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Research Paper

λ_{pc}-open sets and λ_{pc}-separation axioms in topological spaces

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Abstract

The aim of this paper is to introduce a new class of sets called λ_{pc}- open sets and to investigate some of their relationships and properties. Further, by using this set, the notion of λ_{pc}-T_i spaces (i = 0,1/2, 1, 2) and λ_{pc}-R_j spaces (j = 0, 1) are introduced and some of their properties are investigated.

Keywords: *s-operation; λ_{pc}-Open Set; λ_{pc}-T_i, i=0,1,2; λ_{pc}-R_j, j=0,1.*

1. Introduction

The study of pre-open sets and pre-continuity in topological spaces was initiated by Mashhour, El-Monsef and El-Deeb [7]. Analogous to generalized closed sets which was introduced by Levine [8], Maki, Umehara and Nori [2], introduced the concept of pre-generalized closed sets in topological spaces. Kasahara [6], defined the concept of an operation on topological spaces and introduced the concept of closed graphs of a function. Ahmad and Hussain [1], continued studying the properties of operations on topological spaces introduced by Kasahara [6]. Ogata [10], introduced the concept of γ-T_i (i = 0,1/2,1,2) and characterized γ-T_i by the notion of γ- closed sets or γ-open sets. Chattopadhyay [9] defined other new types of separation axioms and Caldas, Jafari and Nori [5], defined pre-R_1, and pre-R_0 spaces.

In this paper, we introduce and study a new class of pre-open sets called λ_{pc}-open sets in topological spaces. By using the notion of λ_{pc}-closed and λ_{pc}-open sets, we introduce the concept of λ_{pc}-T_i $(i = 0, 1/2, 1, 2)$ and λ_{pc}-R_j $(j = 0, 1)$ spaces. several properties and characterizations of these spaces are obtained.

2. Preliminaries

Throughout, X denote a topological space with out any separation axiom. Let A be a subset of X, the closure (interior) of A are denoted by $Cl(A)$ $(Int(A))$ respectively. A subset A of a topological space (X, τ) is said to be pre-open [7] if $A \subseteq Int(Cl(A))$. The complement of a pre-open set is said to be pre-closed [7]. The family of all pre-open (resp. pre-closed) sets in a topological space (X, τ) is denoted by $PO(X, \tau)$ or $PO(X)$ (resp. $PC(X, \tau)$ or $PC(X)$).

Definition 2.1 *[4] Let (X, τ) be a topological space and let A be a subset of X then:*

1. *The pre-interior of A $(pInt(A))$ is the union of all pre-open sets of X contained in A.*
2. *A point $x \in X$ is said to be a pre-limit point of A if every pre-open set containing x contains a point of A different from x, and the set of all pre-limit points of A is called the pre-derived set of A denoted by $pd(A)$.*
3. *The intersection of all pre-closed sets of X containing A is called the pre-closure of A and is denoted by $pCl(A)$.*

Definition 2.2 *[2] A subset A of a space (X, τ) is called a pre-generalized closed set (pg-closed), if $A \subseteq U$ and U is pre-open implies that $pCl(A) \subseteq U$.*

Definition 2.3 *A topological space (X, τ) is said to be:*

1. *pre-T_0 [9] if for any distinct pair of points in X, there is an pre-open set containing one of the points but not the other.*
2. *pre-T_1 [9] if for any distinct pair of points x and y in X, there is a pre-open U in X containing x but not y and a pre-open set V in X containing y but not x.*
3. *pre-T_2 [9] if for any distinct pair of points x and y in X, there exist pre-open sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.*
4. *pre-$T_{1/2}$ [2] if every pg-closed set is pre-closed.*
5. *pre-R_0 [5] if for each $O \in PO(X)$ and $x \in O$, $pCl(\{x\}) \subseteq O$.*
6. *pre-R_1 [5] if for each pair $x, y \in X$ such that $pCl(\{x\}) \neq pCl(\{y\})$, there exist disjoint pre-open sets U and V such that $pCl(\{x\}) \subseteq U$ and $pCl(\{y\}) \subseteq V$.*

Definition 2.4 *[10] Let (X, τ) be any topological space. A mapping $\lambda : \tau \to P(X)$, ($P(X)$ stands for all subsets of X), is called an operation on τ if $V \subseteq \lambda(V)$ for each non-empty open set V and $\lambda(\phi) = \phi$.*

Definition 2.5 *[3] Let (X, τ) be a topological space and let A be a subset of X then:*

1. *The λ-interior of A $({}_\lambda Int(A))$ is the union of all λ-open sets of X contained in A.*
2. *A point $x \in X$ is said to be a λ-limit point of A if every λ-open set containing x contains a point of A different from x, and the set of all λ-limit points of A is called the λ-derived set of A denoted by ${}_\lambda d(A)$.*
3. *The intersection of all λ-closed sets of X containing A is called the λ-closure of A and is denoted by ${}_\lambda Cl(A)$.*

3. λ_{pc}-Open sets

In this section, we introduce a new class of pre-open sets called λ_{pc}-open sets. Further, the notion of λ_{pc}-derived set, λ_{pc}-closure and λ_{pc}-interior are introduced and their properties are discussed.

