λ_{pc}-open sets and λ_{pc}-separation axioms in topological spaces

Alias B. Khalaf *, Alan M. Omar

University of Duhok, Department of Mathematics
*Corresponding author E-mail: aliasbkhalaf@gmail.com

Abstract

The aim of this paper is to introduce a new class of sets called λ_{pc}-open sets and to investigate some of their relationships and properties. Further, by using this set, the notion of λ_{pc}-T_i spaces (i = 0,1/2,1,2) and λ_{pc}-R_j spaces (j = 0,1) are introduced and some of their properties are investigated.

Keywords: s-operation; λ_{pc}-Open Set; λ_{pc}-T_i, i=0,1,2; λ_{pc}-R_j, j=0,1.

1. Introduction

The study of pre-open sets and pre-continuity in topological spaces was initiated by Mashhour, El-Monsef and El-Deeb [7]. Analogous to generalized closed sets which was introduced by Levine [8], Maki, Umehara and Nori [2], introduced the concept of pre-generalized closed sets in topological spaces. Kasahara [6], defined the concept of an operation on topological spaces and introduced the concept of closed graphs of a function. Ahmad and Hussain [1], introduced the concept of γ-T_i (i = 0,1/2,1,2) and characterized γ-T_i by the notion of γ-closed sets or γ-open sets. Chattopadhyay [9] defined other new types of separation axioms and Caldas, Jafari and Nori [5], defined pre-R_1 and pre-R_0.

In this paper, we introduce and study a new class of pre-open sets called λ_{pc}-open sets in topological spaces. By using the notion of λ_{pc}-closed and λ_{pc}-open sets, we introduce the concept of λ_{pc}-T_i (i = 0,1/2,1,2) and λ_{pc}-R_j (j = 0,1) spaces. several properties and characterizations of these spaces are obtained.

2. Preliminaries

Throughout, X denote a topological space with out any separation axiom. Let A be a subset of X, the closure (interior) of A are denoted by Cl(A) (Int(A)) respectively. A subset A of a topological space (X, τ) is said to be pre-open [7] if A ⊆ Int(Cl(A)). The complement of a pre-open set is said to be pre-closed [7]. The family of all pre-open (resp. pre-closed) sets in a topological space (X, τ) is denoted by PO(X, τ) or PO(X) (resp. PC(X, τ) or PC(X)).
Definition 2.1 [4] Let \((X, \tau)\) be a topological space and let \(A\) be a subset of \(X\) then:

1. The pre-interior of \(A\) (\(p\text{Int}(A)\)) is the union of all pre-open sets of \(X\) contained in \(A\).
2. A point \(x \in X\) is said to be a pre-limit point of \(A\) if every pre-open set containing \(x\) contains a point of \(A\) different from \(x\), and the set of all pre-limit points of \(A\) is called the pre-derived set of \(A\) denoted by \(pd(A)\).
3. The intersection of all pre-closed sets of \(X\) containing \(A\) is called the pre-closure of \(A\) and is denoted by \(p\text{Cl}(A)\).

Definition 2.2 [2] A subset \(A\) of a space \((X, \tau)\) is called a pre-generalized closed set (pg-closed), if \(A \subseteq U\) and \(U\) is pre-open implies that \(p\text{Cl}(A) \subseteq U\).

Definition 2.3 A topological space \((X, \tau)\) is said to be:

1. pre-\(T_0\) [9] if for any distinct pair of points in \(X\), there is an pre-open set containing one of the points but not the other.
2. pre-\(T_1\) [9] if for any distinct pair of points \(x, y \in X\), there is a pre-open \(U\) in \(X\) containing \(x\) but not \(y\) and a pre-open set \(V\) in \(X\) containing \(y\) but not \(x\).
3. pre-\(T_2\) [9] if for any distinct pair of points \(x, y \in X\), there exist pre-open sets \(U\) and \(V\) in \(X\) containing \(x\) and \(y\), respectively, such that \(U \cap V = \phi\).
4. pre-\(T_{1/2}\) [2] if every pg-closed set is pre-closed.
5. pre-\(R_0\) [5] if for each \(O \in \text{PO}(X)\) and \(x \in O\), \(p\text{Cl}(\{x\}) \subseteq O\).
6. pre-\(R_1\) [5] if for each pair \(x, y \in X\) such that \(p\text{Cl}({x}) \neq p\text{Cl}({y})\), there exist disjoint pre-open sets \(U\) and \(V\) such that \(p\text{Cl}({x}) \subseteq U\) and \(p\text{Cl}({y}) \subseteq V\).

Definition 2.4 [10] Let \((X, \tau)\) be any topological space. A mapping \(\lambda : \tau \to P(X)\), \((P(X)\) stands for all subsets of \(X)\), is called an operation on \(\tau\) if \(V \subseteq \lambda(V)\) for each non-empty open set \(V\) and \(\lambda(\phi) = \phi\).

Definition 2.5 [3] Let \((X, \tau)\) be a topological space and let \(A\) be a subset of \(X\) then:

1. The \(\lambda\)-interior of \(A\) (\(\lambda\text{Int}(A)\)) is the union of all \(\lambda\)-open sets of \(X\) contained in \(A\).
2. A point \(x \in X\) is said to be a \(\lambda\)-limit point of \(A\) if every \(\lambda\)-open set containing \(x\) contains a point of \(A\) different from \(x\), and the set of all \(\lambda\)-limit points of \(A\) is called the \(\lambda\)-derived set of \(A\) denoted by \(\lambda d(A)\).
3. The intersection of all \(\lambda\)-closed sets of \(X\) containing \(A\) is called the \(\lambda\)-closure of \(A\) and is denoted by \(\lambda\text{Cl}(A)\).

3. \(\lambda_{pc}\)-Open sets

In this section, we introduce a new class of pre-open sets called \(\lambda_{pc}\)-open sets. Further, the notion of \(\lambda_{pc}\)-derived set, \(\lambda_{pc}\)-closure and \(\lambda_{pc}\)-interior are introduced and their properties are discussed.

