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Research Paper

On the Kumaraswamy Kumaraswamy distribution

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Abstract

In this paper, new distribution so called the Kumaraswamy – Kumaraswamy (KW-KW) distribution, as a Special model from the class of Kumaraswamy Generalized (KW-G) distributions, is introduced. the probability density function (pdf), the cumulative distribution function (cdf), moments, quantiles, the median, the mode, the mean deviation, the entropy, order statistics, L-moments and parameters estimation based on maximum likelihood are obtained. A numerical illustration is used to studying the properties of the parameters.

***Keywords**: Kumaraswamy Generalized Distributions, Moments, Order Statistics, L-Moments, Entropy, Maximum Likelihood Estimation.*

1. Introduction

Cordeiroa and Castro (2010) presented the class of KW-G distributions where the CDF of the class of KW-G distributions was presented as follows:

$$F(x) = 1 - \left\{1 - G(x)^a\right\}^b \quad ; a, b, > 0, -\infty < x < \infty \tag{1}$$

and the pdf is:

$$f(x) = ab\, g(x)\, G(x)^{a-1} \left\{1 - G(x)^a\right\}^{b-1} \quad ; a, b > 0, -\infty < x < \infty \tag{2}$$

Then, if ***a*** is real non-integer, an expansion of density function of the class of KW-G distributions will be

$$f(x) = g(x) \sum_{i,j=0}^{\infty} \sum_{k=0}^{j} w_{i,j,k}\, G(x)^k$$

Where,

$$w_{i,j,k}(a,b) = (-1)^{i+j+k}\, ab \binom{a(i+1)-1}{j} \binom{b-1}{i} \binom{j}{k}$$

Simplified expansion of $F(x)^r$ was used, as follows:

$$F(x)^r = \sum_{q=0}^{\infty} d_{q,r}(a,b)\, G(x_u)^q$$

Where,

$$d_{q,r}(a,b) = \sum_{w=0}^{r} \binom{r}{w} (-1)^w \sum_{m=0}^{\infty} \sum_{p=q}^{\infty} (-1)^{m+p+q} \binom{wb}{m} \binom{ma}{p} \binom{p}{q}$$

It is noted that if b > 0 is integer, the index *i* stops at *b-1* and if *a* > 0 is integer the index *j* stops at *a (i+1)-1*

2. The Kumaraswamy Kumaraswamy distribution

The CDF and PDF of the Kumaraswamy (KW) distribution with two shape parameters $\alpha > 0$ and $\beta > 0$, respectively are:

$$G(x) = 1 - \left(1 - x^{\alpha}\right)^{\beta} \;\; ; \; 0 < x < 1 ; \; \alpha, \beta > 0 \tag{3}$$

and

$$g(x) = \alpha \beta x^{\alpha - 1}\left(1 - x^{\alpha}\right)^{\beta - 1} \;\; ; \; 0 < x < 1 ; \; \alpha, \beta > 0 \tag{4}$$

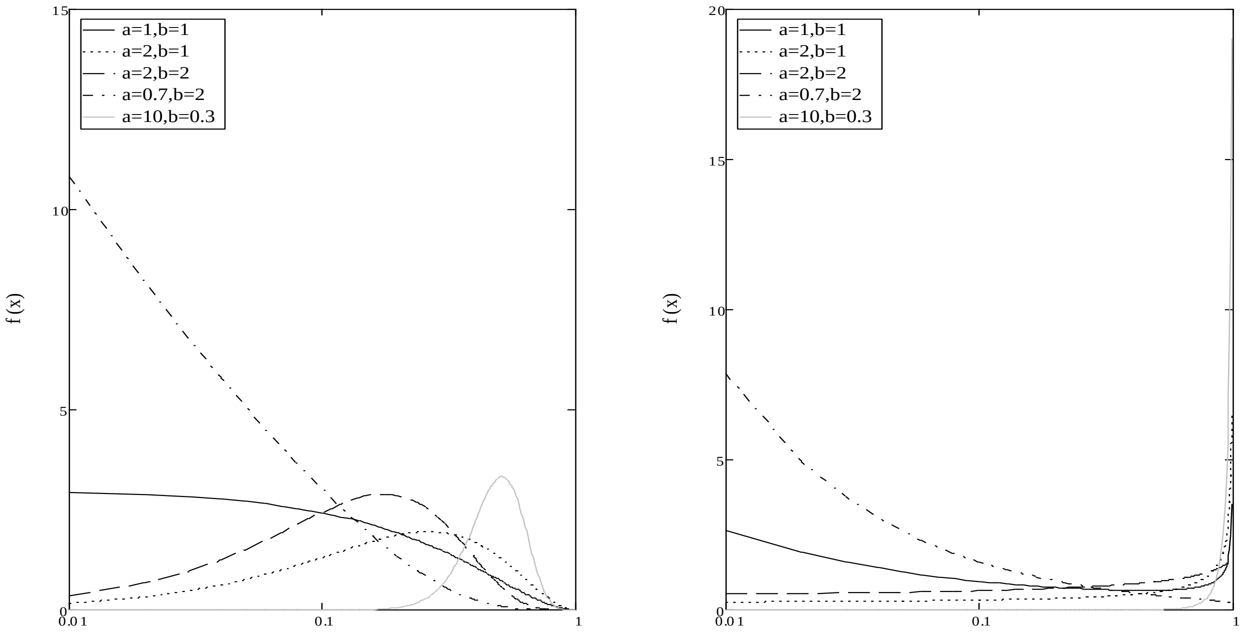
Substituting from equation (3) and (4) into equation (1), yields cumulative distribution function of the KW-KW distribution.