Definition 3.1 *A mapping $\lambda : PO(X) \to P(X)$ is called an p-operation on $PO(X)$ if $V \subseteq \lambda(V)$ for each non-empty pre-open set V and $\lambda(\phi) = \phi$.*

If $\lambda : PO(X) \to P(X)$ is any p-operation, then it is clear that $\lambda(X) = X$.

Definition 3.2 *Let (X, τ) be a topological space and $\lambda : PO(X) \to P(X)$ be an p-operation defined on $PO(X)$, then a subset A of X is λp-open set if for each $x \in A$ there exists a pre-open set U such that $x \in U$ and $\lambda(U) \subseteq A$.*

Definition 3.3 *A λp-open subset A of X is called λ_{pc}-open if for each $x \in A$ there exists a closed subset F of X such that $x \in F \subseteq A$.*

The complement of a λ_{pc}-open set is said to be λ_{pc}-closed. The family of all λ_{pc}-open (resp., λ_{pc}-closed) subsets of a topological space (X, τ) is denoted by $PO_{\lambda_{pc}}(X, \tau)$ or $PO_{\lambda_{pc}}(X)$(resp., $PC_{\lambda_{pc}}(X, \tau)$ or $PC_{\lambda_{pc}}(X)$)

Proposition 3.4 *For any topological space (X, τ), we have $PO_{\lambda_{pc}}(X) \subseteq PO_{\lambda}(X) \subseteq PO(X)$.*

Proof. Obvious

The following examples show that the equality in the above proposition may not be true in general.

Example 3.5 *Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a, c\}$ or A is empty and $\lambda(A) = X$ otherwise. Here, we have $\{a, c\}$ is λp-open set but it is not λ_{pc}-open.*

The following examples shows that τ is incomparable with $\lambda_{pc}O(X)$.

Example 3.6 *Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A \neq \{a\}$ or $\{b\}$ and $\lambda(A) = \{a, b\}$ if $A = \{a\}$ or $\{b\}$. Now, we have $\{a\}$ is open set but not λ_{pc}-open.*

Example 3.7 *Let N be a set of natural numbers. In a topological space (N, τ) with cofinite topology. We define an p-operation $\lambda : PO(N) \to P(N)$ as $\lambda(A) = A$. Then we obtain that $\{1, 3, 5, ...\}$ is λ_{pc}-open set but not open.*

Proposition 3.8 *Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of λ_{pc}-open sets in a topological space (X, τ), then $\bigcup_{\alpha \in I} A_\alpha$ is a λ_{pc}-open set.*

Proof. Since A_α is λ_{pc}-open set for all $\alpha \in I$, then A_α is a λp-open set for all $\alpha \in I$. This implies that there exists a pre-open set U such that $\lambda(U) \subseteq A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$. Therefore, $\bigcup_{\alpha \in I} A_\alpha$ is a λp-open subset of (X, τ). Let $x \in \bigcup_{\alpha \in I} A_\alpha$, then there exists an $\alpha_0 \in I$ such that $x \in A_{\alpha 0}$. Since A_α is a λ_{pc}-open set for all $\alpha \in I$, then there exists a closed set F such that $x \in F \subseteq A_{\alpha 0}$ but $A_{\alpha 0} \subseteq \bigcup_{\alpha \in I} A_\alpha$, then $x \in F \subseteq \bigcup_{\alpha \in I} A_\alpha$. Hence, $\bigcup_{\alpha \in I} A_\alpha$ is λ_{pc}-open.

The following example shows that the intersection of two λ_{pc}-open sets need not be λ_{pc}-open.

