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Research paper

Coupled points for total weakly contraction mappings via ρ-distance

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Abstract

In this paper, the total weakly contraction mappings and T-total weakly contraction mappings are defined with respect to ρ-distance. The concepts of mixed monotone and general mixed monotone are used to prove some theorems about coupled fixed points, common fixed point and coincidence points for these mappings in partially general b-metric spaces which equipped with ρ-distance.

Keywords: *Weak Contractions; Coupled Fixed Points; Coupled Coincidence Points; General Metric Spaces.*

1. Introduction

There have been a number of generalizations of usual metric space. One such generalization is b-metric spaces. Introduced by, Czerwik [14] in 1993. Several results have dealt with fixed point theory in such space such as [17],[23],[24]. In 2000, Branceciri [10] defined a generalized metric space as a metric space in which the triangle inequality is replaced by the rectangular one. And then, many authors, also, proved results in the field of metric fixed point theory such as [2],[3]. In 2006, Mustafa and Sims [21] used another modification of usual metric which known as G-metric space to prove some fixed point results. Saadati et al. [25] proved the existence of fixed point for contractive mappings in partially ordered G-metric space. Lakshmikantham et al. [25],[19] display the notion of coupled coincidence point for a mapping T from $X \times X$ into X and studied coupled fixed point theorems in partially ordered G-metric spaces. Therefore Mustafa and Sims and other researchers extended some previous results and gave new; for instance, see [5],[6],[8],[9],[13]&[4]. Recently, in 2014, Aghajani et al. [5] studied a new generalizations of b-metric and G-metric spaces, denoted by G_b-metric. Mustafa et al. [20] have obtained some coupled coincidence point theorems for, G_b-metric space. On the other hand, Kada et al. [18] introduced the concept of ɣ-distance on a metric space and proved a non-convex minimization theorem and used it to prove a generalization of Caristis fixed theorem. Gassem [15] gave a simple modification of ɣ-distance on Branceciris metric space and proved several results about existence of fixed points. Saadati et al. [25] defined an ρ-distance on a complete G-metric spaces and generalized the concept of ρ-distance in [18]. For recent results in this field, see[11],[12],[16],[22],&[27]. Throughout this work, we define a new type of weak contraction mappings on g_b-m spaces depending on ρ-distance. The mapping $G: X \times X \to X$ is called total weak contraction
If

$$a\rho\big(G(x,y),G(u,v),G(w,z)\big) + b\rho\big(G(y,x),G(v,u),G(z,w)\big) \le \mu\left(\frac{\rho(x,u,w)+\rho(y,v,z)}{2}\right) - 2\psi\big(\rho(x,u,w),\rho(y,v,z)\big)$$

with suitable conditions on a, b, ψ, μ. Here, we prove some five theorems about the existence of coupled fixed point, coupled coincidence point and coupled common fixed point.

2. Preliminaries

Definition 2-1:*[19]Let X be a non-empty set and* $\gamma: X \times X \times X \to R^+$ *be a function satisfying the following property:*

1) $\gamma(x,y,z) = 0$ if $x = y = z$.
2) $\gamma(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$.
3) $\gamma(x,x,y) \le \gamma(x,y,z)$ for all $x,y,z \in X$ with $x \neq y$.
4) $\gamma(x,y,z) = \gamma(p\{x,y,z\})$, p permutation.
5) $\gamma(x,y,z) \le s[\gamma(x,a,a) + \gamma(a,y,z)]$ for all $x,y,z,a \in X, s \ge 1$.

Then the function ɣ is called like Trihedron metric (or generalized b-metric) and the pair (X,γ) is called generalized b-metric space (shortly g_b-m space).

Definition 2-2:*[19]Let X be a* g_b*-m space, a sequence* $\{x_n\}$ *in X is said to be:*

1) ɣ-Cauchy sequence if, $\forall \varepsilon > 0$ there exists $n_0 \in N$ such that for all $m,n,i \ge n_0, \gamma(x_n,x_m,x_i) < \varepsilon$.
2) ɣ-convergent to a point $x \in X$ if for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $n,m \ge n_0$, $\gamma(x_n,x_m,x) < \varepsilon$.

Definition 2-3:*[23] Let* (X,γ) *be a* g_b*-m space and* $\rho: X \times X \times X \to R^+$*. ρ is called an ρ-distance on X iff:*

a) $\rho(x,y,z) \le \rho(x,a,a) + \rho(a,y,z)$, for all $x,y,z,a \in X$.

b) For each $x,y \in X$, $\rho(x,y,.), \rho(x,.,y): X \to R^+$ are lower semi-continuous (l.s.c).

c) $\forall \varepsilon > 0$ There is $\delta > 0$ such that $\rho(x,a,a) \le \delta$ and $\rho(a,y,z) \le \delta$ imply $\text{ɣ}(x,y,z) \le \varepsilon$.

Lemma 2-4:*[23]: Let $(X,\text{ɣ})$ be a g_b-m space and let ρ be an ρ-distance on X. Let $\{x_n\}$, $\{y_n\}$ are sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in R^+ with $lim_{n\to\infty}\alpha_n = lim_{n\to\infty}\beta_n = 0$. If $x,y,z,a \in X$ then*

1) if $\rho(y,x_n,x_n) \le \alpha_n$ and $\rho(x_n,y,z) \le \beta_n$ for $n \in N$ then $\text{ɣ}(y,y,z) < \varepsilon$ and $y = z$.

2) if $\rho(y_n,x_n,x_n) \le \alpha_n$ and $\rho(x_n,y_m,z) \le \beta_n$ for $m > n$ then $\text{ɣ}(y_n,y_m,z) \to 0$, hence $y_n \to z$.

3) if $\rho(x_n,x_m,x_i) \le \alpha_n$ for $i,n,m \in N$ with $n \le m \le i$, then $\{x_n\}$ is a ɣ-Cauchy sequence.

4) if $\rho(x_n,a,a) \le \alpha_n, n \in N$ then $\{x_n\}$ is a ɣ-Cauchy sequence.