Definition 3.1 A mapping \(\lambda : \text{PO}(X) \to P(X)\) is called an \(\text{p}\)-operation on \(\text{PO}(X)\) if \(V \subseteq \lambda(V)\) for each non-empty \text{pre-open} set \(V\) and \(\lambda(\phi) = \phi\).

If \(\lambda : \text{PO}(X) \to P(X)\) is any \(\text{p}\)-operation, then it is clear that \(\lambda(X) = X\).

Definition 3.2 Let \((X, \tau)\) be a topological space and \(\lambda : \text{PO}(X) \to P(X)\) be an \(\text{p}\)-operation defined on \(\text{PO}(X)\), then a subset \(A\) of \(X\) is \(\lambda_{pc}\)-open set if for each \(x \in A\) there exists a pre-open set \(U\) such that \(x \in U\) and \(\lambda(U) \subseteq A\).

Definition 3.3 A \(\lambda_{pc}\)-open subset \(A\) of \(X\) is called \(\lambda_{pc}\)-open if for each \(x \in A\) there exists a closed subset \(F\) of \(X\) such that \(x \in F \subseteq A\).

The complement of a \(\lambda_{pc}\)-open set is said to be \(\lambda_{pc}\)-closed. The family of all \(\lambda_{pc}\)-open (resp., \(\lambda_{pc}\)-closed) subsets of a topological space \((X, \tau)\) is denoted by \(\text{PO}_{\lambda_{pc}}(X, \tau)\) or \(\text{PO}_{\lambda_{pc}}(X)\) (resp., \(\text{PC}_{\lambda_{pc}}(X, \tau)\) or \(\text{PC}_{\lambda_{pc}}(X)\)).

Proposition 3.4 For any topological space \((X, \tau)\), we have \(\text{PO}_{\lambda_{pc}}(X) \subseteq \text{PO}_{\lambda}(X) \subseteq \text{PO}(X)\).
Proof. Obvious
The following examples show that the equality in the above proposition may not be true in general.

Example 3.5 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an $p$-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a, c\}$ or $A$ is empty and $\lambda(A) = X$ otherwise. Here, we have $\{a, c\}$ is $\lambda p$-open set but it is not $\lambda_{pc}$-open.

The following examples shows that $\tau$ is incompatible with $\lambda_{pc}O(X)$.

Example 3.6 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an $p$-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A \neq \{a\}$ or $\{b\}$ and $\lambda(A) = \{a, b\}$ if $A = \{a\}$ or $\{b\}$. Now, we have $\{a\}$ is open set but not $\lambda_{pc}$-open.

Example 3.7 Let $N$ be a set of natural numbers. In a topological space $(N, \tau)$ with cofinite topology. We define an $p$-operation $\lambda : PO(N) \to P(N)$ as $\lambda(A) = A$. Then we obtain that $\{1, 3, 5, \ldots\}$ is $\lambda_{pc}$-open set but not open.

Proposition 3.8 Let $\{A_{\alpha}\}_{\alpha \in I}$ be any collection of $\lambda_{pc}$-open sets in a topological space $(X, \tau)$, then $\bigcup_{\alpha \in I} A_{\alpha}$ is a $\lambda_{pc}$-open set.

Proof. Since $A_{\alpha}$ is $\lambda_{pc}$-open set for all $\alpha \in I$, then $A_{\alpha}$ is a $\lambda p$-open set for all $\alpha \in I$. This implies that there exists a pre-open set $U$ such that $\lambda(U) \subseteq A_{\alpha_{0}} \subseteq \bigcup_{\alpha \in I} A_{\alpha}$. Therefore, $\bigcup_{\alpha \in I} A_{\alpha}$ is a $\lambda p$-open subset of $(X, \tau)$. Let $x \in \bigcup_{\alpha \in I} A_{\alpha}$, then there exists an $\alpha_{0} \in I$ such that $x \in A_{\alpha_{0}}$. Since $A_{\alpha}$ is a $\lambda_{pc}$-open set for all $\alpha \in I$, then there exists a closed set $F$ such that $x \in F \subseteq A_{\alpha_{0}}$ but $A_{\alpha_{0}} \subseteq \bigcup_{\alpha \in I} A_{\alpha}$, then $x \in F \subseteq \bigcup_{\alpha \in I} A_{\alpha}$. Hence, $\bigcup_{\alpha \in I} A_{\alpha}$ is $\lambda_{pc}$-open.

The following example shows that the intersection of two $\lambda_{pc}$-open sets need not be $\lambda_{pc}$-open.

Example 3.9 Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an $p$-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A$ is empty or $A \neq \{b\}$ and $\lambda(A) = X$ otherwise. So we have $\{a, b\}$ and $\{b, c\}$ are $\lambda_{pc}$-open sets but $\{a, b\} \cap \{b, c\} = \{b\}$ is not $\lambda_{pc}$-open.

Proposition 3.10 The set $A$ is $\lambda_{pc}$-open in the space $(X, \tau)$ if and only if for each $x \in A$ there exists a $\lambda_{pc}$-open set $B$ such that $x \in B \subseteq A$.

Proof. Suppose that $A$ is $\lambda_{pc}$-open in $(X, \tau)$. Then for each $x \in A$ we put $B = A$ is a $\lambda_{pc}$-open set such that $x \in B \subseteq A$.

Conversely, Suppose that for each $x \in A$ there exists a $\lambda_{pc}$-open set $B_{x}$ such that $x \in B_{x} \subseteq A$. Thus $A = \bigcup B_{x}$, where $B_{x} \in PO_{\lambda_{pc}}(X)$ for each $x$. Therefore, by Proposition 3.8, $A$ is $\lambda_{pc}$-open.