Example 3.9 *Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A \neq \{b\}$ and $\lambda(A) = X$ otherwise. So we have $\{a, b\}$ and $\{b, c\}$ are λ_{pc}-open sets but $\{a, b\} \cap \{b, c\} = \{b\}$ is not λ_{pc}-open.*

Proposition 3.10 *The set A is λ_{pc}-open in the space (X, τ) if and only if for each $x \in A$ there exists a λ_{pc}-open set B such that $x \in B \subseteq A$.*

Proof. Suppose that A is λ_{pc}-open in (X, τ). Then for each $x \in A$ we put $B = A$ is a λ_{pc}-open set such that $x \in B \subseteq A$.
Conversely, Suppose that for each $x \in A$ there exists a λ_{pc}-open set B_x such that $x \in B_x \subseteq A$. Thus $A = \bigcup B_x$, where $B_x \in PO_{\lambda pc}(X)$ for each x. Therefore, by Proposition 3.8, A is λ_{pc}-open.

Definition 3.11 *Let (X, τ) be a topological space. An p-operation λ is said to be p-regular if for every pre-open sets U and V containing $x \in X$, there exists a pre-open set W containing x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.*

Theorem 3.12 *Let λ be an p-regular p-operation. If A and B are λ_{pc}-open sets in X, then $A \cap B$ is also λ_{pc}-open.*

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are λp-open sets, so there exist pre-open sets U and V such that $x \in U$ and $\lambda(U) \subseteq A$, $x \in V$ and $\lambda(V) \subseteq B$. Since λ is p-regular, this implies that there exists a pre-open set W of x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V) \subseteq A \cap B$. Therefore, $A \cap B$ is λp-open set. Again for each $x \in A \cap B$, we have $x \in A$ and $x \in B$ and since A and B areλ_{pc}-open sets, then there exist closed sets E, F such that $x \in E \subseteq A$ and $x \in F \subseteq B$. Therefore, $x \in E \cap F \subseteq A \cap B$. Since $E \cap F$ is closed, so by Definition 3.2, we obtain that $A \cap B$ is λ_{pc}-open.

Definition 3.13 *Let (X, τ) be a topological space and let A be subset of X, then a point $x \in X$ is called a λ_{pc}-limit point of A if every λ_{pc}-open set containing x contains a point of A different from x.*

The set of all λ_{pc}-limit points of A is called the λ_{pc}-derived set of A denoted by $\lambda_{pc}d(A)$.

Definition 3.14 *Let A be subset of the space (X, τ), then the λ_{pc}-closure of A ($\lambda_{pc}Cl(A)$) is the intersection of all λ_{pc}-closed sets containing A.*

Here we introduce some properties of λ_{pc}-closure of the sets.

Proposition 3.15 *For subsets A, B of a topological space (X, τ), the following statements are true.*

1. $A \subseteq \lambda_{pc}Cl(A)$.
2. $\lambda_{pc}Cl(A)$ *is* λ_{pc}*-closed set in* X.
3. $\lambda_{pc}Cl(A)$ *is smallest* λ_{pc}*-closed set which contain* A.
4. A *is* λ_{pc}*-closed set if and only if* $A = \lambda_{pc}Cl(A)$.
5. $\lambda_{pc}Cl(\phi) = \phi$ *and* $\lambda_{pc}Cl(X) = X$.
6. *If* $A \subseteq B$. *Then* $\lambda_{pc}Cl(A) \subseteq \lambda_{pc}Cl(B)$.
7. $\lambda_{pc}Cl(A) \cup \lambda_{pc}Cl(B) \subseteq \lambda_{pc}Cl(A \cup B)$.
8. $\lambda_{pc}Cl(A \cap B) \subseteq \lambda_{pc}Cl(A) \cap \lambda_{pc}Cl(B)$.

Proof. Obvious.

In general the equalities (7) and (8) of the above proposition is not true, as it is shown in the following examples:

Example 3.16 *Let* $X = \{a, b, c\}$, *and* $\tau = P(X)$. *We define an p-operation* $\lambda : PO(X) \to P(X)$ *as* $\lambda(A) = A$ *if* $A = \phi$, $A = \{a, b\}$ *or* $\{b, c\}$ *and* $\lambda(A) = X$ *otherwise. Now, if* $A = \{b\}$ *and* $B = \{c\}$, *then* $\lambda_{pc}Cl(A) = \{b\}$ *and* $\lambda_{pc}Cl(B) = \{c\}$, *but* $\lambda_{pc}Cl(A \cup B) = X$, *where* $A \cup B = \{b, c\}$. $\lambda_{pc}Cl(A \cup B) \neq \lambda_{pc}Cl(A) \cup \lambda_{pc}Cl(B)$.