Definition 2-5:*[24] Let X be a non-empty set, let $G: X\times X \to X$ and $T: X \to X$ be two mapping. An ordered pair $(x,y) \in X\times X$ is called:*

i) Coupled fixed point of G if $x = G(x,y)$ and $y = G(y,x)$.

ii) Coupled coincidence point of G and T if $T(x) = G(x,y)$ and $T(y) = G(y,x)$.

iii) Common coupled fixed point of G and T if $x = T(x) = G(x,y)$ and $y = T(y) = G(y,x)$.

Definition 2-6:*[1]Let $(X,<)$ be a partially ordered set, the elements x and y in X are said to be comparable elements of X if either $x < y$ or $y < x$.*

Definition 2-7:*[10] Let $(X,\text{ɣ},\preccurlyeq)$ be a partially ordered g_b-m space and $G: X\times X \to X$, G is called mixed monotone if $x_1,x_2 \in X, x_1 \preccurlyeq x_2$ implies that $G(x_1,y) \preccurlyeq G(x_2,y)$ and $y_1,y_2 \in X, y_1 \preccurlyeq y_2$ implies that $G(x,y_1) \succcurlyeq G(x,y_2)$.*

Definition 2-8:*[25]Let $(X,\text{ɣ},\preccurlyeq)$ be a partially ordered g_b-m space and $G: X\times X \to X$ and $T: X \to X$, then G is called mixed T-monotone if*

$x_1, x_2 \in X, Tx_1 \preccurlyeq Tx_2$ implies that $G(x_1,y) \preccurlyeq G(x_2,y)$ and
$y_1,y_2 \in X, Ty_1 \preccurlyeq Ty_2$ implies that $G(x,y_1) \succcurlyeq G(x,y_2)$.

Definition 2-9:*Let $(X,\text{ɣ},\le)$ be a partially ordered g_b-m space. we say that $(X,\text{ɣ},\preccurlyeq)$ is regular if the following hypotheses hold:*

i) If a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$ then $x_n \preccurlyeq x$ for all $n \in N$.

ii) If a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$ as $n \to \infty$ then $y_n \succcurlyeq y$ for all $n \in N$.

Now the following classes are needed
μ be a class of functions $\mu: R^+ \to R^+$ (R^+ is non-negative real numbers) with
1) μ is continuous and non-decreasing
2) $\mu(t) = 0$ if $t = 0$.
3) $\mu(\alpha t) \le \alpha\mu(t)$ for $\alpha \in (0,\infty)$.
4) $\mu(t+s) \le \mu(t) + \mu(s)$ for all $s,t \in [0,\infty)$.

And Ψ be a class of functions $\psi: R^+ \times R^+ \to R^+$ with

$$\lim_{\substack{t_1\to r_1\\ t_2\to r_2}} \psi(t_1,t_2) > 0 \text{ for all } (t_1,t_2) \in R^+\times R^+ \text{ with } t_1+t_2 > 0.$$

Definition 2-10:*Let $(X,\text{ɣ})$ be a g_b-m space and ρ be an ρ-distance on X, for all $a,b \in R^{++}, a+b \ge 1$, all x,y,u,v,z and $w \in X$ and $\mu \in \mu, \psi \in \Psi$. The mapping $G: X\times X \to X$ is called*

i) ρ-total weakly contraction mapping if

$$a\rho\big(G(x,y),G(u,v),G(w,z)\big) + b\rho\big(G(y,x),G(v,u),G(z,w)\big)$$
$$\le \mu\left(\tfrac{\rho(x,u,w)+\rho(y,v,z)}{2}\right) - 2\psi\big(\rho(x,u,w),\rho(y,v,z)\big) \quad (2.1)$$

For which $x \ge u \ge w$ and $y \le v \le z$.

ii) ρ-T-total weakly contraction mapping if

$$a\rho\big(G(x,y),G(u,v),G(w,z)\big) + b\rho\big(G(y,x),G(v,u),G(z,w)\big)$$
$$\le \mu\left(\tfrac{\rho(Tx,Tu,Tw)+\rho(Ty,Tv,Tz)}{2}\right) - 2\psi\big(\rho(Tx,Tu,Tw),\rho(Ty,Tv,Tz)\big) \quad (2.2)$$

For which $Tx \ge Tu \ge Tw$ and $Ty \le Tv \le Tz$.

3. Main results

We start with the following

Coupled fixed point:

Theorem 3-1:*Let $(X,\text{ɣ},\le)$ be a partially ordered complete g_b-m space and ρ be an ρ-distance on X and $G: X\times X \to X$ be a continuous ρ-total weakly contraction mapping with the mixed monotone property. If there exists $x_0,y_0 \in X$ such that $x_0 \le G(x_0,y_0)$ and $y_0 \ge G(y_0,x_0)$ then G has a coupled fixed point in X.*

Proof:

Let $x_0,y_0 \in X$ *such that* $x_0 \le G(x_0,y_0)$ *and* $y_0 \ge G(y_0,x_0)$

Define $x_1 = G(x_0,y_0)$ and $y_1 = G(y_0,x_0)$

then $x_0 \le x_1$ and $y_0 \ge y_1$ Also $x_2 = G(x_1,y_1)$ and $y_2 = G(y_1,x_1)$. Continue in the process, we construct
Two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = G(x_n,y_n) \text{ and } y_{n+1} = G(y_n,x_n), \forall n \ge 0 \quad (3.1)$$

Since G is mixed monotone property, we have

$$x_n \le x_{n+1} \text{ and } y_{n+1} \le y_n, \forall n \ge 0 \quad (3.2)$$

By condition (2.1) we get

$$a\rho(x_n,x_{n+1},x_{n+1}) + b\rho(y_n,y_{n+1},y_{n+1})$$
$$= a\rho\big(G(x_{n-1},y_{n-1}),G(x_n,y_n),G(x_n,y_n)\big) + b\rho\big(G(y_{n-1},x_{n-1}),G(y_n,x_n),G(y_n,x_n)\big)$$
$$\le \mu\left(\frac{\rho(x_{n-1},x_n,x_n)+\rho(y_{n-1},y_n,y_n)}{2}\right) - 2\psi\big(\rho(x_{n-1},x_n,x_n),\rho(y_{n-1},y_n,y_n)\big)$$