Definition 3.11 Let $(X, \tau)$ be a topological space. An $p$-operation $\lambda$ is said to be $p$-regular if for every pre-open sets $U$ and $V$ containing $x \in X$, there exists a pre-open set $W$ containing $x$ such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.

Theorem 3.12 Let $\lambda$ be an $p$-regular $p$-operation. If $A$ and $B$ are $\lambda_{pc}$-open sets in $X$, then $A \cap B$ is also $\lambda_{pc}$-open.

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $A$ and $B$ are $\lambda_{pc}$-open sets, so there exist pre-open sets $U$ and $V$ such that $x \in U$ and $\lambda(U) \subseteq X$, $x \in V$ and $\lambda(V) \subseteq B$. Since $\lambda$ is $p$-regular, this implies that there exists a pre-open set $W$ of $x$ such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V) \subseteq A \cap B$. Therefore, $A \cap B$ is $\lambda_{pc}$-open set. Again for each $x \in A \cap B$, we have $x \in A$ and $x \in B$ and since $A$ and $B$ are $\lambda_{pc}$-open sets, then there exist closed sets $E, F$ such that $x \in E \subseteq A$ and $x \in F \subseteq B$. Therefore, $x \in E \cap F \subseteq A \cap B$. Since $E \cap F$ is closed, so by Definition 3.2, we obtain that $A \cap B$ is $\lambda_{pc}$-open.

Definition 3.13 Let $(X, \tau)$ be a topological space and let $A$ be subset of $X$, then a point $x \in X$ is called a $\lambda_{pc}$-limit point of $A$ if every $\lambda_{pc}$-open set containing $x$ contains a point of $A$ different from $x$.

The set of all $\lambda_{pc}$-limit points of $A$ is called the $\lambda_{pc}$-derived set of $A$ denoted by $\lambda_{pc}d(A)$.

Definition 3.14 Let $A$ be subset of the space $(X, \tau)$, then the $\lambda_{pc}$-closure of $A$ $(\lambda_{pc}Cl(A))$ is the intersection of all $\lambda_{pc}$-closed sets containing $A$.

Here we introduce some properties of $\lambda_{pc}$-closure of the sets.

Proposition 3.15 For subsets $A, B$ of a topological space $(X, \tau)$, the following statements are true.
1. $A \subseteq \lambda_{pc}Cl(A)$.

2. $\lambda_{pc}Cl(A)$ is $\lambda_{pc}$-closed set in $X$.

3. $\lambda_{pc}Cl(A)$ is smallest $\lambda_{pc}$-closed set which contain $A$.

4. $A$ is $\lambda_{pc}$-closed set if and only if $A = \lambda_{pc}Cl(A)$.

5. $\lambda_{pc}Cl(\phi) = \phi$ and $\lambda_{pc}Cl(X) = X$.

6. If $A \subseteq B$. Then $\lambda_{pc}Cl(A) \subseteq \lambda_{pc}Cl(B)$.

7. $\lambda_{pc}Cl(A) \cup \lambda_{pc}Cl(B) \subseteq \lambda_{pc}Cl(A \cup B)$.

8. $\lambda_{pc}Cl(A \cap B) \subseteq \lambda_{pc}Cl(A) \cap \lambda_{pc}Cl(B)$.

**Proof.** Obvious.

In general the equalities (7) and (8) of the above proposition is not true, as it is shown in the following examples:

**Example 3.16** Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$, $A = \{a, b\}$ or $\{b, c\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{b\}$ and $B = \{c\}$, then $\lambda_{pc}Cl(A) = \{b\}$ and $\lambda_{pc}Cl(B) = \{c\}$, but $\lambda_{pc}Cl(A \cup B) = X$, where $A \cup B = \{b, c\}$. $\lambda_{pc}Cl(A \cup B) \neq \lambda_{pc}Cl(A) \cup \lambda_{pc}Cl(B)$.

**Example 3.17** Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$ or $c \in A$ and $\lambda(A) = Cl(A)$ otherwise. Now, if $A = \{a\}$ and $B = \{b\}$ then $\lambda_{pc}Cl(A) = \{a, b\}$ and $\lambda_{pc}Cl(B) = \{b\}$, but $\lambda_{pc}Cl(A \cap B) = \phi$, where $A \cap B = \phi$. Hence $\lambda_{pc}Cl(A \cap B) \neq \lambda_{pc}Cl(A) \cap \lambda_{pc}Cl(B)$.

**Proposition 3.18** Let $A$ be any subset of a space $X$, then $\lambda_{pc}Cl(A) = A \cup \lambda_{pc}d(A)$.

**Proof.** Obvious.

**Proposition 3.19** If $A$ is a subset of $(X, \tau)$, then $x \in \lambda_{pc}Cl(A)$ if and only if $V \cap A \neq \phi$ for every $\lambda_{pc}$-open set $V$ containing $x$.

**Proof.** Let $x \in \lambda_{pc}Cl(A)$ and suppose that $V \cap A = \phi$ for some $\lambda_{pc}$-open set $V$ which contains $x$. This implies that $X \setminus V$ is $\lambda_{pc}$-closed and $A \subseteq (X \setminus V)$, so $\lambda_{pc}Cl(A) \subseteq (X \setminus V)$. But this implies that $x \in (X \setminus V)$ which is contradiction. Therefore, $V \cap A \neq \phi$.

Conversely, Let $A \subseteq X$ and $x \in X$ such that for each $\lambda_{pc}$-open set $V$ containing $x$, $V \cap A \neq \phi$. If $x \notin \lambda_{pc}Cl(A)$, then there is a $\lambda_{pc}$-closed set $S$ such that $A \subseteq S$ and $x \notin S$. Hence, $(X \setminus S)$ is a $\lambda_{pc}$-open set with $x \in (X \setminus S)$ and thus $(X \setminus S) \cap A \neq \phi$ which is a contradiction. Therefore, $x \in \lambda_{pc}Cl(A)$.

