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Research paper

Soliton-like solutions of nonlinear scalar and electromagnetic field equations in gravitational theory

A. E. Yamadjako[1,2,4*], **A. Adomou** [1,4] **and A. C. Houngan**[2,3]

[1] *Department of Physics, University of Abomey-Calavi, Abomey-Calavi, Bénin*
[2] *Department of Industrial and Technical Sciences, ENSET-Lokossa, UNSTIM-Abomey, Abomey, Bénin*
[3] *Laboratory of Thermophysical Characterization of Matérials and Energy Appropriation (Labo-C.T.M.E.), Polytechnic School of Abomey-Calavi (EPAC), UAC, Bénin*
[4] *National Higher Institute of Industrial Technology, National University of Sciences, Technologies, Engineering and Mathematics of Abomey, Abomey, Bénin*
[*] *Corresponding author Email: arnouard@gmail.com*

Abstract

In this work, exact analytical static spherical symmetric solutions to the nonlinear interacting electromagnetic and massless scalar fields equation have been determined taking into account the proper gravitational field of elementary particles. The obtained results prove that all solutions of the Einstein equation and those of the interacting fields are regular with localized energy density. All non-zero components of the 4-vector potential are solutions to the inverted Painlevé-Gambier XI equation. Moreover, the total energy of interacting fields is limited and the total charge of elementary particles is finite. These solutions obtained are soliton-like and can be used as a model to describe the internal structure of elementary particles.

Keywords: *Interaction, scalar, electromagnetic, gravitational fields; description, configuration, elementary particles.*

1. Introduction

An elementary particle is a particle whose internal structure (composition) remains unknown. In the Standard Model (SM), the Grand Unification Theories (GUT) and the Super Symmetry Super Strings (SUSY), the theory of gravitation is absent. From experimental results, elementary particles are extended objects with a complex internal spatial structure. Similarly, the proper gravitational field of elementary particles has an infinite extent. The theories or models (SM, GUT, SUSY) do not solve the problem of the description of the structure of elementary particles, nor of the importance of their proper gravitational fields. This description can only be complete within the framework of the theory of the interaction fields and in quantum mechanics [1,2]. In particle physics, soliton-like solutions to the nonlinear differential field equations are used as models to describe the complex spatial structure of elementary particles [3–12]. Motivated by the importance to the proper gravitational field of elementary particles and thirsty for knowing their internal structures, many works have been done. Shikin [13,14] explored the basis of soliton physics. Rybakov et al. [15–18] established and discussed respectively, exact droplet solutions for the interaction equation to the scalar and electromagnetic fields in different configurations; soliton-like solutions for the equation of the electromagnetic, scalar and proper gravitational fields in spherical and cylindrical symmetric metrics. They especially studied the particular case where the solution of the Liouville equation is taken in the form $S(k,\xi)=\xi$. Bronnikov et al. [19–21] obtained static cylindrical symmetric solutions to the nonlinear Einstein scalar field equation by defining the regularity conditions at the axes and by selecting soliton-like solutions. They also examined the solutions to the electromagnetic, scalar and proper gravitational field equations and determined particular, static, spherical symmetric solutions on the basis of predefined strong and weak criteria, related to singularity, regularity and locality. In the above works related to soliton-like solutions, the 4-vector potential is reduced only to its electric scalar potential and the used function $S(k,\xi)$, lead to gravitational singularities at the level of some metric functions. From the works of Rybakov et al. [22,23] relating to the solutions of the electromagnetic and scalar or spinor field equation of massive induction to explain the phenomenon of the expansion of the universe, it is exploited the 4-vector potential $A(0,\ A_1(t),\ A_2(t),\ A_3(t))$.

In this research work, it is determined exact analytical static spherical symmetric solutions to the nonlinear interacting electromagnetic, massless scalar and proper gravitational fields equation induced by the 4-vector potential $A(A_0(\xi),0,A_2(\xi),A_3(\xi))$. The objective is to prove that the extension of the 4-potential vector to its magnetic components and the consideration of a massless scalar field, would allow to soliton-like solutions, not containing the function or solution of Liouville's equation $S(k,\xi)$, which is often the cause of the observed gravitational singularities. To achieve this aim, Section 2 gives a brief overview of the fundamental equations. The results obtained are presented in Section 3 accompanied by a discussion. In Section 4, a conclusion is presented.