$$F(x) = 1 - \left\{ 1 - \left[1 - \left(1 - x^{\alpha}\right)^{\beta} \right]^{a} \right\}^{b} \;\; ; 0 < x < 1 ; \; a, b, \alpha, \beta > 0 \tag{5}$$

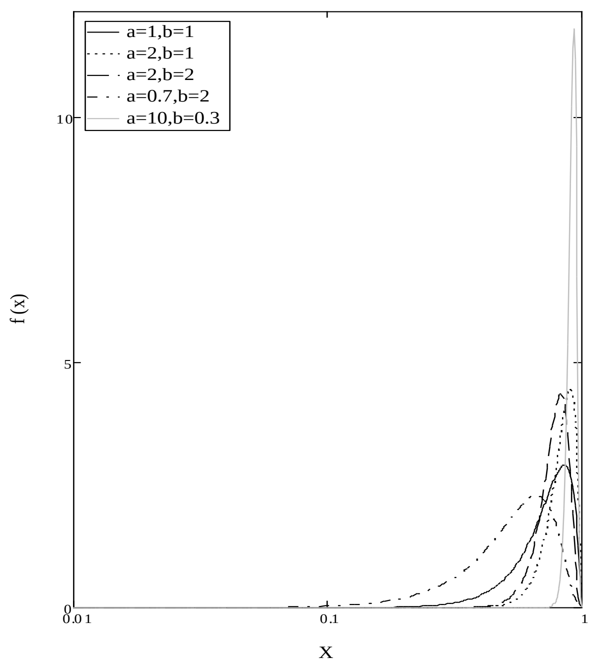
Substituting from equation (3) and (4) into equation (2), yields probability density function of the KW-KW distribution.

$$f(x) = ab\,\alpha \beta x^{\alpha - 1}\left(1 - x^{\alpha}\right)^{\beta - 1} \left[1 - \left(1 - x^{\alpha}\right)^{\beta} \right]^{a-1} \left\{ 1 - \left[1 - \left(1 - x^{\alpha}\right)^{\beta} \right]^{a} \right\}^{b-1} \;\; ; 0 < x < 1 ; \; a, b, \alpha, \beta > 0 \tag{6}$$

It is noted that, at *a*=1 and b=1 equation (5) and (6) give the CDF and PDF of the ordinary KW distribution and when b=1 they give the CDF and PDF of the exponentiated KW distribution, see Cordeiroa *et al.* (2013).



(i) **(ii)**



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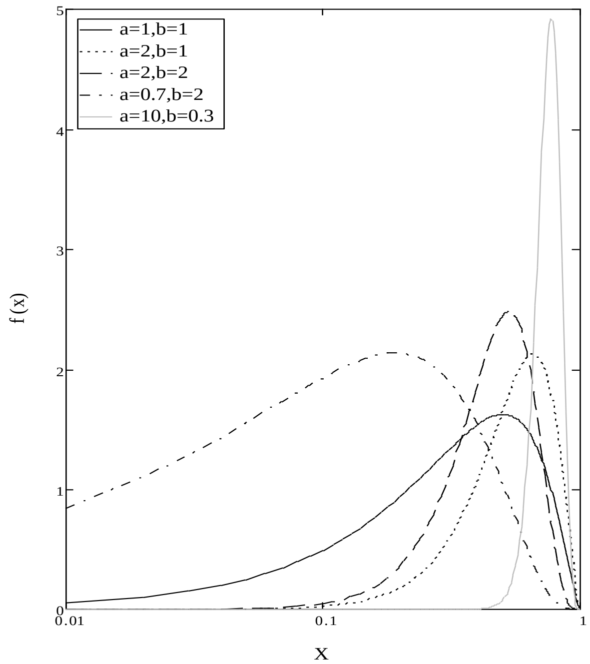