**Proposition 3.20** If $A$ is any subset of a topological space $(X, \tau)$, then $\mu Cl(A) \subseteq \lambda_{pc}Cl(A)$.

**Proof.** Obvious.

The following example shows that the equality in the above proposition is not true in general.

**Example 3.21** Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$ or $A = \{a\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{c\}$, then $\mu Cl(A) = \{c\}$ and $\lambda_{pc}Cl(A) = X$.

**Definition 3.22** Let $(X, \tau)$ be a topological space and let $A$ be subset of $X$, then the $\lambda_{pc}$-interior of $A$ ($\lambda_{pc}Int(A)$) is the union of all $\lambda_{pc}$-open sets of $X$ contained in $A$.

**Proposition 3.23** For subsets $A$, $B$ of a space $X$, the following statements hold.

1. $\lambda_{pc}Int(A)$ is the union of all $\lambda_{pc}$-open sets which are contained in $A$.

2. $\lambda_{pc}Int(A)$ is a $\lambda_{pc}$-open set in $X$.

3. $\lambda_{pc}Int(A) \subseteq A$.

4. $\lambda_{pc}Int(A)$ is the largest $\lambda_{pc}$-open set contained in $A$.

5. $A$ is $\lambda_{pc}$-open set if and only if $\lambda_{pc}Int(A) = A$. 
6. \( \lambda_{pc} \text{Int}(\lambda_{pc} \text{Int}(A)) = \lambda_{pc} \text{Int}(A) \).

7. If \( A \subseteq B \), then \( \lambda_{pc} \text{Int}(A) \subseteq \lambda_{pc} \text{Int}(B) \).

8. \( \lambda_{pc} \text{Int}(\phi) = \phi \) and \( \lambda_{pc} \text{Int}(X) = X \).

9. \( \lambda_{pc} \text{Int}(A) \cup \lambda_{pc} \text{Int}(B) \subseteq \lambda_{pc} \text{Int}(A \cup B) \).

10. \( \lambda_{pc} \text{Int}(A \cap B) \subseteq \lambda_{pc} \text{Int}(A) \cap \lambda_{pc} \text{Int}(B) \).

**Proof.** Obvious.

In general the equalities of (9) and (10) of the above proposition is not true, as it is shown in the following examples:

**Example 3.24** Let \( X = \{a, b, c\} \), and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \). We define an \( p \)-operation \( \lambda : PO(X) \rightarrow P(X) \) as \( \lambda(A) = A \) if \( b \in A \) and \( \lambda(A) = \text{Cl}(A) \) if \( b \notin A \). Now, let \( A = \{a\} \) and \( B = \{c\} \), then \( \lambda_{pc} \text{Int}(A) = \phi \), and \( \lambda_{pc} \text{Int}(B) = \phi \), but \( \lambda_{pc} \text{Int}(A \cup B) = \{a, c\} \). Thus \( \lambda_{pc} \text{Int}(A \cup B) \neq \lambda_{pc} \text{Int}(A) \cup \lambda_{pc} \text{Int}(B) \).

**Example 3.25** Let \( X = \{a, b, c\} \), and \( \tau = P(X) \). We define an \( p \)-operation \( \lambda : PO(X) \rightarrow P(X) \) as \( \lambda(A) = A \) if \( A = \phi \), \( A = \{c\} \), \( \{a, b\} \) or \( \{a, c\} \) and \( \lambda(A) = X \) otherwise. Now, if \( A = \{a, b\} \) and \( B = \{a, c\} \), then \( \lambda_{pc} \text{Int}(A) = \{a, b\} \) and \( \lambda_{pc} \text{Int}(B) = \{a, c\} \), but \( \lambda_{pc} \text{Int}(A \cap B) = \phi \). Hence, \( \lambda_{pc} \text{Int}(A \cap B) \neq \lambda_{pc} \text{Int}(A) \cap \lambda_{pc} \text{Int}(B) \).

**Proposition 3.26** if \( A \) is a subset of a space \( X \), then \( \lambda_{pc} \text{Int}(A) = A \setminus \lambda_{pc} \text{Int}(X \setminus A) \).

**Proof.** Obvious.

**Proposition 3.27** If \( A \) is any subset of a space \( X \), then the following statements are true:

1. \( X \setminus \lambda_{pc} \text{Int}(A) = \lambda_{pc} \text{Cl}(X \setminus A) \).
2. \( \lambda_{pc} \text{Cl}(A) = X \setminus \lambda_{pc} \text{Int}(X \setminus A) \).
3. \( X \setminus \lambda_{pc} \text{Cl}(A) = \lambda_{pc} \text{Int}(X \setminus A) \).
4. \( \lambda_{pc} \text{Int}(A) = X \setminus \lambda_{pc} \text{Cl}(X \setminus A) \).

**Proof.** Obvious.

**Proposition 3.28** If \( A \) is a subset of a topological space \( (X, \tau) \), then \( \lambda_{pc} \text{Int}(A) \subseteq p \text{Int}(A) \).

**Proof.** Obvious.

The equality in the above proposition need not be true in general, as shown by the following example:

**Example 3.29** Let \( X = \{a, b, c\} \), and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \). We define an \( p \)-operation \( \lambda : PO(X) \rightarrow P(X) \) as \( \lambda(A) = A \) if \( A = \phi \) or \( A = \{b\} \) and \( \lambda(A) = X \) otherwise. Now, if \( A = \{a, b\} \), then \( \lambda_{pc} \text{Int}(A) = \phi \) and \( p \text{Int}(A) = \{a, b\} \).

**Theorem 3.30** Let \( A, B \) be subsets of \( X \). If the \( p \)-operation \( \lambda : PO(X) \rightarrow P(X) \) is s-regular, then we have:

1. \( \lambda_{pc} \text{Cl}(A \cup B) = \lambda_{pc} \text{Cl}(A) \cup \lambda_{pc} \text{Cl}(B) \).
2. \( \lambda_{pc} \text{Int}(A \cap B) = \lambda_{pc} \text{Int}(A) \cap \lambda_{pc} \text{Int}(B) \).

