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Excellent – Domination Subdivision Stable Graphs

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Abstract

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. A graph G is said to be excellent if given any vertex v then there is a γ - set of G containing v. An excellent graph G is said to be very excellent (VE), if there is a γ - set D of G such that to each vertex u $\in$ V – D there is a vertex v $\in$ D such that D – { v } $\cup$ { u } is a γ - set of G. In this paper we have proved that very excellent trees are subdivision stable. We also have provided a method of generating an excellent subdivision stable graph from a non - excellent subdivision stable graph.

Keywords: *Domination, Excellent, Very excellent, Subdivision stable.*

1 Introduction

We consider only simple connected undirected graphs G = (V, E). For graph theoretic terminologies we refer to [5]. A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for G is called the domination number of *G* and it is denoted by γ (G). A γ - set denotes a dominating set for G with minimum cardinality. The subgraph of G induced by the vertices in D is denoted by < D >. The open neighborhood of vertex v $\in$ V (G) is denoted by N (v) = { u $\in$ V (G) | uv $\in$ E (G) } while its closed neighborhood is the set N [v] = N (v) $\cup$ { v }. The private neighborhood of v $\in$ D denoted by pn[v, D], is defined by pn[v, D] =

N (v) – N (D – { v }). A vertex v is said to be a, down vertex if $\gamma (G – u) < \gamma (G)$, level vertex if $\gamma (G – u) = \gamma (G)$, up vertex if $\gamma (G – u) > \gamma (G)$. A vertex v is said to be selfish in the γ- set D, if v is needed only to dominate itself. We indicate that u is adjacent to v by writing $u \perp v$.

A vertex v is said to be good if there is a γ – set of G containing v. If there is no γ – set of G containing v, then v is said to be a bad vertex. A vertex in V – D is k – dominated if it is dominated by at least k – vertices in D that is $| N (v) \cap D | \geq k$. If every vertex in V – D is k – dominated then D is called a k – dominating set. A subdivision of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints {u, v} yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, {u, w} and {w, v}.

2 Preliminary Notes

A graph G is said to be excellent if given any vertex v then there is a γ - set of G containing v. In [3], M. Yamuna and N. Sridharan introduced the concept of very excellent and rigid very excellent graphs. An excellent graph G is said to be very excellent (VE), if there is a γ - set D of G such that to each vertex $u \in V – D$ there is a vertex $v \in D$ such that $D – \{ v \} \cup \{ u \}$ is a γ - set of G. A γ - set D of G satisfying this property is called a very excellent γ - set of G. In this case we say that u and v are vertex exchangeable.

In all the figures encircled vertices denote a γ - set for G.

2.1 Example for very excellent graph

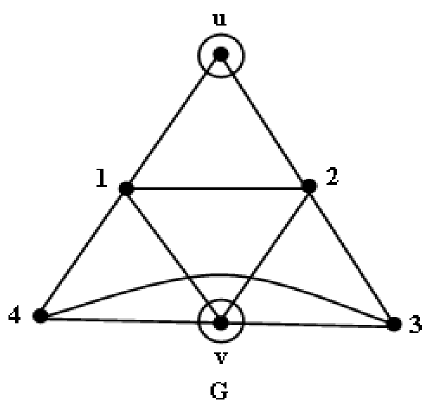

Fig. 1

In Fig. 1 $D – \{ v \} \cup \{ 1 \}$, $D – \{ v \} \cup \{ 2 \}$, $D – \{ u \} \cup \{ 3 \}$, $D – \{ u \} \cup \{ 4 \}$, are γ - sets for G.

Let G be a very excellent graph and D be a very excellent γ - set of G. To each u $\notin$ D, let E (u, D) = { v $\in$ D / (D – v) $\cup$ { u } is a γ - set of G }. If | E (u, D) | = 1, for each vertex u $\notin$ D, D is said to be a rigid very excellent γ - set of G. If G has at least one rigid very excellent γ - set then G is said to be rigid very excellent (RVE).

2.2 Example for rigid very excellent graph

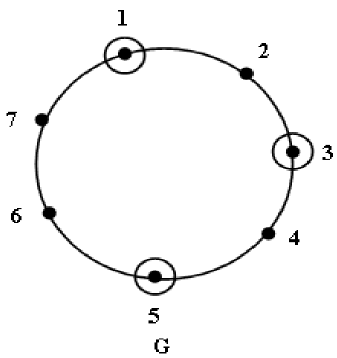

Fig. 2

In Fig. 2, to each vertex u $\notin$ D there is a only one possible way to select a vertex v $\in$ D such that D – { v } $\cup$ { u } is a γ - set for G.