Example 3.17 *Let* $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. *We define an p-operation* $\lambda : PO(X) \to P(X)$ *as* $\lambda(A) = A$ *if* $A = \phi$ *or* $c \in A$ *and* $\lambda(A) = Cl(A)$ *otherwise. Now, if* $A = \{a\}$ *and* $B = \{b\}$ *then* $\lambda_{pc}Cl(A) = \{a, b\}$ *and* $\lambda_{pc}Cl(B) = \{b\}$, *but* $\lambda_{pc}Cl(A \cap B) = \phi$, *where* $A \cap B = \phi$. *Hence* $\lambda_{pc}Cl(A \cap B) \neq \lambda_{pc}Cl(A) \cap \lambda_{pc}Cl(B)$.

Proposition 3.18 *Let* A *be any subset of a space* X, *then* $\lambda_{pc}Cl(A) = A \cup \lambda_{pc}d(A)$.

Proof. Obvious.

Proposition 3.19 *If* A *is a subset of* (X, τ), *then* $x \in \lambda_{pc}Cl(A)$ *if and only if* $V \cap A \neq \phi$ *for every* λ_{pc}*-open set* V *containing* x.

Proof. Let $x \in \lambda_{pc}Cl(A)$ and suppose that $V \cap A = \phi$ for some λ_{pc}-open set V which contains x. This implies that $X \backslash V$ is λ_{pc}-closed and $A \subseteq (X \backslash V)$, so $\lambda_{pc}Cl(A) \subseteq (X \backslash V)$. But this implies that $x \in (X \backslash V)$ which is contradiction. Therefore, $V \cap A \neq \phi$.
Conversely, Let $A \subseteq X$ and $x \in X$ such that for each λ_{pc}-open set V containing x, $V \cap A \neq \phi$. If $x \notin \lambda_{pc}Cl(A)$, then there is a λ_{pc}-closed set S such that $A \subseteq S$ and $x \notin S$. Hence, $(X \backslash S)$ is a λ_{pc}-open set with $x \in (X \backslash S)$ and thus $(X \backslash S) \cap A \neq \phi$ which is a contradiction. Therefore, $x \in \lambda_{pc}Cl(A)$.

Proposition 3.20 *If* A *is any subset of a topological space* (X, τ), *then* ${}_pCl(A) \subseteq \lambda_{pc}Cl(A)$.

Proof. Obvious.

The following example shows that the equality in the above proposition is not true in general.

Example 3.21 *Let* $X = \{a, b, c\}$, *and* $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. *We define an p-operation* $\lambda : PO(X) \to P(X)$ *as* $\lambda(A) = A$ *if* $A = \phi$ *or* $A = \{a\}$ *and* $\lambda(A) = X$ *otherwise. Now, if* $A = \{c\}$, *then* ${}_pCl(A) = \{c\}$ *and* $\lambda_{pc}Cl(A) = X$.

Definition 3.22 *Let* (X, τ) *be a topological space and let* A *be subset of* X, *then the* λ_{pc}*-interior of* A $(\lambda_{pc}Int(A))$ *is the union of all* λ_{pc}*-open sets of* X *contained in* A.

Proposition 3.23 *For subsets* A, B *of a space* X, *the following statements hold.*

1. $\lambda_{pc}Int(A)$ *is the union of all* λ_{pc}*-open sets which are contained in* A.
2. $\lambda_{pc}Int(A)$ *is a* λ_{pc}*-open set in* X.
3. $\lambda_{pc}Int(A) \subseteq A$.
4. $\lambda_{pc}Int(A)$ *is the largest* λ_{pc}*-open set contained in* A.
5. A *is* λ_{pc}*-open set if and only if* $\lambda_{pc}Int(A) = A$.

6. $\lambda_{pc}Int(\lambda_{pc}Int(A)) = \lambda_{pc}Int(A)$.

7. If $A \subseteq B$, then $\lambda_{pc}Int(A) \subseteq \lambda_{pc}Int(B)$.

8. $\lambda_{pc}Int(\phi) = \phi$ and $\lambda_{pc}Int(X) = X$.

9. $\lambda_{pc}Int(A) \cup \lambda_{pc}Int(B) \subseteq \lambda_{pc}Int(A \cup B)$.

10. $\lambda_{pc}Int(A \cap B) \subseteq \lambda_{pc}Int(A) \cap \lambda_{pc}Int(B)$.

Proof. Obvious.