**Proof.** Obvious.
4. $\lambda_{pc}$-Separation axioms

In this section, we define new types of separation axioms called $\lambda_{pc}$-$T_i$ ($i = 0, 1/2, 1, 2$) and $\lambda_{pc}$-$R_j$ ($j = 0, 1$) by using the notion of $\lambda_{pc}$-open and $\lambda_{pc}$-closed sets. First, we begin with the following definition.

**Definition 4.1** A subset $A$ of $(X, \tau)$ is said to be **generalized $\lambda_{pc}$-closed** (briefly $g$-$\lambda_{pc}$-closed) if $\lambda_{pc}Cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is a $\lambda_{pc}$-open set in $(X, \tau)$.

We say that a subset $B$ of $X$ is generalized $\lambda_{pc}$-open (briefly $g$-$\lambda_{pc}$-open) if its complement $X \setminus B$ is generalized $\lambda_{pc}$-closed in $(X, \tau)$.

**Theorem 4.2** If a subset $A$ of $X$ is $g$-$\lambda_{pc}$-closed and $A \subseteq B \subseteq \lambda_{pc}Cl(A)$, then $B$ is a $g$-$\lambda_{pc}$-closed set in $X$.

**Proof.** Let $A$ be $g$-$\lambda_{pc}$-closed set such that $A \subseteq B \subseteq \lambda_{pc}Cl(A)$. Let $U$ be a $\lambda_{pc}$-open set of $X$ such that $B \subseteq U$.

Since $A$ is $g$-$\lambda_{pc}$-closed, we have $\lambda_{pc}Cl(A) \subseteq U$. Now $\lambda_{pc}Cl(A) \subseteq \lambda_{pc}Cl(B) \subseteq \lambda_{pc}Cl(\lambda_{pc}Cl(A)) = \lambda_{pc}Cl(A) \subseteq U$. This implies that $\lambda_{pc}Cl(B) \subseteq U$, where $U$ is $\lambda_{pc}$-open. Therefore, $B$ is a $g$-$\lambda_{pc}$-closed set in $X$.

In the following example, we have two $g$-$\lambda_{pc}$-closed sets $A$ and $B$ such that $A \subseteq B$ but $B \not\subseteq \lambda_{pc}Cl(A)$.

**Example 4.3** Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{a\}, \{a, c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $\lambda : PO(X) \rightarrow P(X)$ be identity p-operation. If $A = \{a\}$ and $B = \{a, c\}$, then $A$ and $B$ are $g$-$\lambda_{pc}$-closed sets in $(X, \tau)$. But $A \subseteq B \not\subseteq \lambda_{pc}Cl(A)$.

**Theorem 4.4** Let $\lambda : PO(X) \rightarrow P(X)$ be an $p$-operation, then for each singleton set $\{x\}$ is $\lambda_{pc}$-closed or $X \setminus \{x\}$ is $g$-$\lambda_{pc}$-closed in $(X, \tau)$.

**Proof.** Suppose that $\{x\}$ is not $\lambda_{pc}$-closed, then $X \setminus \{x\}$ is not $\lambda_{pc}$-open. Let $U$ be any $\lambda_{pc}$-open set such that $X \setminus \{x\} \subseteq U$, then $U = X$. Therefore $\lambda_{pc}Cl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is $g$-$\lambda_{pc}$-closed.

**Proposition 4.5** A subset $A$ of $(X, \tau)$ is $g$-$\lambda_{pc}$-closed if and only if $\lambda_{pc}Cl(\{x\}) \cap A \neq \phi$, for every $x \in \lambda_{pc}Cl(A)$.

**Proof.** Let $U$ be a $\lambda_{pc}$-open set such that $A \subseteq U$ and let $x \in \lambda_{pc}Cl(A)$. By assumption, there exists a $z \in \lambda_{pc}Cl(\{x\})$ and $z \in A \subseteq U$. It follows From Proposition 3.19, that $U \cap \{x\} \neq \emptyset$, hence $x \in U$, implies $\lambda_{pc}Cl(A) \subseteq U$. Therefore $A$ is $g$-$\lambda_{pc}$-closed.

Conversely, suppose that $x \in \lambda_{pc}Cl(A)$ such that $\lambda_{pc}Cl(\{x\}) \cap A = \emptyset$. Since $A \subseteq X \setminus \lambda_{pc}Cl(\{x\})$ and $A$ is $g$-$\lambda_{pc}$-closed implies that $\lambda_{pc}Cl(A) \subseteq X \setminus \lambda_{pc}Cl(\{x\})$ holds, and hence $x \notin \lambda_{pc}Cl(A)$, which is contradiction. Therefore $\lambda_{pc}Cl(\{x\}) \cap A \neq \emptyset$.

**Theorem 4.6** If a subset $A$ of $X$ is $g$-$\lambda_{pc}$-closed set in $X$, then $\lambda_{pc}Cl(A) \setminus A$ does not contain any non-empty $\lambda_{pc}$-closed set in $X$.

**Proof.** Let $A$ be a $g$-$\lambda_{pc}$-closed set in $X$. Let $F$ be a $\lambda_{pc}$-closed set such that $F \subseteq \lambda_{pc}Cl(A) \setminus A$ and $F \neq \emptyset$. Then $F \subseteq X \setminus A$ which implies that $A \subseteq X \setminus F$. Since $A$ is $g$-$\lambda_{pc}$-closed and $X \setminus F$ is a $\lambda_{pc}$-open set, therefore $\lambda_{pc}Cl(A) \subseteq X \setminus F$, so $F \subseteq X \setminus \lambda_{pc}Cl(A)$. Hence $F \subseteq \lambda_{pc}Cl(A) \cap X \setminus \lambda_{pc}Cl(A) = \emptyset$. This shows that, $F = \emptyset$ which is a contradiction. Hence $\lambda_{pc}Cl(A) \setminus A$ does not contains any non empty $\lambda_{pc}$-closed set in $X$.