(iv)

Fig. 1: KW-KW Densities of Selected Values of α and β; (i) α =1 and β = 3; (ii) α =0.5 and β = 0.5; (iii) α =2 and β = 2.5; (iv) α =5 and β = 2

A General Expansion of the KW-KW Density Function:

A general expansion of the KW-KW density function can be yielded, if *a* is real non-integer using binomial Expansion yields:

$$f(x) = ab\alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{b-1}{i}\left[1-\left(1-x^{\alpha}\right)^{\beta}\right]^{a(1+i)-1}$$

Then using the following expansion in the last equation:

$$\left(1-z\right)^{c-1} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\,\Gamma(c)}{\Gamma(c-j)\,j\,!}z^{j}\,;\left|z\right|<1 \tag{7}$$

Will be as follows:

$$f(x) = \sum_{i,j=0}^{\infty} w_{i,j}\, x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta(1+j)-1} \tag{8}$$

Where,

$$w_{i,j} = ab\alpha\beta\frac{\left(-1\right)^{i+j}}{j\,!}\binom{b-1}{i}\frac{\Gamma(a(1+i))}{\Gamma(a(1+i)-j)}$$

and

$$\sum_{i,j=0}^{\infty} w_{i,j}\,\alpha^{-1}\,B\left(1,\beta(1+j)\right) = 1 \tag{9}$$

It is noted that if b > 0 is integer, the index i stops at b-1

3. Some properties of the KW- KW distribution

In this section some properties of the KW- KW distribution will be calculated as follows:

3.1. Moments

If X has the pdf (6), then its r^{th} Moment can be given simply from (8) as follows:

$$\mu^{r} = \sum_{i,j=0}^{\infty} w_{i,j}\int_{0}^{1} x^{r+\alpha-1}\left(1-x^{\alpha}\right)^{\beta(1+j)-1}\,dx$$

Then,

$$\mu^{r} = \sum_{i,j=0}^{\infty} w_{i,j}\,\alpha^{-1} B\left(\frac{r}{\alpha}+1, \beta(1+j)\right)$$

We can see that $\mu^{0} = \sum_{i,j=0}^{\infty} w_{i,j}\,\alpha^{-1} B\left(1, \beta(1+j)\right)$ and from equation (9), $\mu^{0} = 1$,

$\mu = \sum_{i,j=0}^{\infty} w_{i,j}\,\alpha^{-1} B\left(\frac{1}{\alpha}+1, \beta(1+j)\right)$ and $\mu^{2} = \sum_{i,j=0}^{\infty} w_{i,j}\,\alpha^{-1} B\left(\frac{2}{\alpha}+1, \beta(1+j)\right)$

3.2. The moment generating function

The moment generating function of the KW-KW distribution can be got by:

$$M_{x}(t) = \sum_{i,j=0}^{\infty} w_{i,j} \int_{0}^{1} e^{tx}\, x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta(1+j)-1} dx$$

using epansion in equation (7) yieids:

$$M_{x}(t) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_{0}^{1} e^{tx}\, x^{\alpha(k+1)-1}\, dx$$

Then,

$$M_{x}(t) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{1}{\alpha(1+k)} 1F1(\alpha(1+k), \alpha(1+k)+1, t)$$

Where,
$1F1$ is the confluent hypergeometric function; see Prudnikov *et al.* (1986).
And

$$w_{i,j,k} = ab\,\alpha\,\beta \frac{(-1)^{i+j+k}}{j!\,k!} \binom{b-1}{i} \frac{\Gamma(a(1+i))}{\Gamma(a(1+i)-j)} \frac{\Gamma(\beta(1+j))}{\Gamma(\beta(1+j)-k)}$$