2. Fundamental equations

The static spherical symmetric metric is defined by the following expression:

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}d\xi^2 - e^{2\beta}\left[d\theta^2 + \sin^2(\theta)d\varphi^2\right], \tag{1}$$

where the functions α, β and γ depend only on the harmonic radial coordinate ξ defined between $0 < \xi < \xi_c$ [24]. These functions respect [25]:

$$\alpha = 2\beta + \gamma. \tag{2}$$

In general relativity, the Einstein equation is defined by:

$$G_\mu^\nu = -\chi T_\mu^\nu, \tag{3}$$

where G_μ^ν is Einstein's tensor; χ is Einstein's gravitational constant and T_μ^ν is the energy-momentum metric tensor.
From Eq.(1), Eq.(2) and Eq.(3), the non-zero components of Einstein tensor equation are [26–28] :

$$G_0^0 = e^{-2\alpha}(2\beta'' - 2\gamma'\beta' - \beta'^2) - e^{-2\beta} = -\chi T_0^0 \tag{4}$$

$$G_1^1 = e^{-2\alpha}(2\gamma'\beta' + \beta'^2) - e^{-2\beta} = -\chi T_1^1 \tag{5}$$

$$G_2^2 = e^{-2\alpha}(\beta'' + \gamma'' - 2\gamma'\beta' - \beta'^2) = -\chi T_2^2 \tag{6}$$

$$G_2^2 = G_3^3 \tag{7}$$

$$T_2^2 = T_3^3 \tag{8}$$

where $(')$ denotes the first derivative with respect to ξ.
The Lagrangian of the nonlinear scalar and electromagnetic fields taking into account the proper gravitational field of elementary particles is chosen in the form:

$$L = \frac{R}{2\chi} - \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\varphi_{,i}\varphi^{,i}\psi(I) \tag{9}$$

where $I = A_iA^i$ is the invariant; $A(A_0(\xi), 0, A_2(\xi), A_3(\xi))$ denotes the 4-vector potential; $\psi(I) = 1 + \lambda\phi(I)$ is some function characterizing the interaction between the nonlinear electromagnetic and scalar fields and λ represents the parameter of the nonlinearity.
The nonlinear scalar and electromagnetic field equations corresponding to the Lagrangian Eq.(9) are:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\xi^\nu}\left[\sqrt{-g}g^{\nu\mu}\varphi_{,\mu}\psi(I)\right] = 0 \tag{10}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\xi^\mu}\left[\sqrt{-g}F^{\nu\mu}\right] - \varphi_{,i}\varphi^{,i}\psi_I(I)A^\nu = 0. \tag{11}$$

Solving the scalar field equation Eq.(10), we obtain the particular solution:

$$\frac{d\varphi}{d\xi} = \frac{1}{\psi(I)} = P(I). \tag{12}$$

Developing the equation of nonlinear electromagnetic field Eq.(11), we find:

$$\begin{cases} (e^{-2\gamma}A_0')' - e^{-2\gamma}P_I(I)A_0 = 0, & \text{(a)} \\ A_1 = 0, & \text{(b)} \\ (e^{-2\beta}A_2')' - e^{-2\beta}P_I(I)A_2 = 0, & \text{(c)} \\ (e^{-2\beta}A_3')' - e^{-2\beta}P_I(I)A_3 = 0 & \text{(d).} \end{cases} \tag{13}$$

The invariant I turns into:

$$I = e^{-2\gamma}A_0^2 + e^{-2\beta}\left(A_2^2 + \frac{A_3^2}{\sin^2(\theta)}\right). \tag{14}$$