**Definition 4.7** Let $(X, \tau)$ be a topological space then $(X, \tau)$ is said to be:

1. a $\lambda_{pc}$-$T_0$ space if for each distinct points $x, y \in X$ there exists a $\lambda_{pc}$-open set $U$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.
2. a $\lambda_{pc}$-$T_{1/2}$ space if every $g$-$\lambda_{pc}$-closed set in $(X, \tau)$ is $\lambda_{pc}$-closed.
3. a $\lambda_{pc}$-$T_1$ space if for each distinct points $x, y \in X$, there exists a $\lambda_{pc}$-open set, containing and respectively such that $y \notin U$ and $x \notin V$.
4. a $\lambda_{pc}$-$T_2$ space if for each $x, y \in X$ there exists a $\lambda_{pc}$-open sets $U, V$ such that $x \in U$ and $y \in V$ and $U \cap V \neq \emptyset$.

**Proposition 4.8** Each $\lambda_{pc}$-$T_i$ space is $pre-T_i$ ($i = 0, 1/2, 1, 2$).

**Proof.** Obvious.

The following example show that every $pre-T_i$ space need not be $\lambda_{pc}$-$T_i$ ($i = 0, 1/2, 1, 2$).
Example 4.9 Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an $p$-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a\}$ and $\lambda(A) = X$ otherwise. Then the space $X$ is a pre-$T_0$ but it is not $\lambda_p$-$T_0$ space. Moreover a space is pre-$T_i$, for $i = 0, 1/2, 1, 2$.

Theorem 4.10 A space $X$ is $\lambda_p$-$T_0$ if and only if for each distinct points $x$ and $y$ in $X$, either $x \notin \lambda_p Cl(\{y\})$ or $y \notin \lambda_p Cl(\{x\})$.

**Proof.** Let $x \neq y$ in a $\lambda_p$-$T_0$ space $X$. Then there exists an $\lambda_p$-open set $U$ containing one of them but not the other, without loss of generality, we assume that $U$ contains $x$ but not $y$. Then $U \cap \{y\} = \emptyset$, this implies that $x \notin \lambda_p Cl(\{y\})$.

Conversely, Let $x$ and $y$ be two distinct points of $X$, then by hypothesis, either $x \notin \lambda_p Cl(\{y\})$ or $y \notin \lambda_p Cl(\{x\})$. With out loss of generality, we assume that $y \notin \lambda_p Cl(\{x\})$. Then $X \setminus \lambda_p Cl(\{x\})$ is an $\lambda_p$-open subset of $X$ containing $y$ but not $x$. Therefore, $X$ is $\lambda_p$-$T_0$.

Theorem 4.11 Let $\lambda : PO(X) \to P(X)$ be an $p$-operation, then the following statements are equivalent:

1. $(X, \tau)$ is $\lambda_p$-$T_{1/2}$.
2. Each singleton $\{x\}$ of $X$ is $\lambda_p$-closed or $\lambda_p$-open.

**Proof.** $(1) \Rightarrow (2)$ : Suppose that $\{x\}$ is not $\lambda_p$-closed. Then by Theorem 4.4, $X\setminus \{x\}$ is $g$-$\lambda_p$-closed. Since $(X, \tau)$ is $\lambda_p$-$T_{1/2}$, then $X\setminus \{x\}$ is $\lambda_p$-closed. Hence, $\{x\}$ is $\lambda_p$-open.

$(2) \Rightarrow (1)$ : Let $A$ be any $g$-$\lambda_p$-closed set in $(X, \tau)$ and $x \in \lambda_p Cl(A)$. By (2), we have $\{x\}$ is $\lambda_p$-closed or $\lambda_p$-open. If $\{x\}$ is $\lambda_p$-closed and $x \notin A$ will imply $x \in \lambda_p Cl(A) \setminus A$ which is not true by Theorem 4.6, so $x \in A$. Therefore, $\lambda_p Cl(A) = A$, so $A$ is $\lambda_p$-closed. Therefore, $(X, \tau)$ is $\lambda_p$-$T_{1/2}$.

On the other hand, if $\{x\}$ is $\lambda_p$-open, then as $x \in \lambda_p Cl(A)$, we have $\{x\} \cap A \neq \emptyset$. Hence $x \in A$, so $A$ is $\lambda_p$-closed.

Corollary 4.12 Each $\lambda_p$-$T_{1/2}$ space is $\lambda_p$-$T_0$ space.

**Proof.** Follows from Theorem 4.11 and Theorem 4.10.

Example 4.13 Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an $p$-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A$ is empty, $A = \{a\}$ or $\{a, b\}$ and $\lambda(A) = X$ otherwise. Then $(X, \tau)$ is a $\lambda_p$-$T_0$ space but not $\lambda_p$-$T_{1/2}$ space because $\{a, b\}$ is $g$-$\lambda_p$-closed but not $\lambda_p$-closed.

Theorem 4.14 Each $\lambda_p$-$T_1$ space is $\lambda_p$-$T_{1/2}$ space.

**Proof.** Follows from Theorem 4.6.

Example 4.15 $X = \{a, b\}$, and $\tau = P(X)$. We define an $p$-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A$ is empty and $\lambda(A) = X$ otherwise. Then $(X, \tau)$ is a $\lambda_p$-$T_1$ space but not $\lambda_p$-$T_1$ space.

Definition 4.16 A topological space $(X, \tau)$ is called a $\lambda_p$-symmetric space if for $x$ and $y$ in $X$, $x \in \lambda_p Cl(\{y\})$ implies that $y \in \lambda_p Cl(\{x\})$.