3.3. Quantile function and the median

Starting with the well-known definition of the 100 q-th quantile, it is clear that

$$q = P(X \le x_q) = F(x_q)\,;\; x_q > 0,\, 0 < q < 1$$

Quantiles of the KW-KW distribution can be yielded by equating the cdf with q:

$$q = 1 - \left\{ 1 - \left[1 - \left(1 - x^{\alpha}\right)^{\beta} \right]^{a} \right\}^{b} \;;\; 0 < x < 1\,;\; a, b, \alpha, \beta > 0$$

Then,

$$x_q = \sqrt[\alpha]{1 - \sqrt[\beta]{1 - \sqrt[a]{1 - \sqrt[b]{1-q}}}}$$

Obviously, by substituting q by 1/2 (the second quartile) the median will be got. So, the median is

$$x = \sqrt[\alpha]{1 - \sqrt[\beta]{1 - \sqrt[a]{1 - \sqrt[b]{0.5}}}}$$

3.4. The mode

The mode of the KW-KW distribution can be got by:
Differentiating the normal logarithm of equation (7) with respect to x as follows:

$$\frac{d\left[\log f(x)\right]}{d}=\frac{\alpha-1}{x}-\frac{\alpha(\beta-1)x^{\alpha-1}}{1-x^{\alpha}}+\frac{\alpha\beta(a-1)x^{\alpha-1}(1-x^{\alpha})^{\beta-1}}{1-(1-x^{\alpha})^{\beta}}$$

$$-\frac{a\alpha\beta(b-1)x^{\alpha-1}(1-x^{\alpha})^{\beta-1}(1-(1-x^{\alpha})^{\beta})^{a-1}}{1-(1-(1-x^{\alpha})^{\beta})^{a}}=0$$

, but the last equation is a nonlinear equation and does not have an analytic solution with respect to x therefore, we have to solve it numerically, If x_0 is a root of the last equation it must be $f''(x_0)<0$.

3.5. The mean deviation

Basically, the mean deviation is a measure of the amount of scatter in X; generally, the mean deviation about the mean and about the median is expressed respectively by

$$\delta_1(x)=\int_{-\infty}^{\infty}\left|x-\mu\right|f(x)dx \quad \text{and} \quad \delta_2(x)=\int_{-\infty}^{\infty}\left|x-M\right|f(x)dx$$

Where, μ is the mean and M is the median.
These measures can be expressed easily by:

$$\delta_1(X)=2\mu F(\mu)-2T(\mu) \quad \textit{and} \quad \delta_2(X)=\mu-2T(M)$$

Where, $T(q)=\int_{-\infty}^{q} x\,f(x)\,dx$

From equation (8) the last equation will be:

$$T(q)=\sum_{i,j=0}^{\infty} w_{i,j}\int_{0}^{q} x^{\alpha}(1-x^{\alpha})^{\beta(j+1)-1}dx$$

Then,

$$T(q)=\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{q^{\alpha(1+k)+1}}{\alpha(1+k)+1}$$

4. Order statistics and L-moments

A simple random sample X_1, X_2,..,X_w given from the KW-KW distribution where X´s are i.i.d random variables, see Arnold *et al.*(1992), The density $f_{u:w}(x_u)$ of the u-th order statistic, for u = 1,2,......,w is given by:

$$f_{u:w}(x_u)=\frac{f(x_u)}{B(u,w-u+1)}F(x_u)^{u-1}\left\{1-F(x_u)\right\}^{w-u}$$

Using binomial expansion and the expansion in equation (7), yields:

$$f_{u:w}(x_u)=\frac{1}{B(u,w-u+1)}\sum_{k,l,m,n=0}^{\infty}h_{k,l,m,n}\sum_{i,j=0}^{\infty}w_{i,j}\,x_u^{\alpha-1}\left(1-x_u^{\alpha}\right)^{\beta(1+j)-1} \tag{10}$$

Where,

$$w_{i,j}=ab\alpha\beta\frac{(-1)^{i+j}}{j!}\binom{b-1}{i}\frac{\Gamma(a(1+i))}{\Gamma(a(1+i)-j)}$$

and

$$h_{k,l,m,n}=(-1)^{k+l+m+n}\binom{w-u}{k}\binom{u+k-1}{l}\binom{bl}{m}\frac{\Gamma(am+1)}{n!\,\Gamma(am-n+1)}$$