Let

$$z_{n+1}^x = \rho(x_n,x_{n+1},x_{n+1}) \text{ and } z_{n+1}^y = \rho(y_n,y_{n+1},y_{n+1}), \forall n \ge 0$$

Then $az_{n+1}^x + bz_{n+1}^y \le \mu\left(\frac{z_n^x+z_n^y}{2}\right) - 2\psi\left(z_n^x, z_n^y\right)$

$\psi(t_1,t_2) \geq 0$ for all $(t_1,t_2) \in R^+ \times R^+$, we have

$$az_{n+1}^x + bz_{n+1}^y \leq az_n^x + bz_n^y, \forall n \geq 0 \quad (3.3)$$

Then the sequence $\{az_n^x + bz_n^y\}$ is decreasing and bounded below therefore there exists $z \geq 0$ such that

$$\lim_{n\to\infty}[\, az_n^x + bz_n^y] = (a+b)z\, , z\ must\ be\ 0. To\ prove\ this$$

Suppose that, $z > 0$

The sequences $\{\rho(x_n, x_{n+1}, x_{n+1})\}\ and\ \{\rho(y_n, y_{n+1}, y_{n+1})\}$ have convergent subsequences which are

$\left\{\rho\left(x_{n_j}, x_{n_j+1}, x_{n_j+1}\right)\right\}$ And $\{\rho\left(y_{n_j}, y_{n_j+1}, y_{n_j+1}\right)\}$ respectively

Assume that $lim_{j\to\infty}\, az_{n_j+1}^x = lim_{j\to\infty}\, \rho\left(x_{n_j}, x_{n_j+1}, x_{n_j+1}\right) = az_1$

And $lim_{j\to\infty}\, bz_{n_j+1}^y = lim_{j\to\infty}\, \rho\left(y_{n_j}, y_{n_j+1}, y_{n_j+1}\right) = bz_2$

Which gives that $az_1 + bz_2 = (a+b)z$ from (3.3) we have

$$az_{n_j+1}^x + bz_{n_j+1}^y \leq \mu\left(\frac{z_{n_j}^x + z_{n_j}^y}{2}\right) - 2\psi(z_{n_j}^x, z_{n_j}^y\,)$$

Then taking the limit as $j \to \infty$ in the above inequality, we obtain

$$(a+b)z \leq \mu\left(\frac{z}{2}\right) - \lim_{j\to\infty} 2\psi(z_{n_j}^x, z_{n_j}^y) < (a+b)z\ which\ is\ contradiction, thus\ z = 0\ that\ is$$

$$lim_{n\to\infty}[\rho(x_n, x_{n+1}, x_{n+1}) + \rho(y_n, y_{n+1}, y_{n+1})] = 0 \quad (3.4)$$

Similarly $lim_{n\to\infty}[\rho(x_{n+1}, x_n, x_n) + \rho(y_{n+1}, y_n, y_n)] = 0$ (3.5)

Now, we show that $\{x_n\}$ and $\{y_n\}$ are ɣ-Cauchy sequence

Assume that at least one of $\{x_n\}$ or $\{y_n\}$ is not a ɣ-Cauchy sequence, so, there is an $\varepsilon > 0$ and $\{x_{n_k}\}$, $\{x_{m_k}\}$ subsequences of $\{x_n\}$ and $\{y_{n_k}\}, \{y_{m_k}\}$ subsequences of $\{y_n\}$ with $n_k \geq m_k \geq k$ such that

$$\rho(x_{n_k}, x_{m_k}, x_{m_k}) + \rho(y_{n_k}, y_{m_k}, y_{m_k}) \geq \varepsilon \quad (3.6)$$

$$\rho(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) + \rho(y_{n_k-1}, y_{m_k-1}, y_{m_k-1}) < \varepsilon \quad (3.7)$$

From (3.6) and (3.7) we have

$$\varepsilon \leq \rho(x_{n_k}, x_{m_k}, x_{m_k}) + \rho(y_{n_k}, y_{m_k}, y_{m_k})$$

$$\leq \rho(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + \rho(x_{n_k-1}, x_{m_k}, x_{m_k}) + \rho(y_{n_k}, y_{n_k-1}, y_{n_k-1}) + \rho(y_{n_k-1}, y_{m_k}, y_{m_k})$$

$$\leq [\rho(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + \rho(y_{n_k}, y_{n_k-1}, y_{n_k-1})] + [\rho(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) + \rho(y_{n_k-1}, y_{m_k-1}, y_{m_k-1})]$$

$$+ [\, \rho(x_{m_k-1}, x_{m_k}, x_{m_k}) + \rho(y_{m_k-1}, y_{m_k}, y_{m_k})]$$

$$< [\rho(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + \rho(y_{n_k}, y_{n_k-1}, y_{n_k-1})] + [\rho(x_{m_k-1}, x_{m_k}, x_{m_k}) + \rho(y_{m_k-1}, y_{m_k}, y_{m_k})] + \varepsilon$$

Then letting $k \to \infty$ in the above inequality and using (3.4) and (3.5), we have

$$lim_{k\to\infty}[\, \rho(x_{n_k}, x_{m_k}, x_{m_k}) + \rho(y_{n_k}, y_{m_k}, y_{m_k})] = \varepsilon \quad (3.8)$$

Where,

$$a\rho(x_{n_k}, x_{m_k}, x_{m_k}) + b\rho(y_{n_k}, y_{m_k}, y_{m_k})$$

$$\leq a\rho(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + a\rho(x_{n_k+1}, x_{m_k}, x_{m_k}) + b\rho(y_{n_k}, y_{n_k+1}, y_{n_k+1}) + b\rho(y_{n_k+1}, y_{m_k}, y_{m_k})$$

$$\leq a\rho(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + a\rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + a\rho(x_{m_k+1}, x_{m_k}, x_{m_k})$$

$$+ b\rho(y_{n_k}, y_{n_k+1}, y_{n_k+1}) + b\rho(y_{n_k+1}, y_{m_k+1}, y_{m_k+1}) + b\rho(y_{m_k+1}, y_{m_k}, y_{m_k}) \quad (3.9)$$