Theorem 4.17 Let $(X, \tau)$ be a $\lambda_p$-symmetric space, then the following are equivalent:

1. $(X, \tau)$ is $\lambda_p$-$T_0$.
2. $(X, \tau)$ is $\lambda_p$-$T_{1/2}$.
3. $(X, \tau)$ is $\lambda_p$-$T_1$.

**Proof.** It is enough to prove only the necessity of $(1) \Leftrightarrow (2)$. Let $x \neq y$ and since $(X, \tau)$ is $\lambda_p$-$T_0$, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in PO_{\lambda_p}(X)$. Then $x \notin \lambda_p Cl(\{y\})$ and hence $y \notin \lambda_p Cl(\{x\})$. Therefore, there exists $V \in PO_{\lambda_p}(X)$ such that $y \in V \subseteq X \setminus \{x\}$ and $(X, \tau)$ is a $\lambda_p$-$T_1$ space.

Remark 4.18 From the definitions of $\lambda_p$-$T_i$, $(i = 0, 1/2, 1, 2)$ and previous results, we get the following diagram of implications:

$\lambda_p$-$T_2 \Rightarrow \lambda_p$-$T_1 \Rightarrow \lambda_p$-$T_{1/2} \Rightarrow \lambda_p$-$T_0$
Definition 4.19 Let $\lambda : PO(X) \rightarrow P(X)$ be an $p$-operation, a topological space $(X, \tau)$ is called $\lambda_{pc}$-$R_0$ if $U \in PO_{\lambda_{pc}}(X)$ and $x \in U$, then $\lambda_{pc}Cl\{x\} \subseteq U$.

Theorem 4.20 For any topological space $X$ and any $s$-operation $\lambda$, the following are equivalent:

1. $X$ is $\lambda_{pc}$-$R_0$.
2. $F \in PC_{\lambda_{pc}}(X)$ and $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in PO_{\lambda_{pc}}(X)$.
3. $F \in PC_{\lambda_{pc}}(X)$ and $x \notin F$ implies $F \cap \lambda_{pc}Cl\{x\} = \phi$.
4. For any two distinct points $x$, $y$ of $X$, either $\lambda_{pc}Cl\{x\} = \lambda_{pc}Cl\{y\}$ or $\lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$.

Proof. (1) $\Rightarrow$ (2): $F \in PC_{\lambda_{pc}}(X)$ and $x \notin F$ implies $x \in X \setminus F \in PO_{\lambda_{pc}}(X)$ then $\lambda_{pc}Cl\{x\} \subseteq X \setminus F$. By (1), if we put $U = X \setminus \lambda_{pc}Cl\{x\}$, then $x \notin U \in PO_{\lambda_{pc}}(X)$ and $F \subseteq U$.

(2) $\Rightarrow$ (3) : if $F \in PC_{\lambda_{pc}}(X)$ and $x \notin F$, then there exists $U \in PO_{\lambda_{pc}}(X)$ such that $x \notin U$ and $F \subseteq U$. By (2), we have $U \cap \lambda_{pc}Cl\{x\} = \phi$, so $F \cap \lambda_{pc}Cl\{x\} = \phi$.

(3) $\Rightarrow$ (4): Suppose that for any two distinct points $x, y$ of $X$, $\lambda_{pc}Cl\{x\} \neq \lambda_{pc}Cl\{y\}$. Then suppose, without loss of generality, that there exists some $z \in \lambda_{pc}Cl\{x\}$ such that $z \notin \lambda_{pc}Cl\{y\}$. Thus there exists a $\lambda_{pc}$-open set $V$ such that $z \in V$ and $y \notin V$ but $x \notin V$. Thus $x \notin \lambda_{pc}Cl\{y\}$. Hence by (3), $\lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$.

(4) $\Rightarrow$ (1): Let $U \in PO_{\lambda_{pc}}(X)$ and $x \in U$. Then for each $y \notin U$, $x \notin \lambda_{pc}Cl\{y\}$. Thus $\lambda_{pc}Cl\{x\} \neq \lambda_{pc}Cl\{y\}$. Hence by (4), $\lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$, for each $y \in X \setminus U$. So $\lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$. Now, $U \in PO_{\lambda_{pc}}(X)$ and $y \notin X \setminus U$, then $\{y\} \subseteq \lambda_{pc}Cl\{y\} \subseteq \lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$. Thus $X \setminus U = \bigcup \lambda_{pc}Cl\{y\} : y \in \lambda_{pc}Cl\{y\}$. Hence, $\lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$, so $\lambda_{pc}Cl\{x\} \subseteq \lambda_{pc}Cl\{y\}$. This implies that $(X, \tau)$ is $\lambda_{pc}$-$R_0$.

Theorem 4.21 Let $(X, \tau)$ be a topological space and $\lambda : PO(X) \rightarrow P(X)$ be any $p$-operation, then the following are equivalent:

1. $X$ is $\lambda_{pc}$-$T_1$.
2. $\lambda_{pc}Cl\{x\} = \{x\}$ for all $x \in X$.
3. $X$ is $\lambda_{pc}$-$R_0$ and $\lambda_{pc}$-$T_0$.

Proof. (1) $\Rightarrow$ (2): Let $y \notin \{x\}$, then there exists $U \in PO_{\lambda_{pc}}(X)$ such that $y \in U$, $x \notin U$, so $U \cap \{x\} = \phi$. Hence $y \notin \lambda_{pc}Cl\{x\}$ implies $\lambda_{pc}Cl\{x\} \subseteq \{x\}$ also $\{x\} \subseteq \lambda_{pc}Cl\{x\}$ always, hence $\lambda_{pc}Cl\{x\} = \{x\}$ for all $x \in X$.