If G is a graph, then S (G) denotes the subdivision graph of G obtained from G by subdividing each edge of G once. A graph H is said to be a subdivision of G, if it is obtained from G by subdividing each edge of G at most once. In other words, to obtain H, it is not necessary to subdivide each edge of G.

In [6], they have proved the following results

1. **Type 1 operation:** If G is very excellent and u is a level vertex of G, then the graph H obtained from G by attaching a path P_2 at u is very excellent.
2. **Type 2 operation:** Let u be a vertex of a graph G. A graph H is obtained from G by attaching a tree path P_3 at u. Then H is very excellent if and only if G is very excellent and there is a very excellent γ - set D of G such that u $\in$ D and D – u dominates G – u.
3. **Type 3 operation:** Let G be a very excellent graph and u be a vertex of G. Assume that there exist a very excellent γ - set D of G such that for some v $\in$ D, (D − v) is a γ - set for G − u. Then the graph H obtained from G and a disjoint path w_1 w_2 w_3 w_4 w_5 of length four by joining w_3 and u by an edge is very excellent.
4. A tree T is very excellent if and only if T is obtained by applying type 1, 2 and 3 operation any number of times.
5. Every very excellent tree is rigid very excellent.

6. If a graph G is not excellent, then there is a subdivision graph H of G which is excellent.

In [7], M. Yamuna and K. Karthika have introduced the concept of domination subdivision stable graphs. A graph G is said to be domination subdivision stable (DSS) if the γ - value of G does not change by subdividing any edge of G. We shall denote the graph obtained by subdividing any edge e = (u v) of a graph G, by G $_{sd}$ uv. Let w be a vertex introduced by subdividing uv. We shall denote this by G $_{sd}$ uv = w.

2.3 Example for DSS graph

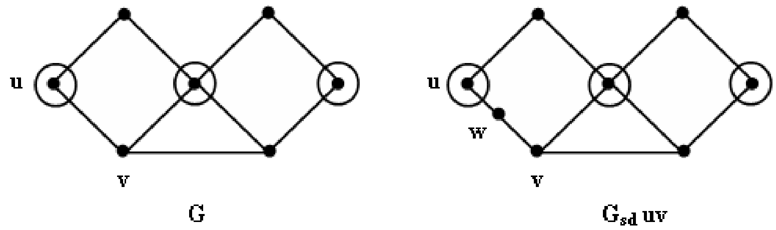

Fig. 3

In Fig. 3, γ (G) = γ (G $_{sd}$ uv) = 3. This is true for all e = (a b)$\in$ E (G) which implies that G is a DSS graph.

In [7], they have proved the following results

1. A graph G is DSS if and only if for every u, v $\in$ V (G), either there is a γ - set containing u and v or, there is a γ - set D such that either
   - pn[u, D] = { v } or
   - v is 2 – dominated.
2. For any graph G, γ (G $_{sd}$ uv) $\geq$ γ (G) for all e = (u v) $\in$ E (G).
3. Every graph is an induced subgraph of a DSS graph.

They also have discussed properties satisfied by the vertices of a DSS graph and obtained methods of generating new DSS graphs from existing ones.

3 Main Results

In the following theorem we provide a method of generating an excellent DSS graph from a non – excellent DSS graph.

Theorem 3.1 *Let G be a DSS and non – excellent graph. Then there is a subdivision graph H of G which is excellent and DSS.*

Proof: Let G be a non – excellent and DSS graph. Let B be the set of all bad vertex of G. Let $x \in B$. Among the γ - sets of G select one γ - set S_1 such that $| N (x) \cap S_1 |$ is maximum. For each $y \in N (x) \cap S_1$, subdivided the edge xy. Let w_y be the vertex introduced while subdividing edge xy. Let H_1 be the graph thus obtained.
In [4] it has been proved that,

1. H_1 is a subdivision of G.
2. The set of all bad vertices of H_1 is a proper subset of the set of all bad vertices of G.
3. In H_1, $\{ w_y / y \in N (x) \cap S_1$ in G $\}$ are good vertices. x is also a good vertex in H_1.
4. $D \cup \{ x \}$ is a γ - set for H_1.