Moments of Order Statistics

Basically, moments of order statistics, see Arnold *et al.* (1992), is defined by:

$$E_{u:w}(X_u^r)=\int_{-\infty}^{\infty}x_u^r f_{u:w}(x_u)dx_u$$

Substituting from equation (10) into the last equation yields:

$$E_{u:w}(X_u^r)=\frac{1}{B(u,w-u+1)}\sum_{k,l,m,n=0}^{\infty}h_{k,l,m,n}\sum_{i,j=0}^{\infty}w_{i,j}\int_{0}^{1}x_u^{\alpha+r-1}\left(1-x_u^{\alpha}\right)^{\beta(1+j+n)-1}dx_u$$

Then,

$$E_{u,w}(X_u^r)=\frac{1}{B(u,w-u+1)}\sum_{k,l,m,n=0}^{\infty}h_{k,l,m,n}\sum_{i,j=0}^{\infty}w_{i,j}\frac{B(1+\frac{r}{\alpha},\beta(1+j+n))}{\alpha}$$

L-Moments

L-moments are linear functions of expected order statistics, and L-moments is defined by, see Hosking (1990):

$$\lambda_{\theta+1}=(\theta+1)^{-1}\sum_{t=0}^{\theta}(-1)^t\binom{\theta}{t}E(x_{\theta+1-t:\theta+1}),\theta=0,1,....$$

The first four L-moments are:

$$\lambda_1=E(X_{1:1}),\ \lambda_2=\frac{1}{2}E(X_{2:2}-2X_{1:2}),\ \lambda_3=\frac{1}{3}E(X_{3:3}-2X_{2:3}+X_{1:3})$$

$$\lambda_4=\frac{1}{4}E(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4})$$

Therefore,

$$\lambda_1=\sum_{k,l,m,n=0}^{\infty}h_{k,l,m,n}\sum_{i,j=0}^{\infty}w_{i,j}\frac{1}{\alpha}B(1+\frac{1}{\alpha},\beta(1+j+n))$$

Where,

$$w_{i,j}=ab\alpha\beta\frac{(-1)^{i+j}}{j!}\binom{b-1}{i}\frac{\Gamma(a(1+i))}{\Gamma(a(1+i)-j)}$$

and

$$h_{k,l,m,n}=(-1)^{k+l+m+n}\binom{k}{l}\binom{bl}{m}\frac{\Gamma(am+1)}{n!\,\Gamma(am-n+1)}$$

Similarly, λ_2, λ_3 and λ_n can be yielded

5. Rényi entropy

According to Meeker and Escobar (1998), the entropy of a random variable is a measure of variation of the uncertainty, the entropy has been used in various situations in science and engineering, the Rényi entropy is defined by:

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\int_{-\infty}^{\infty}f(x)^{\rho}dx\right]$$

Substituting from equation (6) into the last equation yields:

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\int_0^1(ab\alpha\beta)^{\rho}x^{\rho(\alpha-1)}\left(1-x^{\alpha}\right)^{\rho(\beta-1)}\left[1-\left(1-x^{\alpha}\right)^{\beta}\right]^{\rho(a-1)}\left\{1-\left[1-\left(1-x^{\alpha}\right)^{\beta}\right]^{a}\right\}^{\rho(b-1)}dx\right]$$

Using binomial expansion and the expansion in equation (7) yields:

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[(ab\alpha\beta)^{\rho}\sum_{i,j=0}^{\infty}\binom{\rho(b-1)}{i}\frac{(-1)^{i+j}}{j!}\frac{\Gamma(\rho(a-1)+ai+1)}{\Gamma(\rho(a-1)+ai+1-j)}\int_0^1x^{\rho(\alpha-1)}\left(1-x^{\alpha}\right)^{\rho(\beta-1)+j\beta}\left(1-x^{\alpha}\right)^{\beta}dx\right]$$

Then,

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\frac{(ab\alpha\beta)^{\rho}}{\alpha}\sum_{i,j=0}^{\infty}\binom{\rho(b-1)}{i}\frac{(-1)^{i+j}}{j!}\frac{\Gamma(\rho(a-1)+ai+1)}{\Gamma(\rho(a-1)+ai+1-j)}\mathrm{B}\left(\frac{\rho(\alpha-1)+1}{\alpha},\rho(\beta-1)+j\beta+1\right)\right]$$

6. Parameters estimation

The purpose of the present section is to estimate the KW-KW distribution parameters based on maximum likelihood method.

