
$$\left.[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|\right)\le 1, \tag{2.5}$$

where $n,\eta\in N_0$, $\rho\ge 0, 0\le\alpha<1$, $\mu,\lambda,w\ge 0, 0\le\eta\le\mu w^{\lambda}$.

Proof: Since $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)\subset B_H(n,\eta,\alpha,\rho,\lambda,w)$, we just need to prove the 'only if 'part of the theorem.

For functions f_n of the form (1.4), we notice that the condition (1.3) is correspondent to

$$\mathrm{Re}\left[(1+\rho e^{ir})\frac{D^{n+1}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}-\rho e^{ir}\right]>\alpha$$

$$\mathrm{Re}\left[\left((1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\right.\right.$$
$$\left.\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|z^{k-1}\right)/$$
$$\left(1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\right.$$
$$\left.\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\overline{z}^{k-1}\right)-$$
$$\left(-(-1)^{2n}\frac{\overline{z}}{z}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\right.$$
$$\left.\left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\overline{z}^{k-1}\right)/$$
$$\left(1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\right.$$
$$\left.\left.\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\overline{z}^{k-1}\right)\right]\ge 0 \tag{2.6}$$

The above condition (2.6) must hold for all values of z on the positive real axis, where $0\le|z|=\gamma<1$, we must have

$$\mathrm{Re}\left[\frac{\left(\begin{array}{l}(1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\\ \left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|\gamma^{k-1}\end{array}\right)}{\left(\begin{array}{l}1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+\\ (-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\end{array}\right)}-\right.$$
$$\left.\frac{\left(\begin{array}{l}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\\ \left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\gamma^{k-1}\end{array}\right)}{\left(\begin{array}{l}1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+\\ (-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\end{array}\right)}\right]\ge 0$$

Since $\mathrm{Re}(e^{ir})=|e^{ir}|=1$, the above inequality reduces to

$$\left\{\begin{array}{l}\left((1-\alpha)z-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\right.\\ \left.\left[(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho\right]|a_k|\gamma^{k-1}\right)\\ -\left(\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\right.\\ \left.\left[(1+\rho)\left((k+1)(\mu w^{\lambda}+\eta)+k\right)+\alpha+\rho\right]|b_k|\gamma^{k-1}\right)\end{array}\right\}$$
$$\times\left\{\begin{array}{l}1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+\\ \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\end{array}\right\}^{-1}\ge 0 \tag{2.7}$$

In the event that the condition (2.5) does not hold, at that point the numerator in (2.7) is negative for γ adequately near 1. Henceforth there exist a, $z_0=\gamma_0$ in $(0,1)$ for which the quotient in (2.7) is negative. This negates the condition for $f_n\in\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ and thus the verification is finished.

3. Convolution

In this section, we prove that the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ is closed with reference to convolution.
For harmonic functions,

$$f_n(z) = z - \sum_{k=2}^{\infty} |\, a_k \,|\, z^k + (-1)^n \sum_{k=1}^{\infty} |\, b_k \,|\, \overline{z}^k \quad \text{and}$$

$$\mathrm{F}_n(z) = z - \sum_{k=2}^{\infty} |\, \mathrm{A}_k \,|\, z^k + (-1)^n \sum_{k=1}^{\infty} |\, \mathrm{B}_k \,|\, \overline{z}^k .$$

The convolution of f_n and F_n is given by

$$(f_n * \mathrm{F}_n)(z) = f_n(z) * \mathrm{F}_n(z) =$$

$$z - \sum_{k=2}^{\infty} |\, a_k \mathrm{A}_k \,|\, z^k + (-1)^n \sum_{k=1}^{\infty} |\, b_k \mathrm{B}_k \,|\, \overline{z}^k \tag{3.1}$$

Theorem 3.1: For $n, \eta \in N_0$, $\rho \geq 0$, $\mu, \lambda, w \geq 0$, $0 \leq \eta \leq \mu w^{\lambda}$, $0 \leq \beta \leq \alpha < 1$,

Let $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ and $F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w)$.

Then $(f_n * \mathrm{F}_n) \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w) \subset F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w)$.

Proof: We desire to demonstrate that the coefficient $f_n * \mathrm{F}_n$ satisfy the condition given in theorem (2.2).

For $F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w)$, we notice that $|\, \mathrm{A}_k \,| \leq 1$ and $|\, \mathrm{B}_k \,| \leq 1$. Now, for the convolution function $f_n * \mathrm{F}_n$, we obtain

$$\frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] |\, a_k \,|\, |\, \mathrm{A}_k \,|$$

$$+ \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] |\, b_k \,|\, |\, B_k \,|$$

$$\leq \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] |\, a_k \,|$$

$$+ \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n$$

$$\left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] |\, b_k \,|$$

$$\leq 1 .$$

Since $0 \leq \beta \leq \alpha < 1$, and $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$.