Defined in the form:

$$T_{\mu}^{\nu} = \varphi_{,\mu}\varphi^{,\nu}\psi(I) - F_{\mu i}F^{\nu i} + \varphi_{,i}\varphi^{,i}\psi_I(I)A^{\nu}A_{\mu} - \delta_{\mu}^{\nu}\left[-\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}(\varphi_{,i}\varphi^{,i})\psi(I)\right], \tag{15}$$

the energy-momentum metric tensor has the following non-zero components:

$$T_0^0 = \frac{1}{2}e^{-2\alpha}\left[P(I) + e^{-2\gamma}A_0'^2 + e^{-2\beta}\left(A_2'^2 + \frac{A_3'^2}{\sin^2(\theta)}\right) + 2e^{-2\gamma}P_I(I)A_0^2\right]; \tag{16}$$

$$T_1^1 = -\frac{1}{2}e^{-2\alpha}\left[P(I) - e^{-2\gamma}A_0'^2 + e^{-2\beta}\left(A_2'^2 + \frac{A_3'^2}{\sin^2(\theta)}\right)\right]; \tag{17}$$

$$T_2^2 = \frac{1}{2}e^{-2\alpha}\left[P(I) - e^{-2\gamma}A_0'^2 - e^{-2\beta}\left(A_2'^2 - \frac{A_3'^2}{\sin^2(\theta)}\right) - 2e^{-2\beta}P_I(I)A_2^2\right]; \tag{18}$$

$$T_3^3 = \frac{1}{2}e^{-2\alpha}\left[P(I) - e^{-2\gamma}A_0'^2 + e^{-2\beta}\left(A_2'^2 - \frac{A_3'^2}{\sin^2(\theta)}\right) - 2e^{-2\beta}P_I(I)\frac{A_3^2}{\sin^2(\theta)}\right]; \tag{19}$$

$$T_0^2 = -e^{-2(\alpha+\beta)}\left[A_0'A_2' + P_I(I)A_2A_0\right]; \tag{20}$$

$$T_0^3 = -\frac{e^{-2(\alpha+\beta)}}{\sin^2(\theta)}\left[A_0'A_3' + P_I(I)A_3A_0\right]; \tag{21}$$

$$T_2^3 = -\frac{e^{-2(\alpha+\beta)}}{\sin^2(\theta)}\left[A_2'A_3' + P_I(I)A_3A_2\right]. \tag{22}$$

The energy-momentum metric tensor being symmetric, the relations Eq.(20), Eq.(21) and Eq.(22) allow to establish:

$$\frac{A_0'A_2'}{A_0A_2} = \frac{A_2'A_3'}{A_2A_3} = \frac{A_0'A_3'}{A_0A_3} = -P_I(I). \tag{23}$$

From Eq.(23), we get:

$$\begin{cases} A_2 = k_2A_0, & k_2 = const \\ A_3 = k_3A_0, & k_3 = const \end{cases} \tag{24}$$

and

$$\begin{cases} A_0'^2 = -P_I(I)A_0^2, \\ A_2'^2 = -P_I(I)A_2^2, \\ A_3'^2 = -P_I(I)A_3^2. \end{cases} \tag{25}$$

Using Eq.(24) and Eq.(25), the expressions Eq.(16), Eq.(17), Eq.(18) and Eq.(19) reduce to:

$$T_0^0 = -T_1^1 = T_2^2 = T_3^3 = \frac{1}{2}e^{-2\alpha}\left[P(I) + IP_I(I)\right]. \tag{26}$$

Summing Eq.(5) and Eq.(6), we obtain the Liouville type equation [25]:

$$(\beta+\gamma)'' = e^{2(\beta+\gamma)}. \tag{27}$$

Adding Eq.(4) and Eq.(5), we find:

$$\beta'' = e^{2(\beta+\gamma)}. \tag{28}$$

The relation Eq.(28) introduced in Eq.(27) leads to:

$$\gamma(\xi)'' = 0, \tag{29}$$

which has as a solution:

$$\gamma(\xi) = C\xi + C_0. \tag{30}$$

For $C = C_0 = 0$, we have:

$$\gamma = 0. \tag{31}$$

The relation (31) is equivalent to one obtained with a massless scalar field. Under this condition, Eq.(28) admits as solution:

$$\beta(\xi)=\ln\left(\frac{\eta}{\sinh(\eta\xi+\tau_0)}\right),\quad \eta>0 \quad and \quad \tau_0>0. \tag{32}$$