Given the iid random variables X_1, X2… Xn from the KW-KW (a, b, α, β) distribution, Henceforce, let θ is the parameters vector, then The likelihood function will be:

$$L(\theta; x_i) = \left(ab\,\alpha\,\beta\right)^n \prod_{i=1}^{n}\left(x_i^{\alpha-1}\right)\left(1-x_i^{\alpha}\right)^{\beta-1} \prod_{i=1}^{n}\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a-1} \prod_{i=1}^{n}\left\{1-\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b-1}$$

Hence, the log likelihood function will be:

$$\mathrm{l}(\theta; x) = n\log\left(ab\,\alpha\,\beta\right) + \sum_{i=1}^{n}\log\left(x_i^{\alpha-1}\right)\left(1-x_i^{\alpha}\right)^{\beta-1} + \sum_{i=1}^{n}\log\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a-1} + \sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b-1}$$

Differentiating $\lambda(\theta; x)$ with respect to *a* yields:

$$\frac{\partial \mathrm{l}(\theta; x)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right] - \sum_{i=1}^{n}\frac{\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a}}{1-\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a}}(b-1)\log\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right] \tag{11}$$

Differentiating $\lambda(\theta; x)$ with respect to b yields:

$$\frac{\partial \mathrm{l}(\theta; x)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left[1-\left[1-\left(1-x_i^{\alpha}\right)^{\beta}\right]^{a}\right] \tag{12}$$

Differentiating $\lambda(\theta; x)$ with respect to α yields:

$$\frac{\partial \mathrm{l}(\theta; x)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{(\beta-1)}{(1-x_i^{\alpha})^{\beta-1}}x_i^{\alpha}\log x_i + \sum_{i=1}^{n}\frac{\beta(a-1)}{(1-(1-x_i^{\alpha})^{\beta})}x_i^{\alpha}(1-x_i^{\alpha})^{\beta-1}\log x_i$$

$$-\sum_{i=1}^{n}\frac{a\beta(b-1)}{1-(1-(1-x_i^{\alpha})^{\beta})^{a}}x_i^{\alpha}(1-x_i^{\alpha})^{\beta-1}\log x_i \tag{13}$$

Differentiating $\lambda(\theta; x)$ with respect to β yields:

$$\frac{\partial \mathrm{l}(\theta; x)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1-x_i^{\alpha}) - \sum_{i=1}^{n}\frac{(a-1)}{1-(1-x_i^{\alpha})^{\beta}}(1-x_i^{\alpha})^{\beta}\log(1-x_i^{\alpha})$$

$$+\sum_{i=1}^{n}\frac{a\,(b-1)}{1-(1-(1-x_i^{\alpha})^{\beta})^{a}}(1-(1-x_i^{\alpha})^{\beta})^{a-1}(1-x_i^{\alpha})^{\beta}\log(1-x_i^{\alpha}) \tag{14}$$

Equating equations 11-14 to zero, yielding a system of nonlinear equations that system needs to be solved numerically to obtain parameters estimation values.

The Variance Covariance Matrix:

Let θ is the vector of the unknown parameters (*a*, b, α, β), The element of the 4 x 4 information matrix I (*a*, b, α, β) can be approximated by:

$$I_{ij}(\hat{\theta}) = -\frac{\partial^2 \mathrm{l}(\theta)}{\partial\theta_i\,\partial\theta_j}\bigg|_{\theta=\hat{\theta}}$$

I^{-1} (*a*, b, α, β) is the variance covariance matrix of the unknown parameters

The asymptotic distributions of the MLE parameters

$$\sqrt{n}\,(\hat{\theta}_i - \theta_i) \approx N_4(0, I^{-1}(\hat{\theta}_i))\ ,\ i = 1, ..., 4$$

the approximation 100 (1- α) % confidence intervals of the unknown parameters based on the asymptotic distribution of the KW-KW (*a*, b, α, β) are determined, respectively, as

$$\hat{\theta}_i \pm z_{\frac{\alpha}{2}}\sqrt{I^{-1}(\hat{\theta}_i)}\ ,\ i = 1, ..., 4$$

Where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ th percentile of the standard normal distribution

7. Simulation study

A simulation experiment is given to illustrate new results of the KW-KW distribution; this example is about MLEs of parameters of the KW-KW distribution. The algorithm of obtaining the parameters estimates is described in the following steps:

Step (1): Generate a random sample of size n as follows: u1, u2… $UN_{,\ by}$ using the uniform distribution (0, 1)

Step (2): transform the uniform random numbers to random numbers of the KW-KW distribution by using the following Quantile function for the KW-KW distribution:

$$x_u = \sqrt[\alpha]{1-\sqrt[\beta]{1-\sqrt[a]{1-\sqrt[b]{1-u}}}}\ ,0 < u < 1$$

Step (3): Solve the equations (11-14) by iteration to get the maximum likelihood estimators via iterative techniques such as Newton-Raphson algorithm and repeat it many times.