In general the equalities of (9) and (10) of the above proposition is not true, as it is shown in the following examples:

Example 3.24 *Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $b \in A$ and $\lambda(A) = Cl(A)$ if $b \notin A$. Now, let $A = \{a\}$ and $B = \{c\}$, then $\lambda_{pc}Int(A) = \phi$, and $\lambda_{pc}Int(B) = \phi$, but $\lambda_{pc}Int(A \cup B) = \{a, c\}$. Thus $\lambda_{pc}Int(A \cup B) \neq \lambda_{pc}Int(A) \cup \lambda_{c}Int(B)$.*

Example 3.25 *Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$, $A = \{c\}$, $\{a, b\}$ or $\{a, c\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{a, b\}$ and $B = \{a, c\}$, then $\lambda_{pc}Int(A) = \{a, b\}$ and $\lambda_{pc}Int(B) = \{a, c\}$, but $\lambda_{pc}Int(A \cap B) = \phi$. Hence, $\lambda_{pc}Int(A \cap B) \neq \lambda_{pc}Int(A) \cap \lambda_{pc}Int(B)$.*

Proposition 3.26 *if A is a subset of a space X, then $\lambda_{pc}Int(A) = A \backslash \lambda_{pc}d(X \backslash A)$.*

Proof. Obvious.

Proposition 3.27 *If A is any subset of a space X, then the following statements are true:*

1. $X \backslash \lambda_{pc}Int(A) = \lambda_{pc}Cl(X \backslash A)$.

2. $\lambda_{pc}Cl(A) = X \backslash \lambda_{pc}Int(X \backslash A)$.

3. $X \backslash \lambda_{pc}Cl(A) = \lambda_{pc}Int(X \backslash A)$.

4. $\lambda_{pc}Int(A) = X \backslash \lambda_{pc}Cl(X \backslash A)$.

Proof. Obvious.

Proposition 3.28 *If A is a subset of a topological space (X, τ), then $\lambda_{pc}Int(A) \subseteq {}_{p}Int(A)$.*

Proof. Obvious.

The equality in the above proposition need not be true in general, as shown by the following example:

Example 3.29 *Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A = \{b\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{a, b\}$, then $\lambda_{pc}Int(A) = \phi$ and ${}_{p}Int(A) = \{a, b\}$.*

Theorem 3.30 *Let A, B be subsets of X. If the p-operation $\lambda : PO(X) \to P(X)$ is s-regular, then we have:*

1. $\lambda_{pc}Cl(A \cup B) = \lambda_{pc}Cl(A) \cup \lambda_{pc}Cl(B)$.

2. $\lambda_{pc}Int(A \cap B) = \lambda_{pc}Int(A) \cap \lambda_{pc}Int(B)$.

Proof. Obvious.

4. λ_{pc}-Separation axioms

In this section, we define new types of separation axioms called λ_{pc}-T_i $(i = 0, 1/2, 1, 2)$ and λ_{pc}-R_j $(j = 0, 1)$ by using the notion of λ_{pc}-open and λ_{pc}-closed sets. First, we begin with the following definition.

Definition 4.1 *A subset A of (X, τ) is said to be generalized λ_{pc}-closed (briefly g-λ_{pc}-closed) if $\lambda_{pc}Cl(A) \subseteq U$, whenever $A \subseteq U$ and U is a λ_{pc}-open set in (X, τ).*

We say that a subset B of X is generalized λ_{pc}-open (briefly. g-λ_{pc}-open) if its complement $X \backslash B$ is generalized λ_{pc}-closed in (X, τ).

Theorem 4.2 *If a subset A of X is g-λ_{pc}-closed and $A \subseteq B \subseteq \lambda_{pc}Cl(A)$, then B is a g-λ_{pc}-closed set in X.*

Proof. Let A be g-λ_{pc}-closed set such that $A \subseteq B \subseteq \lambda_{pc}Cl(A)$. Let U be a λ_{pc}-open set of X such that $B \subseteq U$. Since A is g-λ_{pc}-closed, we have λ_{pc}Cl(A) $\subseteq U$. Now $\lambda_{pc}Cl(A) \subseteq \lambda_{pc}Cl(B) \subseteq \lambda_{pc}Cl(\lambda_{pc}Cl(A)) = \lambda_{pc}Cl(A) \subseteq U$. This implies that $\lambda_{pc}Cl(B) \subseteq U$, where U is λ_{pc}-open. Therefore, B is a g-λ_{pc}-closed set in X.

In the following example, we have two g-λ_{pc}-closed sets A and B such that $A \subseteq B$ but $B \not\subseteq \lambda_{pc}Cl(A)$.