Since $n_k \geq m_k$, then $x_{n_k} \geq x_{m_k} and\ y_{n_k} \leq y_{m_k}$ and by (3.1)

$$a\rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + b\rho(y_{n_k+1}, y_{m_k+1}, y_{m_k+1})$$

$$= a\rho\left(G(x_{n_k}, y_{n_k}), G(x_{m_k}, y_{m_k}), G(x_{m_k}, y_{m_k})\right) + b\rho\left(G(y_{n_k}, x_{n_k}), G(y_{m_k}, x_{m_k}), G(y_{m_k}, x_{m_k})\right)$$

$$\leq \mu\left(\frac{\rho(x_{n_k}, x_{m_k}, x_{m_k}) + \rho(y_{n_k}, y_{m_k}, y_{m_k})}{2}\right)$$

$$- 2\psi\left(\rho(x_{n_k}, x_{m_k}, x_{m_k}), \rho(y_{n_k}, y_{m_k}, y_{m_k})\right) \quad (3.10)$$

In view of (3.9) and (3.10) we have

$$a\rho(x_{n_k}, x_{m_k}, x_{m_k}) + b\rho(y_{n_k}, y_{m_k}, y_{m_k}) - a\rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) - b\rho(y_{n_k+1}, y_{m_k+1}, y_{m_k+1})$$

$$\leq a\rho(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + b\rho(y_{n_k}, y_{n_k+1}, y_{n_k+1}) + a\rho(x_{m_k+1}, x_{m_k}, x_{m_k}) + b\rho(y_{m_k+1}, y_{m_k}, y_{m_k})$$

$$a\rho(x_{n_k}, x_{m_k}, x_{m_k}) + b\rho(y_{n_k}, y_{m_k}, y_{m_k}) - \mu\left(\frac{\rho(x_{n_k}, x_{m_k}, x_{m_k}) + \rho(y_{n_k}, y_{m_k}, y_{m_k})}{2}\right)$$

$$+2\psi(\rho(x_{n_k}, x_{m_k}, x_{m_k}), \rho(y_{n_k}, y_{m_k}, y_{m_k}))$$

$$\leq a\rho(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + b\rho(y_{n_k}, y_{n_k+1}, y_{n_k+1}) + a\rho(x_{m_k+1}, x_{m_k}, x_{m_k}) + b\rho(y_{m_k+1}, y_{m_k}, y_{m_k}) \quad (3.11)$$

$$2\psi(\rho(x_{n_k}, x_{m_k}, x_{m_k}), \rho(y_{n_k}, y_{m_k}, y_{m_k}))$$

$$\leq a\rho(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + b\rho(y_{n_k}, y_{n_k+1}, y_{n_k+1}) + a\rho(x_{m_k+1}, x_{m_k}, x_{m_k}) + b\rho(y_{m_k+1}, y_{m_k}, y_{m_k}) \quad (3.12)$$

From (3.8) the sequences $\{\rho(x_{n_k}, x_{m_k}, x_{m_k})\}$ and $\{\rho(y_{n_k}, y_{m_k}, y_{m_k})\}$ have subsequences converging to say $\varepsilon_1$and ε_2 respectively and $\varepsilon_1 + \varepsilon_2 = \varepsilon > 0$

We do not lose the generalization when assume that

$$\lim_{k\to\infty} \rho\left(x_{n_k}, x_{m_k}, x_{m_k}\right) = \varepsilon_1 \text{ and } \lim_{k\to\infty} \rho\left(y_{n_k}, y_{m_k}, y_{m_k}\right) = \varepsilon_2$$

Taking $k \to \infty$ in (3.11) and (3.12) we have

$$0 < \lim_{k\to\infty} 2\psi(\rho\left(x_{n_k}, x_{m_k}, x_{m_k}\right), \rho\left(y_{n_k}, y_{m_k}, y_{m_k}\right))$$

$$< \lim_{k\to\infty}[a\rho\left(x_{n_k}, x_{n_k+1}, x_{n_k+1}\right) + b\rho\left(y_{n_k}, y_{n_k+1}, y_{n_k+1}\right) + a\rho\left(x_{m_k+1}, x_{m_k}, x_{m_k}\right) + b\rho\left(y_{m_k+1}, y_{m_k}, y_{m_k}\right)]$$

$$= 0$$

Which is a contradiction
Therefore by lemma (2.4) pent (3) $\{x_n\}$ and $\{y_n\}$ are ɣ-Cauchy sequence. Since X is ɣ-complete, there exists $u, v \in X$ such that

$$\lim_{n\to\infty} x_n = u \text{ and } \lim_{n\to\infty} y_n = v$$

since $x_{n+1} = G(x_n, y_n)$ and $y_{n+1} = G(y_n, x_n)$ to gather with the continually of G, we get

$$u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} G(x_{n-1}, y_{n-1}) = G(u, v))$$

Similarly, we have

$$v = \lim_{n\to\infty} y_n = \lim_{n\to\infty} G(y_{n-1}, x_{n-1}) = G(v, u)$$

Hence (u, v) is coupled fixed point of G.
To obtain another coupled fixed point result we replace the continuity of G by regularity of X and use the following condition:
Condition (I): If u, v in X with $G(u, v) \neq u$ or $G(v, u) \neq v$ then $\inf\{\rho(x, G(x, y), u) + \rho(y, G(y, x), v) : x, y \in X\} > 0$.