(2) $\Rightarrow$ (3) : Let $x$, $y \in X$ with $x \neq y$. Then $\{x\} \cap \{y\}$ are $\lambda_{pc}$-closed and hence $X \setminus \{x\}$ is a $\lambda_{pc}$-open set containing $y$ but not $x$. This shows that $X$ is $\lambda_{pc}$-$T_2$. Again, $x$, $y \in X$ with $x \neq y$, then $\lambda_{pc}Cl\{x\} \neq \lambda_{pc}Cl\{y\}$. Also, $\lambda_{pc}Cl\{x\} \cap \lambda_{pc}Cl\{y\} = \phi$. Thus, by Theorem 4.20, $X$ is $\lambda_{pc}$-$R_0$.

(3) $\Rightarrow$ (1): Let $x$, $y \in X$ with $x \neq y$. there exists $U \in PO_{\lambda_{pc}}(X)$ such that $x \in U$ and $y \notin U$ then, $\lambda_{pc}Cl\{x\} \subseteq U$ (as $X$ is $\lambda_{pc}$-$R_0$) and so $y \notin \lambda_{pc}Cl\{x\}$. Hence $x \in U \in PO_{\lambda_{pc}}(X)$, $y \notin U$ and $y \in X \setminus \lambda_{pc}Cl\{x\} \subseteq PO_{\lambda_{pc}}(X)$, $x \notin X \setminus \lambda_{pc}Cl\{x\}$. Therefore, $X$ is a $\lambda_{pc}$-$T_1$ space.

Definition 4.22 Let $(X, \tau)$ be a topological space $\lambda : PO(X) \rightarrow P(X)$ be an $p$-operation. The space $X$ is said to be $\lambda_{pc}$-$R_1$ if for $x$, $y \in X$ with $\lambda_{pc}Cl\{x\} \neq \lambda_{pc}Cl\{y\}$, there exist disjoint $\lambda_{pc}$-open sets $U$ and $V$ such that $\lambda_{pc}Cl\{x\} \subseteq U$ and $\lambda_{pc}Cl\{y\} \subseteq V$.

Theorem 4.23 If $\lambda : PO(X) \rightarrow P(X)$ is an $p$-operation and $X$ is $\lambda_{pc}$-$R_1$, then $X$ is $\lambda_{pc}$-$R_0$.

Proof. Let $U \in PO_{\lambda_{pc}}(X)$ and $x \in U$. If $y \notin U$, then $\lambda_{pc}Cl\{x\} \neq \lambda_{pc}Cl\{y\}$ (as $x \notin \lambda_{pc}Cl\{y\}$). Hence there exists $V \in PO_{\lambda_{pc}}(X)$ such that $\lambda_{pc}Cl\{y\} \subseteq V$ and $x \notin V$. This gives that $y \notin \lambda_{pc}Cl\{x\}$, so $\lambda_{pc}Cl\{x\} \subseteq U$. Hence, $X$ is a $\lambda_{pc}$-$R_0$ space.

By the following examples, we show the converse of above theorem is not true in general, and also we show $\lambda_{pc}$-$R_0$ and pre-$R_0$ are independent.

Example 4.24 Let $X = \{a, b\}$ and $\tau = \{\phi, \{a\}, X\}$. We define an $p$-operation $\lambda : PO(X) \rightarrow P(X)$ as $\lambda(A) = A$ if $A$ is empty and $\lambda(A) = X$ otherwise. Clearly $X$ is $\lambda_{pc}$-$R_0$, but it is neither pre-$R_0$ nor pre-$R_1$.

Example 4.25 Let $X = \{a, b\}$, and $\tau = P(X)$. We define an $p$-operation $\lambda : PO(X) \rightarrow P(X)$ as $\lambda(A) = A$ if $A$ is empty or $A = \{a\}$ and $\lambda(A) = X$ otherwise. Clearly $X$ is pre-$R_0$ and pre-$R_1$, but it is not $\lambda_{pc}$-$R_0$. 

International Journal of Basic and Applied Sciences
Theorem 4.26 Let $(X, \tau)$ be a topological space $\lambda : PO(X) \rightarrow P(X)$ be an $p$-operation. Then the following are equivalent:

1. $X$ is $\lambda_{pc}-T_2$.
2. $X$ is $\lambda_{pc}-R_1$ and $\lambda_{pc}-T_1$.
3. $X$ is $\lambda_{pc}-R_1$ and $\lambda_{pc}-T_0$.

Proof. (1) $\Rightarrow$ (2) : Let $X$ be $\lambda_{pc}-T_2$, then $X$ is clearly $\lambda_{pc}-T_1$. Now if $x, y \in X$ with $\lambda_{pc} Cl({x}) \neq \lambda_{pc} Cl({y})$ then $x \neq y$, so there exist $U, V \in PO_{\lambda_{pc}}(X)$ such that $x \in U, y \in V$ and $U \cap V = \phi$. Hence by Theorem 4.21, $\lambda_{pc} Cl({x}) = \{x\} \subseteq U$ and $\lambda_{pc} Cl({y}) = \{y\} \subseteq V$ and $U \cap V = \phi$. Therefore, $X$ is $\lambda_{pc}-R_1$.

(2) $\Rightarrow$ (3) : It is obvious.

(3) $\Rightarrow$ (1) : Let $X$ be $\lambda_{pc}-R_1$ and $\lambda_{pc}-T_0$, then by Theorem 4.23, $X$ is $\lambda_{pc}-R_0$ and $\lambda_{pc}-T_0$. Hence, by Theorem 4.21, $X$ is $\lambda_{pc}-T_1$. If $x, y \in X$ with $x \neq y$, then $\lambda_{pc} Cl({x}) = \{x\} \neq \{y\} = \lambda_{pc} Cl({y})$. Since $X$ is $\lambda_{pc}-R_1$, so there exist $U, V \in PO_{\lambda_{pc}}(X)$ such that $\lambda_{pc} Cl({x}) = \{x\} \subseteq U, \lambda_{pc} Cl({y}) = \{y\} \subseteq V$ and $U \cap V = \phi$. Hence, $X$ is $\lambda_{pc}-T_2$.

References