We observe that $\{ w_y / y \in N (x) \cap S_1$ in G $\}$ is 2 – dominated. Hence $\gamma (H_{1\ sd}\ xw_y) = \gamma (H_1) = \gamma (G) + 1$. Let $\{ S_y \subseteq S_1 / y \in N (x) \cap S_1 \}$ be the set of vertices in $S_1 \perp x$ in G. $\gamma (H_{1\ sd}\ w_y s_y) = \gamma (H_1) = \gamma (G) + 1$. Since G is DSS graph, H_1 is also a DSS graph.
If x_0 is a bad vertex of H_1 and S_2 is γ - set of H_1 such that $| N (x_0) \cap S_2 |$ is maximum, then obtain a subdivision H_2 of H_1 by subdividing the edge x_0y, $y \in N (x_0) \cap S_2$, $N (x_0) \subset V (G)$.
The number of bad vertices in H_2 < the number of bad vertices in H_1 < the number of bad vertices in G.
By the above discussion we know that H_2 is a DSS graph.
Proceeding like this, we obtain finite sequences $H_1, H_2, \ldots, H_k$ of subdivision of G with the following property,

1. Each H_{i+1} is a subdivision of H_i.
2. The number of bad vertices of H_{i+1} < the number of bad vertices of H_i.
3. Each H_{i+1} is DSS.

For some k, ($\leq | B |$), we obtain an excellent subdivision stable graph H_k. Denote this H_k by H. ■

We prove the following results to prove that VE trees are DSS

Theorem 3.2 *Let T be a VE tree. Then every pendant vertex can be exchanged with its support vertex only.*

Proof: Let T be a VE tree and D be a VE γ - set for T. Let v be a pendant vertex and u be its support vertex. In [3], it has been proved that, in a VE tree, every support vertex has exactly one pendant vertex adjacent to it. Suppose there is one $y \in D$, $y \neq u$ such that $D' = D – \{ y \} \cup \{ v \}$ is a γ - set for G, then $D'' = D' – \{ v \}$ is also a dominating set for G such that $| D'' | < | D' |$ which is a contradiction. Hence every pendant vertex can be exchanged with its support vertex only. ■

Theorem 3.3 *Let T be a VE tree* and *D* be *a VE* γ *- set for T. Then for each vertex u* $\in$ *D either pn[u, D] = 1 or pn[u, D] =* ϕ*.*

Proof: Let T be a VE tree and D be a VE γ - set for T and if possible, let pn[u, D] $\geq$ 2 for some u $\in$ D, say pn[u, D] = 2 = x, y.

Claim 1
x and y are internal vertices only.

Proof:
In [6] it has been proved that, both x and y cannot be pendant vertices. Let us assume that, x is pendant and y an internal vertex. By theorem 3.2, we know that $D' = D - \{ u \} \cup \{ x \}$ is a γ - set for T. But y is not dominated by D′ which is not possible. Hence if pn[u, D] = 2, then both x and y are internal vertices only.

Claim 2
x and y cannot be interchanged with u.

Proof:
If possible, let us assume that $D' = D - \{ u \} \cup \{ x \}$ is a γ - set for G. Then $x \perp u$, y that is $u \perp x$, $x \perp y$ and $y \perp u$ which is not possible, since T is a tree that is x and y cannot be interchanged with u.

Since T is VE, there is one $z \neq u$ [by claim (2)] such that $D' = D - \{ z \} \cup \{ x \}$ is a γ - set for T which is a contradiction, since $D' - \{ x \}$ itself is a γ - set for T. Hence pn[u, D] is not greater than 2, that is either pn[u, D] = ϕ or pn[u, D] = 1.

■

Theorem 3.4 *VE trees are DSS.*

Proof: Let T be a VE tree and D be a VE γ - set for T. By theorem 3.3, we know that any $v \in V (T)$ is either 2 – dominated or pn[u, D] = v for some $u \in D$. Let $u, v \in V (T)$.

Case 1 $u, v \in D$
Let T_{sd} uv = w. D itself is a γ - set for T_{sd} uv.

Case 2 $u \in D, v \notin D$
Let T_{sd} uv = w. If pn[u, D] = v, then $D' = D - \{ u \} \cup \{ w \}$ is γ - set for T_{sd} uv. If pn[u, D] = ϕ, then D itself is a γ - set for T_{sd} uv.