Step (4): Calculating variance covariance matrix and confidence intervals of the KW-KW distribution

Numerical illustration:

In this example 50, 30 and 10 random numbers were generated by the Mathcad package which calculated MLEs and variance covariance matrix of the KW-KW distribution starting with parameters values : $a = 4$, $b = 2$, $\alpha = 3$, $\beta = 1$ for 1000 times as follows

Sample size	Parameters	Bises	RMSEs
10	a	9.947	15.012
	b	3.404	7.804
	α	0.627	2.765
	β	3.113	5.62
30	a	3.868	6.131
	b	2.202	4.975
	α	-0.146	1.611
	β	1.17	2.711
50	a	2.468	4.408
	b	1.504	3.916
	α	0.135	1.349
	β	0.78	2.05

We see that the more sample size increases the more Bises and RMSEs decrease.

Also, variance-covariance matrix ($\inf^{-1}$) and confidence intervals at significance level 0.05 were calculated in case n=50 and we got the following results:

$$\inf^{-1} = \begin{pmatrix} 1.552 & -0.01 & -0.867 & 0.058 \\ -0.01 & 0.368 & -0.067 & -0.094 \\ -0.867 & -0.067 & 0.947 & 0.074 \\ 0.058 & -0.094 & 0.074 & 0.048 \end{pmatrix}$$

and confidence intervals are:

Parameters	upper	Lower
a	6.442	1.558
b	3.189	0.811
α	4.907	1.093
β	1.429	0.571

8. Conclusion

A new model so called the Kumaraswamy - Kumaraswamy (KW-KW) distribution generalized the exponentiated Kumaraswamy distribution and the Kumaraswamy distribution and it will be used (like the Kumaraswamy distribution and the exponentiated Kumaraswamy distribution) in hydrological application and related fields, some of general mathematical and statistical properties of the KW-KW distribution were calculated, and simulation study were experimented to estimate its parameters. Finally, the KW-KW distribution provides a rather flexible mechanism of fitting a wide spectrum of world data sets.

9. Appendices

Appendix (i)

The condition of the nonlinear mode equation:

If x_0 is a root of the nonlinear mode equation it must be $f''(x_0) < 0$

Let: $A = (1-x^{\alpha})$, $B = (1-(1-x^{\alpha})^{\beta})$ and $C = \left(1-(1-(1-x^{\alpha})^{\beta})^{a}\right)$

Then,

$$f''(x) = \frac{1-\alpha}{x^2} - \frac{\alpha(\beta-1)x^{\alpha-2}\left((\alpha-1)A+\alpha x^{\alpha}\right)}{A^2} + \frac{\alpha\beta(a-1)x^{\alpha-2}A^{\beta-1}}{B^2}\left\{B\left[(\alpha-1)-\alpha(\beta-1)x^{\alpha}A^{-1}\right]\right.$$

$$\left.-\left[\alpha\beta x^{\alpha}A^{\beta-1}\right]\right\} - \frac{a\alpha\beta(b-1)x^{\alpha-2}A^{\beta-1}B^{a-1}}{C^2}\left\{C\left[(\alpha-1)-\alpha(\beta-1)x^{\alpha}A^{-1}\right.\right.$$

$$\left.\left.+\alpha\beta(a-1)x^{\alpha}A^{\beta-1}B^{-1}\right]+a\alpha\beta x^{\alpha}A^{\beta-1}B^{a-1}\right\}$$

Appendix (ii)

The elements of the observed information matrix

Let: $A = (1-x^{\alpha})$, $B = (1-(1-x^{\alpha})^{\beta})$ and $C = \left(1-(1-(1-x^{\alpha})^{\beta})^{a}\right)$