From Eq.(2), Eq.(31) and Eq.(32), we derive the expressions of metric functions:

$$g_{00}(\xi)=1, \tag{33}$$

$$g_{11}(\xi)=-\frac{\eta^4}{\sinh^4(\eta\xi+\tau_0)}, \tag{34}$$

$$g_{22}(\xi)=\frac{g_{33}(\xi)}{\sin^2(\theta)}=-\frac{\eta^2}{\sinh^2(\eta\xi+\tau_0)}. \tag{35}$$

Putting Eq.(33) into Eq.(13) (a), we have:

$$A_0''-P_I(I)A_0=0. \tag{36}$$

Multiplying Eq.(36) by A_0, we obtain:

$$A_0A_0''-P_I(I)A_0^2=0. \tag{37}$$

From Eq.(25) and Eq.(37), we deduce the differential equation:

$$A_0A_0''+A_0'^2=0. \tag{38}$$

The expression Eq.(38) can be rewritten in the form:

$$A_0''+\frac{1}{A_0}A_0'^2=0. \tag{39}$$

The equation (39) represents the inverted Painlevé-Gambier XI equation whose solution was recently obtained by Koudahoun et al. [29]. The general form of this solution is:

$$A_0(\xi)=\sqrt{q_0+q_1\xi},\quad q_0=const \quad and \quad q_1=\text{const}. \tag{40}$$

From Eq.(24) and Eq.(40), we express:

$$A_0(\xi)=\frac{A_2}{k_2}=\frac{A_3}{k_3}=\sqrt{q_0+q_1\xi},\quad A_1=0. \tag{41}$$

From Eq.(25), Eq.(33), Eq.(34), Eq.(35) and Eq.(41), the invariant Eq.(14) and the energy density Eq.(26) transform into:

$$I(\xi)=(q_0+q_1\xi)\left[1+\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\frac{\sinh^2(\eta\xi+\tau_0)}{\eta^2}\right], \tag{42}$$

$$T_0^0(\xi)=\left[P(I)+(q_0+q_1\xi)P_I(I)\left[1+\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\frac{\sinh^2(\eta\xi+\tau_0)}{\eta^2}\right]\right]\frac{\sinh^4(\eta\xi+\tau_0)}{2\eta^4}. \tag{43}$$

The energy density per unit invariant volume $T(\xi)$ and the total energy of interacting fields E_f defined by:

$$T(\xi)=T_0^0(\xi)\sqrt{{}^3_-g}, \tag{44}$$

$$E_f=\int_0^{\xi_c}T_0^0\sqrt{{}^3_-g}\,d\xi \tag{45}$$

become:

$$T(\xi)=\frac{1}{2}\left[P(I)+(q_0+q_1\xi)P_I(I)\left[1+\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\frac{\sinh^2(\eta\xi+\tau_0)}{\eta^2}\right]\right]\sin(\theta), \tag{46}$$

$$E_f=\int_0^{\xi_c}\left[P(I)+(q_0+q_1\xi)P_I(I)\left[1+\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\frac{\sinh^2(\eta\xi+\tau_0)}{\eta^2}\right]\right]\frac{\sin(\theta)}{2}d\xi. \tag{47}$$

From Eq.(11), we derive the expression of the 4-vector potential j^ν [18]:

$$j^\nu=-\varphi_{,i}\varphi^{,i}\psi_I(I)A^\nu. \tag{48}$$

Taking into account Eq.(48), we obtain:

$$j^0 = -P_I(I)\frac{\sinh^4(\eta\xi+\tau_0)}{\eta^4}\sqrt{q_0+q_1\xi}, \quad (49)$$

$$j^1 = 0, \quad (50)$$

$$j^2 = P_I(I)k_2\frac{\sinh^6(\eta\xi+\tau_0)}{\eta^6}\sqrt{q_0+q_1\xi}, \quad (51)$$

$$j^3 = P_I(I)\frac{k_3^2}{\sin^2(\theta)}\frac{\sinh^6(\eta\xi+\tau_0)}{\eta^6}\sqrt{q_0+q_1\xi}. \quad (52)$$