Case 3 $u, v \notin D$
Let T_{sd} uv = w. Since T is VE there is one z such that $D' = D - \{ z \} \cup \{ u \}$ is a γ - set for T. Here D′ is γ - set for T_{sd} uv also.
In all cases $\gamma (T_{sd} uv) = \gamma (T)$ that is VE trees are DSS. ■

Theorem 3.5 *Let G be DSS and y be a level vertex of G. Then the graph H obtained from G by attaching a path P_2 at y is DSS.*

Proof: Let G be DSS and y be level vertex for G. Consider a path P_2 : $w_1w_2w_3$. Attach a path P_2 to a vertex y, to obtain a graph H. Label the vertex yw_1 as u.
$\gamma (H) = \gamma (G) \cup \{ w_2 \}$ that is $\gamma (H) = \gamma (G) + 1$.
Consider $H_{sd}\, uw_2$. Let $H_{sd}\, uw_2 = w$. Hence $\gamma (H) = \gamma (H_{sd}\, uw_2)$.
Consider $H_{sd}\, w_2w_3$. Let $H_{sd}\, w_2w_3 = w$. Hence $\gamma (H_{sd}\, w_2w_3) = \gamma (G) \cup \{ w \}$ that is $\gamma (H) = \gamma (H_{sd}\, w_2w_3)$.
Also $\gamma (H_{sd}\, uv) = \gamma (H)$, $u, v \in V (G)$. In general $\gamma (H) = \gamma (H_{sd}\, ab) = \gamma (G) + 1$ for all $a, b \in V (H)$. Hence H is DSS. ■

Example

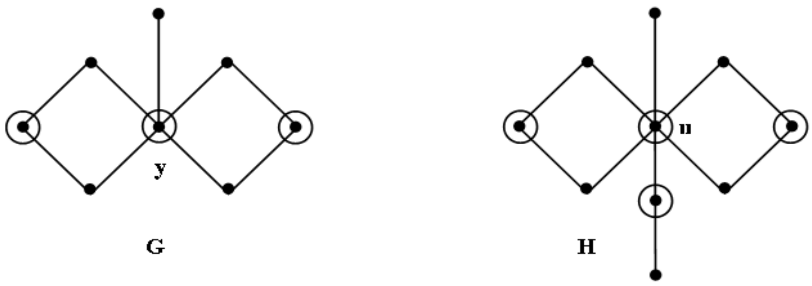

Fig. 4

In Fig. 4, G is DSS and y is a level vertex of G, $\gamma (G) = 3$, $\gamma (H) = 4 = \gamma (H_{sd}\, uv)$. This is true for all $e = (a\ b) \in E (H)$.

Theorem 3.6 *Let u be a vertex of a graph G. Let H be the graph obtained from G by attaching a path P_3 at u. Then H is DSS if G is DSS and there is a γ – set D of G such that $u \in D$ and D – u dominates G – u.*

Proof: Let G be DSS and D be γ – set for G. Let $u \in D$ such that $D - \{ u \}$ dominates $G - \{ u \}$. Consider P_3 : $w_1w_2w_3w_4$. Attach a path P_3 to u, to obtain a new graph H. Label the vertex uw_1 as y. $\gamma (H) = \gamma (G) \cup \{ w_3 \}$. $\gamma (H)$ includes y which is a selfish vertex.
Consider $H_{sd}\, yw_2$. Let $H_{sd}\, yw_2 = w$. $\gamma (H_{sd}\, yw_2) = \gamma (H)$.
Consider $H_{sd}\, w_2w_3$. Let $H_{sd}\, w_2w_3 = w$. $\gamma (H_{sd}\, w_2w_3) = \gamma (H)$.
Consider $H_{sd}\, w_3w_4$. Let $H_{sd}\, w_3w_4 = w$. Hence $\gamma (H_{sd}\, w_2w_3) = \gamma (H) - \{ w_3 \} \cup \{ w \}$. Also $\gamma (H_{sd}\, uv) = \gamma (H)$, for all $u, v \in V (G)$.
In all cases $\gamma (H_{sd}\, uv) = \gamma (H)$ for all $u, v \in V (H)$. Hence H is DSS. ■

Example

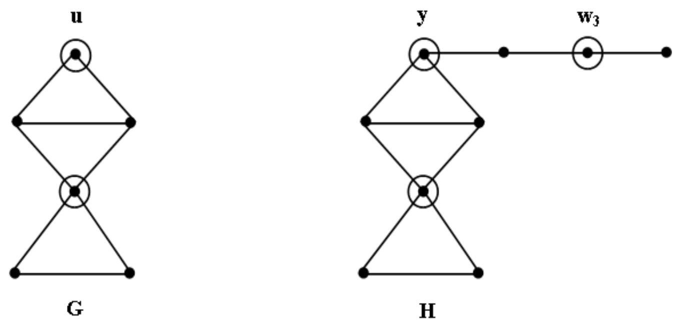

Fig. 5

In Fig. 5, u is a selfish vertex in G, γ (G) = 2, γ (H) = 3 = γ (H_{sd} uv). This is true for all e = (a b) $\in$ E (H) which implies H is DSS.