Example 4.3 *Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Let $\lambda : PO(X) \to P(X)$ be identity p-operation. If $A = \{a\}$ and $B = \{a, c\}$, then A and B are g-λ_{pc}-closed sets in (X, τ). But $A \subseteq B \not\subseteq \lambda_{pc}Cl(A)$.*

Theorem 4.4 *Let $\lambda : PO(X) \to P(X)$ be an p-operation, then for each singleton set $\{x\}$ is λ_{pc}-closed or $X \backslash \{x\}$ is g-λ_{pc}-closed in (X, τ).*

Proof. Suppose that $\{x\}$ is not λ_{pc}-closed, then $X \backslash \{x\}$ is not λ_{pc}-open. Let U be any λ_{pc}-open set such that $X \backslash \{x\} \subseteq U$, then $U = X$. Therefore $\lambda_{pc}Cl(X \backslash \{x\}) \subseteq U$. Hence $X \backslash \{x\}$ is g-λ_{pc}-closed.

Proposition 4.5 *A subset A of (X, τ) is g-λ_{pc}-closed if and only if $\lambda_{pc}Cl(\{x\}) \cap A \neq \phi$, for every $x \in \lambda_{pc}Cl(A)$.*

Proof. Let U be aλ_{pc}-open set such that $A \subseteq U$ and let $x \in \lambda_{pc}Cl(A)$. By assumption, there exists a $z \in \lambda_{pc}Cl(\{x\})$ and $z \in A \subseteq U$. It follows From Proposition 3.19, that $U \cap \{x\} \neq \phi$, hence $x \in U$, implies $\lambda_{pc}Cl(A) \subseteq U$. Therefore A is g-λ_{pc}-closed.
Conversely, suppose that $x \in \lambda_{pc}Cl(A)$ such that $\lambda_{pc}Cl(\{x\}) \cap A = \phi$. Since $A \subseteq X \backslash \lambda_{pc}Cl(\{x\})$ and A is g-λ_{pc}-closed implies that $\lambda_{pc}Cl(A) \subseteq X \backslash \lambda_{pc}Cl(\{x\})$ holds, and hence $x \notin \lambda_{pc}Cl(A)$, which is contradiction. Therefore $\lambda_{pc}Cl(\{x\}) \cap A \neq \phi$.

Theorem 4.6 *If a subset A of X is g-λ_{pc}-closed set in X, then $\lambda_{pc}Cl(A) \backslash A$ does not contain any non empty λ_{pc}-closed set in X.*

Proof. Let A be a g-λ_{pc}-closed set in X. Let F be a λ_{pc}-closed set such that $F \subseteq \lambda_{pc}Cl(A) \backslash A$ and $F \neq \phi$. Then $F \subseteq X \backslash A$ which implies that $A \subseteq X \backslash F$. Since A is g-λ_{pc}-closed and $X \backslash F$ is a λ_{pc}-open set, therefore $\lambda_{pc}Cl(A) \subseteq X \backslash F$, so $F \subseteq X \backslash \lambda_{pc}Cl(A)$. Hence $F \subseteq \lambda_{pc}Cl(A) \cap X \backslash \lambda_{pc}Cl(A) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $\lambda_{pc}Cl(A) \backslash A$ does not contains any non empty λ_{pc}-closed set in X.

Definition 4.7 *Let (X, τ) be a topological space then (X, τ) is said to be:*

1. *a λ_{pc}-T_0 space if for each distinct points $x, y \in X$ there exists a λ_{pc}-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

2. *a λ_{pc}-$T_{1/2}$ space if every g-λ_{pc}-closed set in (X, τ) is λ_{pc}-closed.*

3. *a λ_{pc}-T_1 space if for each distinct points $x, y \in X$, there exists a λ_{pc}-open set, containing and respectively such that $y \notin U$ and $x \notin V$.*

4. *a λ_{pc}-T_2 space if for each $x, y \in X$ there exists a λ_{pc}-open sets U, V such that $x \in U$ and $y \in V$ and $U \cap V \neq \phi$.*

Proposition 4.8 *Each λ_{pc}-T_i space is pre-T_i $(i = 0, 1/2, 1, 2)$.*

Proof. Obvious.

The following example show that every pre-T_i space need not be λ_{pc}-T_i $(i = 0, 1/2, 1, 2)$.