Then,

$$\frac{\partial^2 l(\theta;x)}{\partial a^2} = \frac{-n}{a^2} - \left\{\left[\sum_{i=1}^{n}\frac{(b-1)\left(\log B\right)^2}{C}B^{a}\right] + \frac{1}{C^2}\left[C(b-1)\left(\log B\right)^2 B^{a} + (b-1)\left(\log B\right)^2 B^{2a}\right]\right\},$$

$$\frac{\partial^2 l(\theta;x)}{\partial a\,\partial b} = -\sum_{i=1}^{n}\frac{B^{a}\log B}{C},$$

$$\frac{\partial^2 l(\theta;x)}{\partial a\,\partial \alpha} = \sum_{i=1}^{n}\frac{\beta A^{\beta-1}x^{\alpha}\log x}{B} - \left\{\sum_{i=1}^{n}\frac{b-1}{C^2}\left\{C\left[\left(\log B\right)a B^{a-1}\beta A^{\beta-1}x_i^{\alpha}\log x_i + B^{a-1}\beta A^{\beta-1}x_i^{\alpha}\log x_i\right]\right.\right.$$

$$\left.\left.+\left[a\beta B^{2a-1}\left(\log B\right)A^{\beta-1}x_i^{\alpha}\log x_i\right]\right\}\right\},$$

$$\frac{\partial^2 l(\theta;x)}{\partial a\,\partial \beta} = -\sum_{i=1}^{n}\frac{A^{\beta}\log A}{B} - \left\{\sum_{i=1}^{n}\frac{b-1}{C^2}\left\{-C\left[\left(\log B\right)a B^{a-1}A^{\beta}\log A + B^{a-1}A^{\beta}\log A\right]\right.\right.$$

$$\left.\left.-\left[a B^{2a-1}\left(\log B\right)A^{\beta}\log A^{\beta}\right]\right\}\right\},$$

$$\frac{\partial^2 l(\theta;x)}{\partial b^2} = \frac{-n}{b^2},\quad \frac{\partial^2 l(\theta;x)}{\partial b\,\partial \alpha} = -\sum_{i=1}^{n}\frac{a\beta B^{a-1}A^{\beta-1}x_i^{\alpha}\log x_i}{C},\quad \frac{\partial^2 l(\theta;x)}{\partial b\,\partial \beta} = \sum_{i=1}^{n}\frac{a B^{a-1}A^{\beta}\log A}{C},$$

$$\frac{\partial^2 l(\theta;x)}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{n}\frac{(\beta-1)\log x_i}{A^{2\beta-2}}\left[A^{\beta-1}x_i^{\alpha}\log x_i + x_i^{2\alpha}(\beta-1)A^{\beta-2}\log x_i\right]$$

$$+\sum_{i=1}^{n}\frac{\beta(a-1)x_i^{\alpha}\log x_i}{C^2}\left[B(\beta-1)A^{\beta-2}\left(-x_i^{\alpha}\right)\left(\log x_i\right) - A^{\beta-1}\left(-\beta\right)A^{\beta-1}\left(-x_i^{\alpha}\right)\left(\log x_i\right)\right]$$

$$+\sum_{i=1}^{n}\frac{a\beta(a-1)\log x_i}{C^2}\left\{C\left[x_i^{\alpha}\left(\log x_i\right)A^{\beta-1} + (\beta-1)A^{\beta-2}x_i^{\alpha}\left(\log x_i\right)\right]\right.$$

$$\left.-\left[a\beta x_i^{2\alpha}A^{2\beta-2}B^{a-1}\log x_i\right]\right\},$$

$$\frac{\partial^2 l(\theta;x)}{\partial\alpha\,\partial\beta} = -\sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{A^{2\beta-2}}\left[A^{\beta-1}-(\beta-1)A^{\beta-1}\log A\right]$$

$$+\sum_{i=1}^{n} \frac{(a-1)\,x_i^{\alpha}\log x_i}{B^2}\left\{B\left[A^{\beta-1}+\beta A^{\beta-1}\log A\right]+\left[\beta A^{2\beta-1}\log A\right]\right\}$$

$$-\sum_{i=1}^{n} \frac{a\beta(a-1)\log x_i}{C^2}\left\{C\left[A^{\beta-1}+\beta A^{\beta-1}\log A\right]-\left[a\beta A^{2\beta-1}B^{a-1}\log A\right]\right\}$$

and

$$\frac{\partial^2 l(\theta;x)}{\partial\beta^2} = \frac{-n}{\beta^2}-\sum_{i=1}^{n} \frac{(a-1)\log A}{B^2}\left[A^{\beta}B\log A + A^{2\beta}\log A\right]$$

$$+\sum_{i=1}^{n} \frac{a(b-1)\log A}{C^2}\left\{C\left[A^{\beta}B^{a-1}\log A-(a-1)A^{2\beta}B^{a-2}\log A\right]-\left[a\,A^{2\beta}B^{2a-2}\log A\right]\right\}$$

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