Theorem 3.7 *Let G be a DSS graph and u be a vertex of G. Assume that there exist a γ - set D of G such that for some v $\in$ D, (D − v) is a γ - set for G − u. Then the graph H obtained from G and a disjoint path w_1 w_2 w_3 w_4 w_5 of length four by joining w_3 and u by an edge is a DSS graph.*

Proof Let G be DSS and u $\in$ V (G) such that there is a γ – set D of G so that for some v $\in$ D, D – { v } is a γ – set for G – { u }. Consider a path P_4 : $w_1w_2w_3$ w_4w_5 of length four with V (G) $\cap$ V (P) = ϕ. Let H be a graph obtained from G by joining the vertex u and w_3 by an edge that is G $\cup$ P $\cup$ { e }.
γ (H) = γ (G) $\cup$ { w_2 } $\cup$ { w_4 } that is γ (H) = γ (G) + 2.
Let H_{sd} uw_3 = w. γ (H_{sd} uw_3) = γ (H) – { v } $\cup$ { u } .
Let H_{sd} w_1w_2 = w. γ (H_{sd} w_1w_2) = γ (H) – { w_2 } $\cup$ { w }.
Let H_{sd} w_2w_3 = w. γ (H_{sd} w_2w_3) = γ (H).
Let H_{sd} w_3w_4 = w. γ (H_{sd} w_3w_4) = γ (H).
Let H_{sd} w_4w_5 = w. γ (H_{sd} w_4w_5) = γ (H) – { w_4 } $\cup$ { w }.
Also γ (H_{sd} uv) = γ (H), for all u, v $\in$ V (G). In all cases γ (H_{sd} ab) = γ (H) for all a, b $\in$ V (H). Hence H is DSS. ■

Example

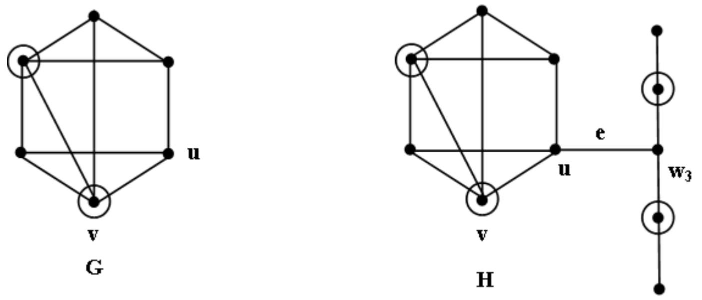

Fig. 6

In Fig. 6, G is DSS, γ (G) = 2, γ (H) = 4 = γ (H_{sd} uv). This is true for all e = (a b) $\in$ E (H) that is H is also DSS.

4 Conclusion

By theorems 3.2 – 3.7, we observe that

1. VE trees are DSS.
2. If G is DSS, then the graph obtained by applying type 1, 2 and 3 operations are DSS.

Using VE characterization and the results in page [3], we conclude that " VE trees and RVE trees are DSS ".

References

[1] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, *Marcel Dekker,* New York, (1998).

[2] K. Karthika, Domination Dot Stable – Domatic dot stable – Domination Subdivision Stable graphs, *M. Phil thesis*, VIT University, Vellore, India, (2011).

[3] N. Sridharan, M. Yamuna, Very excellent graphs and Rigid very excellent graphs, *AKCE J. Graph. Combin.,* (2007), Vol – 4, pp. 211 – 221.

[4] N. Sridharan, M. Yamuna, A note on excellent graphs, *Ars. Combin,* (2006), Vol – 78 , pp. 267 – 276.

[5] D. B. West, Introduction to Graph Theory, second ed., *Prentice-Hall*, Englewood Cliffs, NJ (2001).

[6] M. Yamuna, Excellent – Just Excellent – Very Excellent graphs, *Ph. D thesis*, Alagappa University, Karaikudi, India (2003).

[7] M. Yamuna, K. Karthika, Domination Subdivision Stable Graphs, *International Journal of Mathematical Archive* (2012) - 3(4), Page: 1467-1471.