Theorem 3-2: *Let $(X, ɣ, \leq)$ be regular partially ordered complete g_b-m space and ρ be an ρ-distance on X. and $G: X \times X \to X$ be a ρ-total weakly contraction mapping with the mixed monotone property. If there exists $x_0, y_0 \in X$ such that $x_0 \leq G(x_0, y_0)$ and $y_0 \geq G(y_0, x_0)$. Then G has a coupled fixed point in X.*

Proof:
By similar argument in the first part of proof of theorem (3-1) we have $x_{n+1} = G(x_n, y_n), y_{n+1} = G(y_n, x_n)$ are ɣ-Cauchy and $x_n \leq x_{n+1}, y_{n+1} \leq y_n, \forall n \geq 0$ by caupleleness, suppose that $x_n \to u$ and $y_n \to v$ by regularity $x_n \preccurlyeq u$ and $y_n \succcurlyeq v, \forall n$
Suppose $G(u, v) \neq u$ or $G(v, u) \neq v$
Now, for $\varepsilon > 0$ and by lower semi-continuity of ρ, we get

$$\rho(x_n, x_m, u) \leq lim_{p\to\infty} \inf \rho\left(x_n, x_m, x_p\right) \leq \varepsilon \quad (3.13)$$

$$\rho(y_n, y_m, v) \leq lim_{p\to\infty} \inf \rho\left(y_n, y_m, y_p\right) \leq \varepsilon \quad (3.14)$$

Considering $m = n + 1$ in (3.13) & (3.14), we get

$$\rho(x_n, G(x_n, y_n), u) + \rho(y_n, G(y_n, x_n), v) \leq 2\varepsilon$$

On the other hand, we get

$$0 < \inf\{\rho(x, G(x, y), u) + \rho(y, G(y, x), v) : x, y \in X\}$$

$$\leq \inf\{\rho(x_n, G(x_n, y_n), u) + \rho(y_n, G(y_n, x_n), v) : n \geq n_0\} \leq 2\varepsilon$$

This implies that $\inf\{\rho(x, G(x, y), u) + \rho(y, G(y, x), v) : x, y \in X\} = 0$
Which is contradiction with hypothesis, therefore $G(u, v) = u$ and $G(v, u) = v$.

Coupled coincidence point:

Theorem 3-3: *Let $(X, ɣ, \leq)$ be a partially ordered complete g_b-m space with ρ-distance, $G: X\times X \to X$ and $T: X \to X$ be commuting mappings satisfy (2.2) with the mixed T-monotone property and T, G are continuous. Suppose $G(X \times X) \subseteq TX$ and there exists $x_0, y_0 \in X$ such that $Tx_0 \leq G(x_0, y_0)$ and $Ty_0 \geq G(y_0, x_0)$ then G and T have coupled coincidence point.*

Proof:
Let $x_0, y_0 \in X$ such that $Tx_0 \leq G(x_0, y_0)$ and $Ty_0 \geq G(y_0, x_0)$
since $G(X \times X) \subseteq Tx$, we can choose $x_1, y_1 \in X$ such that $Tx_1 = G(x_0, y_0)$ and $Ty_1 = G(y_0, x_0)$.
Again from $G(X \times X) \subseteq Tx$, we can choose $x_2, y_2 \in X$ such that $Tx_2 = G(x_1, y_1)$ and $Ty_2 = G(y_1, x_1)$.
Continue in the process, we construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Tx_{n+1} = G(x_n, y_n) \text{ and } Ty_{n+1} = G(y_n, x_n), \forall n \geq 0 \quad (3.15)$$

Since G is mixed T-monotone property, we get

$$Tx_n \leq Tx_{n+1} \text{ and } Ty_n \geq Ty_{n+1} \quad (3.16)$$

By contraction (2.2) we get

$$a\rho(Tx_{n+1}, Tx_n, Tx_n) + b\rho(Ty_{n+1}, Ty_n, Ty_n)$$

$$= a\rho\left(G(x_n, y_n), G(x_{n-1}, y_{n-1}), G(x_{n-1}, y_{n-1})\right) + b\rho\left(G(y_n, x_n), G(y_{n-1}, x_{n-1}), G(y_{n-1}, x_{n-1})\right)$$

$$\leq \mu\left(\frac{\rho(Tx_n, Tx_{n-1}, Tx_{n-1}) + \rho(Ty_n, Ty_{n-1}, Ty_{n-1})}{2}\right) - 2\psi\left(\rho(Tx_n, Tx_{n-1}, Tx_{n-1}), \rho(Ty_n, Ty_{n-1}, Ty_{n-1})\right)$$

Let

$z_{n+1}^x = \rho(Tx_{n+1}, Tx_n, Tx_n)$ and $z_{n+1}^y = \rho(Ty_{n+1}, Ty_n, Ty_n), \forall n \geq 0$, then

$$az_{n+1}^x + bz_{n+1}^y \leq \mu\left(\frac{z_n^x + z_n^y}{2}\right) - 2\psi\,(z_n^x, z_n^y)$$

As $\psi(t_1, t_2) \geq 0$ for all $(t_1, t_2) \in R^+ \times R^+$, we have

$$az_{n+1}^x + bz_{n+1}^y \leq az_n^x + bz_n^y, \forall n \geq 0 \quad (3.17)$$

Then the sequence $\{az_n^x + bz_n^y\}$ is decreasing and bounded below therefore there exists $z \geq 0$ such that

$$\lim_{n\to\infty}[az_n^x + bz_n^y] = (a + b)z$$

Suppose that $z > 0$ the
es $\{\rho(Tx_{n+1}, Tx_n, Tx_n)\}$ and $\{\rho(Ty_{n+1}, Ty_n, Ty_n)\}$ have convergent subsequences

$\{\rho\left(Tx_{n_j+1}, Tx_{n_j}, Tx_{n_j}\right)\}$ and $\{\rho\left(Ty_{n_j+1}, Ty_{n_j}, Ty_{n_j}\right)\}$ respectively

Assume that

$$\lim_{j\to\infty} az_{n_j}^x = \lim_{j\to\infty} \rho\left(Tx_{n_j+1}, Tx_{n_j}, Tx_{n_j}\right) = az_1 \text{ and } \lim_{j\to\infty} bz_{n_j}^y = \lim_{j\to\infty} \rho\left(Ty_{n_j+1}, Ty_{n_j}, Ty_{n_j}\right) = bz_2$$

Which gives that $az_1 + bz_2 = (a + b)z$
From (3.17) we have

$$az_{n_j+1}^x + bz_{n_j+1}^y \le \mu\left(\frac{z_{n_j}^x + z_{n_j}^y}{2}\right) - 2\psi(z_{n_j}^x, z_{n_j}^y)$$