Example 4.9 *Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a\}$ and $\lambda(A) = X$ otherwise. Then the space X is a pre-T_0 but it is not λ_{pc}-T_0 space. Moreover a space is pre-T_i, for $i = 0, 1/2, 1, 2$.*

Theorem 4.10 *A space X is λ_{pc}-T_0 if and only if for each distinct points x and y in X, either $x \notin \lambda_{pc}Cl(\{y\})$ or $y \notin \lambda_{pc}Cl(\{x\})$, .*

Proof. Let $x \neq y$ in a λ_{pc}-T_0 space X. Then there exists an λ_{pc}-open set U containing one of them but not the other, without loss of generality, we assume that U contains x but not y. Then $U \cap \{y\} = \phi$, this implies that $x \notin \lambda_{pc}Cl(\{y\})$.
Conversely, Let x and y be two distinct points of X, then by hypothesis, either $x \notin \lambda_{pc}Cl(\{y\})$ or $y \notin \lambda_{pc}Cl(\{x\})$. With out loss of generality, we assume that $y \notin \lambda_{pc}Cl(\{x\})$. Then $X \backslash \lambda_{pc}Cl(\{x\})$ is an λ_{pc}-open subset of X containing y but not x. Therefore, X is λ_{pc}-T_0.

Theorem 4.11 *Let $\lambda : PO(X) \to P(X)$ be an p-operation, then the following statements are equivalent:*

1. *(X, τ) is λ_{pc}-$T_{1/2}$.*
2. *Each singleton $\{x\}$ of X is either λ_{pc}-closed or λ_{pc}-open.*

Proof. (1) $\Rightarrow$ (2) : Suppose that $\{x\}$ is not λ_{pc}-closed. Then by Theorem 4.4, $X \backslash \{x\}$ is g-λ_{pc}-closed. Since (X, τ) is λ_{pc}-$T_{1/2}$, then $X \backslash \{x\}$ is λ_{pc}-closed. Hence, $\{x\}$ is λ_{pc}-open.
(2) $\Rightarrow$ (1) : Let A be any g-λ_{pc}-closed set in (X, τ) and $x \in \lambda_{pc}Cl(A)$. By (2), we have $\{x\}$ is λ_{pc}-closed or λ_{pc}-open. If $\{x\}$ is λ_{pc}-closed and $x \notin A$ will imply $x \in \lambda_{pc}Cl(A) \backslash A$ which is not true by Theorem 4.6, so $x \in A$. Therefore, $\lambda_{pc}Cl(A) = A$, so A is λ_{pc}- closed. Therefore, (X, τ) is λ_{pc}-$T_{1/2}$.
On the other hand, if $\{x\}$ is λ_{pc}-open, then as $x \in \lambda_{pc}Cl(A)$,, we have $\{x\} \cap A \neq \phi$. Hence $x \in A$, so A is λ_{pc}-closed.

Corollary 4.12 *Each λ_{pc}-$T_{1/2}$ space is λ_{pc}-T_0 space.*

Proof. Follows from Theorem 4.11 and Theorem 4.10.

Example 4.13 *Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty, $A = \{a\}$ or $\{a, b\}$ and $\lambda(A) = X$ otherwise. Then (X, τ) is a λ_{pc}-T_0 space but not λ_{pc}-$T_{1/2}$ space because $\{a, b\}$ is g-λ_{pc}- closed but not λ_{pc}- closed.*

Theorem 4.14 *Each λ_{pc}-T_1 space is λ_{pc}-$T_{1/2}$ space.*

Proof. Follows from Theorem 4.6.

Example 4.15 *$X = \{a, b\}$, and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty and $\lambda(A) = X$ otherwise. Then (X, τ) is a λ_{pc}-$T_{1/2}$ space but not λ_{pc}-T_1 space.*

Definition 4.16 *A topological space (X, τ) is called a λ_{pc}-symmetric space if for x and y in X, $x \in \lambda_{pc}Cl(\{y\})$ implies that $y \in \lambda_{pc}Cl(\{x\})$.*

Theorem 4.17 *Let (X, τ) be a λ_{pc}-symmetric space, then the following are equivalent:*

1. *(X, τ) is λ_{pc}-T_0.*
2. *(X, τ) is λ_{pc}-$T_{1/2}$.*
3. *(X, τ) is λ_{pc}-T_1.*

Proof. It is enough to prove only the necessity of (1) $\Leftrightarrow$ (2). Let $x \neq y$ and since (X, τ) is λ_{pc}-T_0, we may assume that $x \in U \subseteq X \backslash \{y\}$ for some $U \in PO_{\lambda pc}(X)$. Then $x \notin \lambda_{pc}Cl(\{y\})$ and hence $y \notin \lambda_{pc}Cl(\{x\})$. Therefore, there exists $V \in PO_{\lambda \mathrm{pc}}(X)$ such that $y \in V \subseteq X \backslash \{x\}$ and (X, τ) is a λ_{pc}-T_1 space.