Therefore

$(f_n * \mathrm{F}_n) \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w) \subset F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w)$.

4. Convex Combination

In this section, we illustrate that the class $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ is closed with regard to convex combination of its members.

Let the functions $f_{n_i}(z)$ be defined, for $i = 1, 2, \ldots\ldots, m$ by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} |\, a_{k,i} \,|\, z^k + (-1)^n \sum_{k=1}^{\infty} |\, b_{k,i} \,|\, \overline{z}^k \tag{4.1}$$

Theorem4.1: Let the functions $f_{n_i}(z)$ defined by (4.1) be in the class $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ for every $i = 1, 2, \ldots\ldots, m$. Then the functions $t_i(z)$ defined by

$t_i(z) = \sum_{i=1}^{\infty} c_i f_{n_i}(z)$, $0 \leq c_i \leq 1$, are also in the class $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$, where $\sum_{i=1}^{\infty} c_i = 1$.

Proof: As indicated by the definition of t_i, we can write

$$t_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i \,|\, a_{k,i} \,| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i \,|\, b_{k,i} \,| \right) \overline{z}^k .$$

Further, since $f_{n_i}(z)$ are in $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$, for every $i = 1, 2, \ldots\ldots, m$, then

$$\sum_{k=2}^{\infty} \left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right]$$

$$\left(\sum_{i=1}^{\infty} c_i \,|\, a_{k,i} \,| \right)$$

$$+ \sum_{k=1}^{\infty} \left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right]$$

$$\left(\sum_{i=1}^{\infty} c_i \,|\, b_{k,i} \,| \right)$$

$$\leq \sum_{i=1}^{\infty} c_i (1-\alpha) \leq (1-\alpha) .$$

5. Extreme Points

In this section, we attain extreme points for the class $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$.

Theorem5.1: Let f_n be given by (1.4).

Then $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$,

if and only if $f_n(z) = \sum_{k=1}^{\infty} \left(\mathrm{X}_k h_k(z) + \mathrm{Y}_k g_{n_k}(z) \right)$, (5.1)

where

$h_1(z) = z$,

$h_k(z) =$

$$z - \left[\frac{(1-\alpha)z^k}{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)(k-1)(\mu w^{\lambda} - \eta) + k\right]} \right]$$

$g_{n_k}(z) =$

$$z + \left[\frac{(-1)^n (1-\alpha)\overline{z}^k}{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)(k+1)(\mu w^{\lambda} - \eta) + k\right]} \right]$$

$k = 2, 3, \ldots\ldots$, and $k = 1, 2, \ldots\ldots$, respectively, with

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0 .$$

Especially, the extreme points of $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof: For the functions f_n of the kind (5.1), we have

$$f_n(z) = \sum_{k=1}^{\infty} X_k h_k(z) + Y_k g_{n_k}(z)$$

$$= \sum_{k=1}^{\infty} X_k \left\{ z - \left(\frac{1-\alpha}{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right) z^k \right\}$$

$$+ \sum_{k=1}^{\infty} Y_k \left\{ z + \left(\frac{(-1)^n (1-\alpha)}{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right) \overline{z}^k \right\}$$

$$= z - \sum_{k=2}^{\infty} \left[\frac{1-\alpha}{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right] X_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} \left[\frac{1-\alpha}{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right] Y_k \overline{z}^k$$

Therefore

$$\frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left(\frac{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n}{\left[(\rho+\alpha) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right) |a_k|$$

$$+ \frac{1}{1-\alpha} \sum_{k=2}^{\infty} \left(\frac{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n}{\left[(\rho+\alpha) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right) |b_k|$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$$

$= 1 - X_1 \leq 1$, and also $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$.

Conversely, assume that $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$.

Setting,

$$X_k = \frac{1}{1-\alpha} \left(\frac{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n}{\left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right) |a_k|,$$

$0 \leq X_k \leq 1, k = 2, 3, \ldots\ldots$

$$Y_k = \frac{1}{1-\alpha} \left(\frac{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n}{\left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right]} \right) |b_k|,$$

$0 \leq Y_k \leq 1, k = 1, 2, \ldots\ldots$

and $X_1 = 1 - \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$.

Therefore f_n can be written as,

$$f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

$$= z - \sum_{k=2}^{\infty} \frac{(1-\alpha)X_k}{\left(\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] \right)} z^k +$$

$$(-1)^n \sum_{k=1}^{\infty} \frac{(1-\alpha)Y_k}{\left(\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] \right)} \overline{z}^k$$

$$= \sum_{k=2}^{\infty} h_k(z) X_k + \sum_{k=1}^{\infty} g_{n_k}(z) Y_k + z X_1$$

$$= \sum_{k=1}^{\infty} (h_k(z) X_k + g_{n_k}(z) Y_k), \text{ as required.}$$

This completes the proof the theorem 5.1.

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