Then taking the limit as $j \to \infty$ in the above inequality, we obtain

$$(a+b)z \le \mu\left(\frac{z}{2}\right) - \lim_{j\to\infty} 2\psi(z_{n_j}^x, z_{n_j}^y) < (a+b)z \text{ which is contradiction, thus } z = 0 \text{ that is}$$

$$lim_{n\to\infty}[\rho(Tx_{n+1}, Tx_n, Tx_n) + \rho(Ty_{n+1}, Ty_n, Ty_n)] = 0 \quad (3.18)$$

Similarly

$$lim_{n\to\infty}\begin{bmatrix}\rho(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ +\rho(Ty_n, Ty_{n+1}, Ty_{n+1})\end{bmatrix} = 0 \quad (3.19)$$

Now, we show that $\{Tx_n\}$ and $\{Ty_n\}$ are ɣ-Cauchy sequence

Assume that at least one of $\{Tx_n\}$ or $\{Ty_n\}$ is not a ɣ-Cauchy sequence, so, there is an $\varepsilon > 0$ and $\{Tx_{n_k}\}$, $\{Tx_{m_k}\}$Subsequences of $\{Tx_n\}$ and $\{Ty_{n_k}\}$,$\{Ty_{m_k}\}$ subsequences of $\{Ty_n\}$ with $n_k \ge m_k \ge k$ such that

$$\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}) \ge \varepsilon \quad (3.20)$$

$$\rho(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}) + \rho(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1}) < \varepsilon \quad (3.21)$$

From (3.20) and (3.21) we have

$$\varepsilon \le \rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})$$

$$\le \rho(Tx_{n_k}, Tx_{n_k-1}, Tx_{n_k-1}) + \rho(Tx_{n_k-1}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{n_k}, Ty_{n_k-1}, Ty_{n_k-1}) + \rho(Ty_{n_k-1}, Ty_{m_k}, Ty_{m_k})$$

$$\le [\rho(Tx_{n_k}, Tx_{n_k-1}, Tx_{n_k-1}) + \rho(Ty_{n_k}, Ty_{n_k-1}, Ty_{n_k-1})] + [\rho(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1})$$

$$+\rho(Ty_{n_k-1}, Ty_{m_k-1}, Ty_{m_k-1})] + [\rho(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{m_k-1}, Ty_{m_k}, Ty_{m_k})]$$

$$\le [\rho(Tx_{n_k}, Tx_{n_k-1}, Tx_{n_k-1}) + \rho(Ty_{n_k}, Ty_{n_k-1}, Ty_{n_k-1})] + [\rho(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k})$$

$$+\rho(Ty_{m_k-1}, Ty_{m_k}, Ty_{m_k})] + \varepsilon$$

Then letting $k \to \infty$ in the above inequality and using (3.18) and (3.19), we have

$$lim_{k\to\infty}[\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})] = \varepsilon \quad (3.22)$$

Where,

$$a\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + b\rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})$$

$$\le a\rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + a\rho(Tx_{n_k+1}, Tx_{m_k}, Tx_{m_k}) + b\rho(Ty_{n_k}, Ty_{n_k+1}, Ty_{n_k+1})$$

$$+b\rho(Ty_{n_k+1}, Ty_{m_k}, Ty_{m_k})$$

$$\le a\rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + a\rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1}) + a\rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

$$+b\rho(Ty_{n_k}, Ty_{n_k+1}, Ty_{n_k+1}) + b\rho(Ty_{n_k+1}, Ty_{m_k+1}, Ty_{m_k+1}) + b\rho(Ty_{m_k+1}, Ty_{m_k}, Ty_{m_k}) \quad (3.23)$$

Since $n_k \ge m_k$ then $x_{n_k} \ge x_{m_k} and\ y_{n_k} \le y_{m_k}$ and by (3.15)

$$a\rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1}) + b\rho(Ty_{n_k+1}, Ty_{m_k+1}, Ty_{m_k+1})$$

$$= a\rho\left(G(x_{n_k}, y_{n_k}), G(x_{m_k}, y_{m_k}), G(x_{m_k}, y_{m_k})\right) + b\rho\left(G(y_{n_k}, x_{n_k}), G(y_{m_k}, x_{m_k}), G(y_{m_k}, x_{m_k})\right)$$

$$\le \mu\left(\frac{\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})}{2}\right) -$$

$$2\psi\left(\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})\right) \quad (3.24)$$

In view of (3.23) and (3.24) we have

$$a\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + b\rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}) - a\rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1})$$

$$-b\rho(Ty_{n_k+1}, Ty_{m_k+1}, Ty_{m_k+1})$$

$$\le a\rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + b\rho(Ty_{n_k}, Ty_{n_k+1}, Ty_{n_k+1}) + a\rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

$$+b\rho(Ty_{m_k+1}, Ty_{m_k}, Ty_{m_k})$$

$$a\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + b\rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}) - \mu\left(\frac{\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) + \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})}{2}\right)$$

$$+2\psi(\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}))$$

$$\le a\rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + b\rho(Ty_{n_k}, Ty_{n_k+1}, Ty_{n_k+1}) + a\rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

$$+b\rho(Ty_{m_k+1}, Ty_{m_k}, Ty_{m_k})$$

$$2\psi(\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}))$$

$$\le a\rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + b\rho(Ty_{n_k}, Ty_{n_k+1}, Ty_{n_k+1}) + a\rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

$$+b\rho(Ty_{m_k+1}, Ty_{m_k}, Ty_{m_k}) \quad (3.25)$$

From (3.22) the sequences $\{\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k})\}$ and $\{\rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k})\}$ have subsequences converging to say $\varepsilon_1$and ε_2 respectively and $\varepsilon_1 + \varepsilon_2 = \varepsilon > 0$

We do not lose the generalization when assume that

$$\lim_{k\to\infty} \rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) = \varepsilon_1 \text{ and } \lim_{k\to\infty} \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}) = \varepsilon_2$$