The charge density $\rho_e(\xi)$, the charge density per unit invariant volume $\rho(\xi)$ and the total charge of elementary particles Q of expressions:

$$\rho_e(\xi) = \frac{j^0}{\sqrt{g^{00}}}, \quad (53)$$

$$\rho(\xi) = \rho_e(\xi)\sqrt{{}^3\_g} \quad (54)$$

and

$$Q = \int_0^{\xi_c}\rho(\xi)d\xi \quad (55)$$

verify:

$$\rho_e(\xi) = -P_I(I)\frac{\sinh^4(\eta\xi+\tau_0)}{\eta^4}\sqrt{q_0+q_1\xi}, \quad (56)$$

$$\rho(\xi) = -P_I(I)\sqrt{q_0+q_1\xi}\sin(\theta), \quad (57)$$

$$Q = -\int_0^{\xi_c}P_I(I)\sqrt{q_0+q_1\xi}\sin(\theta)d\xi. \quad (58)$$

The expressions established in this section depend mostly on the concrete form of the calibrated invariance function $P(I)$. We opt in the following section for:

$$P(I) = -\lambda I. \quad (59)$$

3. Results and discussions

The metric functions $(g_{00}(\xi), g_{11}(\xi), g_{22}(\xi), g_{33}(\xi))$ and the components of the 4-vector potential are all regular and independent of the calibrated invariance function but also of the solution of Liouville's equation $S(k,\xi)$. These metric functions are equivalent to a particular case of the Fisher metric having source is a massless scalar field and corresponding to a zero Schwarzschild mass. This equivalence also follows from the relation Eq.(26) of the energy-momentum tensor which is the same as that known for a massless scalar field $\varphi(\xi)$
Fig 1 gives a graphical illustration of the asymptotic properties to these functions.
Substituting Eq.(59) into Eq.(42), Eq.(46), Eqs.(49)-(52) and Eqs.(56) -(58), we obtain respectively:

$$T_0^0(\xi) = -\lambda\,(q_0+q_1\xi)\left[1+\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\frac{\sinh^2(\eta\xi+\tau_0)}{\eta^2}\right]\frac{\sinh^4(\eta\xi+\tau_0)}{\eta^4}, \quad (60)$$

$$T(\xi) = -\lambda\,(q_0+q_1\xi)\left[1+\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\frac{\sinh^2(\eta\xi+\tau_0)}{\eta^2}\right]\sin(\theta), \quad (61)$$

$$j^0 = \lambda\frac{\sinh^4(\eta\xi+\tau_0)}{\eta^4}\sqrt{q_0+q_1\xi}, \quad (62)$$