Taking $k \to \infty$ in (3.24) and (3.25) we have

$$0 < \lim_{k\to\infty} 2\psi(\rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), \rho(Ty_{n_k}, Ty_{m_k}, Ty_{m_k}))$$

$$< \lim_{k\to\infty}[a\rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + b\rho(Ty_{n_k}, Ty_{n_k+1}, Ty_{n_k+1}) + a\rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

$$+b\rho(Ty_{m_k+1}, Ty_{m_k}, Ty_{m_k})] = 0$$

Which is a contradiction

Therefore by lemma (2.4) part (3) $\{Tx_n\}$ and $\{Ty_n\}$ are ɣ-Cauchy sequence since X is ɣ-complete, there exists $u, v \in X$ such that

$$\lim_{n\to\infty} Tx_n = u \text{ and } \lim_{n\to\infty} Ty_n = v$$

And since T continuous then there exists Tx and Ty such that

$$\lim_{n\to\infty} T(Tx_n) = Tu \text{ and } \lim_{n\to\infty} T(Ty_n) = Tv$$

$$\text{since } Tx_{n+1} = G(x_n, y_n) \text{ and } Ty_{n+1} = G(y_n, x_n)$$

Together with the continually of G and since G and T commutative, we have

$$Tu = \lim_{n\to\infty} T(Tx_{n+1}) = \lim_{n\to\infty} T(G(x_n, y_n)) = \lim_{n\to\infty} G(Tx_n, Ty_n)) = G(u, v)$$

Similarly, we have

$$Tv = \lim_{n\to\infty} T(Ty_{n+1}) = \lim_{n\to\infty} T(G(y_n, x_n)) = \lim_{n\to\infty} G(Ty_n, Tx_n)) = G(v, u)$$

Hence (u, v) is coupled coincidence point of G and T.
To obtain another coupled coincidence point result we replace the continuity of G by regularity of X and completeness of X by completeness of TX also employ the following condition:

Condition(II): If u, v in X with $G(u, v) \neq Tu$ or $G(v, u) \neq Tv$ then

$$\inf\{\rho(Tx, G(x, y), Tu) + \rho(Ty, G(y, x), Tv) : x, y \in X\} > 0.$$

Theorem 3-4: *Let $(X, \gamma, \leq)$ be regular partially ordered g_b-m space with ρ-distance, G:X×X→X and T:X→X be mappings satisfy (2.2) with the mixed T-monotone property and TX is complete. Suppose $G(X \times X) \subseteq TX$ and there exists $x_0, y_0 \in X$ such that $Tx_0 \leq G(x_0, y_0)$ and $Ty_0 \geq G(y_0, x_0)$ then G and T have coupled coincidence point.*

Proof:
By similar argument in the first part of proof of theorem (3-3) we have $Tx_{n+1} = G(x_n, y_n), Ty_{n+1} = G(y_n, x_n)$ are ɣ-Cauchy and $Tx_n \leq Tx_{n+1}, Ty_{n+1} \leq Ty_n, \forall n \geq 0$ by completeness of TX, suppose that $Tx_n \to Tu$ and $Ty_n \to Tv$, u and v in X. By regularity $Tx_n \preccurlyeq Tu$ and $Ty_n \succcurlyeq Tv, \forall n$
Suppose $G(u, v) \neq Tu$ or $G(v, u) \neq Tv$
Now, for $\varepsilon > 0$ and by lower semi-continuity of ρ, we get

$$\rho(Tx_n, Tx_m, Tu) \leq \lim_{p\to\infty} \inf \rho(Tx_n, Tx_m, Tx_p) \leq \varepsilon \tag{3.26}$$

$$\rho(Ty_n, Ty_m, Tv) \leq \lim_{p\to\infty} \inf \rho(Ty_n, Ty_m, Ty_p) \leq \varepsilon \tag{3.27}$$

Considering $m = n + 1$ in (3.26) and (3.27), we get

$$\rho(Tx_n, G(x_n, y_n), Tu) + \rho(Ty_n, G(y_n, x_n), Tv) \leq 2\varepsilon$$

On the other hand, we get

$$0 < \inf\{\rho(Tx, G(x, y), Tu) + \rho(Ty, G(y, x), Tv) : x, y \in X\}$$

$$\leq \inf\{\rho(Tx_n, G(x_n, y_n), Tu) + \rho(Ty_n, G(y_n, x_n), Tv) : n \geq n_0\} \leq 2\varepsilon$$

This implies that

$$\inf\{\rho(Tx, G(x, y), Tu) + \rho(Ty, G(y, x), Tv) : x, y \in X\} = 0$$

Which is contradiction with hypothesis, therefore

$$G(u, v) = Tu \text{ and } G(v, u) = Tv.$$

Coupled common fixed point:

Theorem 3-5: *Adding the hypothesis of theorem (3-3), suppose that for all $(u, v), (u^*, v^*) \in X \times X$ there exists $(h, r) \in X \times X$ such that $(G(h, r), G(r, h))$ is comparable with $\big(G(u, v), G(v, u)\big)$ and $(G(u^*, v^*), G(v^*, u^*))$ then G and T have a unique coupled common fixed point.*

Proof:
From theorem (3-3) the set of coupled coincidence is non-empty
Assume that (u, v) and (u^*, v^*) are coupled coincidence point of G and T

We shall show that $Tu = Tu^*$ and $Tv = Tv^*$ (3.28)

By assumption there exists $(h, r) \in X \times X$ such that $(G(h, r), G(r, h))$ is comparable with $\big(G(u, v), G(v, u)\big)$ and $(G(u^*, v^*), G(v^*, u^*))$
Putting $h_0 = h$ and $r_0 = r$ and choosing $h_1, r_1 \in X$ such that $Th_1 = G(h_0, r_0)$ and $Tr_1 = G(r_0, h_0)$
We can inductively define sequences $\{Th_n\}$ and $\{Tr_n\}$ in X by $Th_{n+1} = G(h_n, r_n)$ and $Tr_{n+1} = G(r_n, h_n), \forall n$ since $\big(G(u^*, v^*), G(v^*, u^*)\big) = (Tu^*, Tv^*)$ and $\big(G(h, r), G(r, h)\big) = (Th_1, Tr_1)$ are comparable, we may assume that

$$\big(G(u^*, v^*), G(v^*, u^*)\big) = (Tu^*, Tv^*) \leq \big(G(h, r), G(r, h)\big) = (Th_1, Tr_1)$$

And

$$\big(G(u, v), G(v, u)\big) = (Tu, Tv) \leq \big(G(h, r), G(r, h)\big) = (Th_1, Tr_1)$$

$$\text{This means that } Tu^* \leq Th_1, Tv^* \geq Tr_1 \text{ and } Tu \leq Th_1, Tv \geq Tr_1$$

Using the fact that G is mixed T-monotone mapping we can inductively show that

$$Tu^* \leq Th_n, Tv^* \geq Tr_n \text{ and } Tu \leq Th_n, Tv \geq Tr_n, \forall n \geq 1$$

Thus from (2.2) we get

$$a\rho\big(G(h_n, r_n), G(u, v), G(u, v)\big) + b\rho\big(G(r_n, h_n), G(v, u), G(v, u)\big)$$
$$\leq \mu\left(\frac{\rho(Th_n, Tu, Tu) + \rho(Tr_n, Tv, Tv)}{2}\right) - 2\psi\big(\rho(Th_n, Tu, Tu), \rho(Tr_n, Tv, Tv)\big) \tag{3.29}$$

Which implies that

$$a\rho(Th_{n+1}, Tu, Tu) + b\rho(Tr_{n+1}, Tv, Tv)$$
$$\leq a\rho(Th_n, Tu, Tu) + b\rho(Tr_n, Tv, Tv)$$

That is the sequences $\{\rho(Th_n, Tu, Tu) + \rho(Tr_n, Tv, Tv)\}$ is decreasing therefore there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} [\rho(Th_n, Tu, Tu) + \rho(Tr_n, Tv, Tv)] = \delta$$

Suppose $\delta > 0$ therefore

$$\rho(Th_n, Tu, Tu) \text{ and } \rho(Tr_n, Tv, Tv) \text{ have subsequences converging to } \delta_1, \delta_2 \text{ respectively with } \delta_1 + \delta_2 = \delta > 0$$

Taking the limit up to subsequences as $n \to \infty$ in (3.29) we have

$$\delta \leq \delta - \lim_{n\to\infty} 2\psi\,[\rho(Th_n, Tu, Tu), \rho(Tr_n, Tv, Tv)]$$

Which is a contradiction. Thus $\delta = 0$ that is

$$\lim_{n\to\infty}[\rho(Th_n,Tu,Tu)+\rho(Tr_n,Tv,Tv)]=0$$

Which implies that

$$lim_{n\to\infty}\,\rho(Th_n,Tu,Tu)=lim_{n\to\infty}\,\rho(Tr_n,Tv,Tv)=0 \qquad (3.30)$$

Similarly

$$lim_{n\to\infty}\,\rho(Tu,Th_n,Tu)=lim_{n\to\infty}\,\rho(Tv,Tr_n,Tv)=0 \qquad (3.31)$$

Taking into account (3.30) and (3.31) and the lemma (2.4) pent (1), we get

$$Tu=Th_n \text{ and } Tv=Tr_n \qquad (3.32)$$

Similarly we can show that

$$Tu^*=Th_n \text{ and } Tv^*=Tr_n \qquad (3.33)$$

Using (3.32) and (3.33), we get

$$Tu=Tu^* \text{and } Tv=Tv^*$$

$$Since\ Tu=G(u,v)\ and\ Tv=G(v,u)$$

By commuting of G and T we have

$$T(Tu)=T\big(G(u,v)\big)=G(Tu,Tv)\ and\ T(Tv)=T\big(G(v,u)\big)=G(Tv,Tu) \qquad (3.34)$$

Denote $Tu=z\ and\ Tv=w$

We get

$$Tz=G(z,w)\ and\ Tw=G(w,z) \qquad (3.35)$$

Thus (z,w) is coincidence point

$$Then\ form\ (3.28)\ with\ Tu^*=Tz\ and\ Tv^*=Tw$$

$$We\ have\ Tu=Tz\ and\ Tv=Tw$$

$$That\ is\ Tz=z\ and\ Tw=w \qquad (3.36)$$

From (3.35) and (3.36) we get

$$Tz=G(z,w)=z\ and\ Tw=G(w,z)=w \qquad (3.37)$$

Then (z,w) is a coupled common fixed point of G and T.
To prove the uniqueness
Assume that (p,q) is another coupled fixed point, then by (3.37) we have

$$Tz=Tp=z\ and\ Tw=Tq=w$$

$$then\ p=z\ and\ q=w.$$

The following remark refer to some corollaries of theorem (3-1).

Remark 3-6: *As special cases of condition (2.1) we get:*

1) $if a=1,b=0,\mu=kt, where\ k\in(0,1), T=I_X(\ identity\ mapping)\ and\ \psi(t_1,t_2)=0$

$$\rho\big(G(x,y),G(u,v),G(w,z)\big)\le k\left(\frac{\rho(x,u,w)+\rho(y,v,z)}{2}\right).$$

2) $if\ a=1,b=0,\psi(t_1,t_2)=0$

$$\rho\big(G(x,y),G(u,v),G(w,z)\big)\le \mu\left(\frac{\rho(x,u,w)+\rho(y,v,z)}{2}\right).$$

3) $if\ a=1,b=1,\mu=2kt\ for\ k\in\left[0,\frac{1}{2}\right)\ and\ \psi(t_1,t_2)=0$

$$\rho\big(G(x,y),G(u,v),G(w,z)\big)+\rho\big(G(y,x),G(v,u),G(z,w)\big)\le 2k\left(\frac{\rho(x,u,w)+\rho(y,v,z)}{2}\right).$$

4) $if\ a=1,b=0,\mu=2t$

$$\rho\big(G(x,y),G(u,v),G(w,z)\big)\le 2\left(\frac{\rho(x,u,w)+\rho(y,v,z)}{2}\right).$$

5) $if\ a=1,b=0,T=I_X(identity\ mapping),\mu=t$

$$\rho\big(G(x,y),G(u,v),G(w,z)\big)\le \left(\frac{\rho(x,u,w)+\rho(y,v,z)}{2}\right).$$

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