$$j^1 = 0, \quad (63)$$

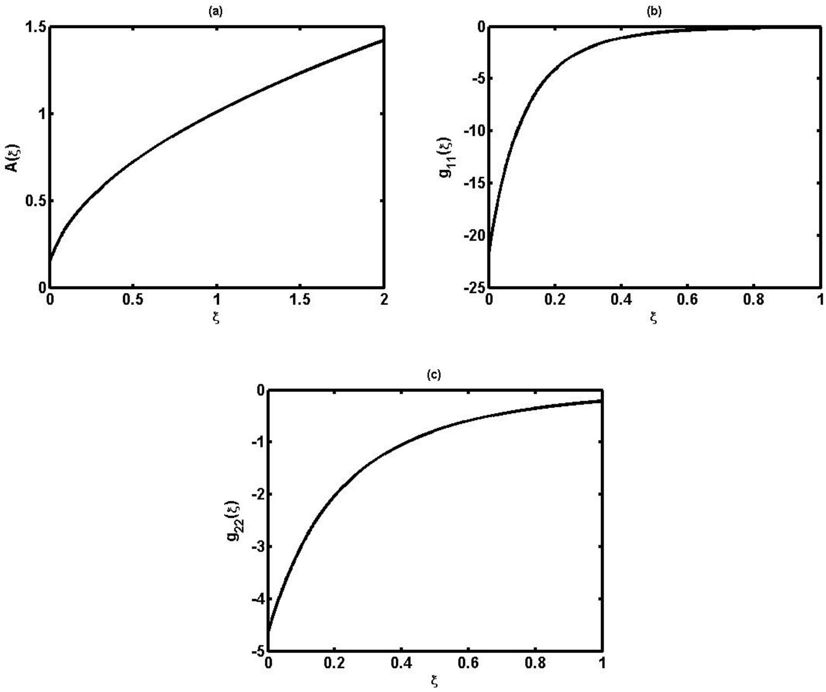

Figure 1: (a) - Electric scalar potential $A(\xi)$, (b) - Component $g_{11}(\xi)$ of the metric tensor, (c) - Component $g_{22}(\xi)$ of the metric tensor. The values of the parameters used for this simulation are: $\eta = 1.1255$; $\tau_0 = 0.5$, $\lambda = 10$, $q_0 = 0.02$; $q_1 = 1$ and $\theta = \pi/2$

.

$$j^2 = -\lambda k_2 \frac{\sinh^6(\eta\xi+\tau_0)}{\eta^6}\sqrt{q_0+q_1\xi}, \tag{64}$$

$$j^3 = -\lambda \frac{k_3^2}{\sin^2(\theta)} \frac{\sinh^6(\eta\xi+\tau_0)}{\eta^6}\sqrt{q_0+q_1\xi}, \tag{65}$$

$$\rho(\xi) = \lambda\sqrt{q_0+q_1\xi}\,\frac{\sinh^4(\eta\xi+\tau_0)}{\eta^4}, \tag{66}$$

$$\rho(\xi) = \lambda\sqrt{q_0+q_1\xi}\,\sin(\theta), \tag{67}$$

$$Q = \frac{2\lambda}{3q_1}\left[(q_0+q_1\xi_c)^{3/2} - q_0^{3/2}\right]. \tag{68}$$

The expressions Eqs.(60)-(67) are regular and asymptotic functions. In Fig.2 a numerical simulation is presented showing the properties of the components Eq.(60), Eq.(61), Eq.(62), Eq.(64) and Eq.(67). In Fig 2 (b), we note that the energy density $T_0^0(\xi)$ is a localized function and varies asymptotically as follows:

$$T_0^0(\xi)\begin{cases} -\lambda q_0\left[1+\left(k_2^2+\dfrac{k_3^2}{\sin^2(\theta)}\right)\dfrac{\sinh^2(\tau_0)}{\eta^2}\right]\dfrac{\sinh^4(\tau_0)}{\eta^4}, & \xi \longrightarrow 0 \\ -\lambda\left(q_0+q_1\xi_c\right)\left[1+\left(k_2^2+\dfrac{k_3^2}{\sin^2(\theta)}\right)\dfrac{\sinh^2(\eta\xi_c+\tau_0)}{\eta^2}\right]\dfrac{\sinh^4(\eta\xi_c+\tau_0)}{\eta^4}, & \xi \longrightarrow \xi_c \end{cases}. \tag{69}$$

The relation (68) shows that the total charge of elementary particles is a finite quantity. From Eq.(47) and Eq.(59), the total energy of interacting fields is written:

$$E_f = -\lambda\left[q_0\xi_c + \frac{1}{2}q_1\xi_c^2 + \frac{1}{\eta^2}\left(k_2^2+\frac{k_3^2}{\sin^2(\theta)}\right)\left[q_0E_{f_1}+q_3E_{f_2}\right]\right]\sin(\theta) \tag{70}$$

where

$$E_{f_1} = \int_0^{\xi_c}\sinh^2(\eta\xi+\tau_0)d\xi$$

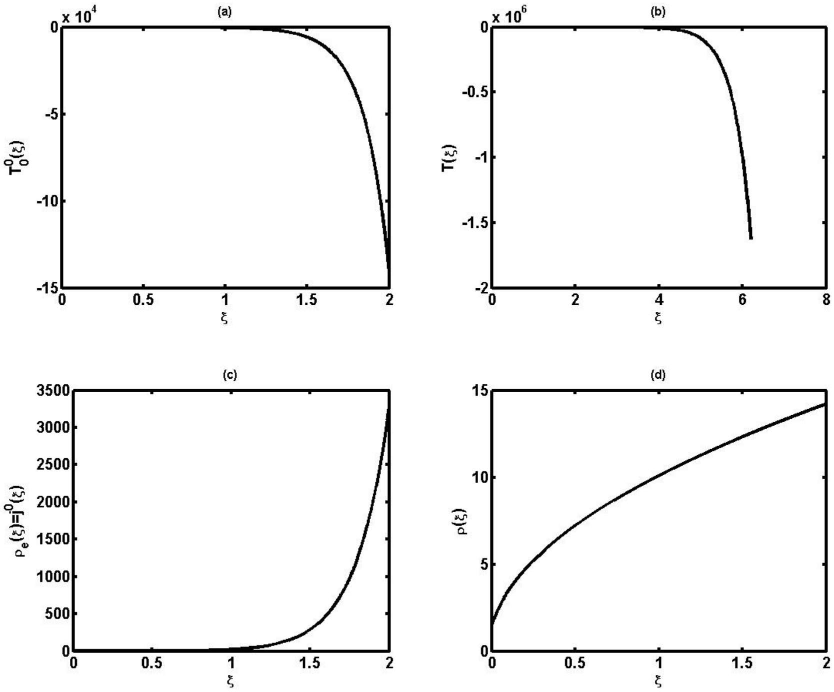

Figure 2: (a) - Energy density $T_0^0(\xi)$, (b) - Energy density per unit volume, (c) - Charge density $\rho_e(\xi)$, (d) - Charge density per unit volume. The values of the parameters used for this simulation are: $\eta = 1.1255$; $k_2 = 0.05$; $k_3 = 0.2$; $\tau_0 = 0.5$, $\lambda = 10$, $q_0 = 0.02$; $q_1 = 1$ and $\theta = \pi/2$

.

and

$$E_{f_2} = \int_0^{\xi_c} \xi \sinh^2(\eta\xi + \tau_0) d\xi.$$

Putting $x = \eta\xi + \tau_0$ and using the appropriate relations of the standard table of integrals [30], we obtain:

$$E_{f_1} = \frac{1}{\eta}\left[\frac{\sinh\left[2(\eta\xi_c + \tau_0)\right]}{4} - \frac{\sinh(2\tau_0)}{4} - \frac{\eta\xi_c}{2}\right], \tag{71}$$

$$E_{f_2} = \frac{1}{\eta^2}\left[\eta\xi_c\frac{\sinh\left[2(\eta\xi_c + \tau_0)\right]}{4} + \frac{\cosh(2\tau_0)}{8} - \frac{\cosh\left[2(\eta\xi_c + \tau_0)\right]}{4} - \frac{\eta^2\xi_c^2}{4}\right]. \tag{72}$$

From Eq.(70), Eq.(71) and Eq.(72), the total energy of interacting fields E_f is limited.

4. Conclusion

In this work, the equation to the nonlinear interacting electromagnetic, massless scalar and proper gravitational fields induced by the 4-vector potential $A(A_0(\xi), 0, k_2A_0(\xi), k_3A_0(\xi))$ has been solved with the consideration of proper gravitational field of elementary particles. The results obtained show that with the calibrated invariance function $P(I) = -\lambda I$, the metric functions, solutions of Einstein equation as well as those of considered fields are all regular, independent of the function $S(k,\xi)$, with localized energy density. The total energy of interacting fields is limited and the total charge of elementary particles is a finite quantity. These analytical solutions are soliton-like and constitute a model capable of describing the complex internal structure of elementary particles. In perspective, we will study the particular case of a direct interaction of fields using the Born-Infeld Lagrangien and the invariant $I = F_{ij}F^{ij}$.

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