Remark 4.18 *From the definitions of λ_{pc}-T_i, ($i = 0, 1/2, 1, 2$) and previous results, we get the following diagram of implications:*
λ_{pc}-$T_2 \Rightarrow \lambda_{pc}$-$T_1 \Rightarrow \lambda_{pc}$-$T_{1/2} \Rightarrow \lambda_{pc}$-$T_0$

Definition 4.19 *Let $\lambda : PO(X) \to P(X)$ be an p-operation, a topological space (X, τ) is called λ_{pc}-R_0 if $U \in PO_{\lambda pc}(X)$ and $x \in U$, then $\lambda_{pc}Cl(\{x\}) \subseteq U$.*

Theorem 4.20 *For any topological space X and any s-operation λ, the following are equivalent:*

1. *X is λ_{pc}-R_0.*
2. *$F \in PC_{\lambda pc}(X)$ and $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in PO_{\lambda pc}(X)$.*
3. *$F \in PC_{\lambda pc}(X)$ and $x \notin F$ implies $F \cap \lambda_{pc}Cl(\{x\}) = \phi$.*
4. *For any two distinct points x, y of X, either $\lambda_{pc}Cl(\{x\}) = \lambda_{pc}Cl(\{y\})$ or $\lambda_{pc}Cl(\{x\}) \cap \lambda_{pc}Cl(\{y\}) = \phi$.*

Proof. (1) $\Rightarrow$ (2): $F \in PC_{\lambda \mathrm{pc}}(X)$ and $x \notin F$ implies $x \in X \backslash F \in PO_{\lambda \mathrm{pc}}(X)$ then $\lambda_{pc}Cl(\{x\}) \subseteq X \backslash F$. By (1), if we put $U = X \backslash \lambda_{pc}Cl(\{x\})$, then $x \notin U \in PO_{\lambda \mathrm{pc}}(X)$ and $F \subseteq U$.
(2) $\Rightarrow$ (3) : if $F \in PC_{\lambda \mathrm{pc}}(X)$ and $x \notin F$, then there exists $U \in PO_{\lambda \mathrm{pc}}(X)$ such that $x \notin U$ and $F \subseteq U$. By (2), we have $U \cap \lambda_{pc}Cl(\{x\}) = \phi$, so $F \cap \lambda_{pc}Cl(\{x\}) = \phi$.
(3) $\Rightarrow$ (4) : Suppose that for any two distinct points x, y of X, $\lambda_{pc}Cl(\{x\}) \neq \lambda_{pc}Cl(\{y\})$. Then suppose, without loss of generality, that there exists some $z \in \lambda_{pc}Cl(\{x\})$ such that $z \notin \lambda_{pc}Cl(\{y\})$. Thus there exists a λ_{pc}-open set V such that $z \in V$ and $y \notin V$ but $x \in V$. Thus $x \notin \lambda_{pc}Cl(\{y\})$. Hence by (3), $\lambda_{pc}Cl(\{x\}) \cap \lambda_{pc}Cl(\{y\}) = \phi$.
(4) $\Rightarrow$ (1): Let $U \in PO_{\lambda \mathrm{pc}}(X)$ and $x \in U$. Then for each $y \notin U$, $x \notin \lambda_{pc}Cl(\{y\})$. Thus $\lambda_{pc}Cl(\{x\}) \neq \lambda_{pc}Cl(\{y\})$. Hence by (4), $\lambda_{pc}Cl(\{x\}) \cap \lambda_{pc}Cl(\{y\}) = \phi$, for each $y \in X \backslash U$. So $\lambda_{pc}Cl(\{x\}) \cap [\cup\{\lambda_{pc}Cl(\{y\}) : y \in X \backslash U\}] = \phi$. Now, $U \in PO_{\lambda \mathrm{pc}}(X)$ and $y \in X \backslash U$, then $\{y\} \subseteq \lambda_{pc}Cl(\{y\}) \subseteq \lambda_{pc}Cl(X \backslash U) = X \backslash U$. Thus $X \backslash U = \bigcup\{\lambda_{pc}Cl(\{y\}) : y \in X \backslash U\}$. Hence, $\lambda_{pc}Cl(\{x\}) \cap X \backslash U = \phi$, so $\lambda_{pc}Cl(\{x\}) \subseteq U$. This implies that (X, τ) is λ_{pc}-R_0.

Theorem 4.21 *Let (X, τ) be a topological space and $\lambda : PO(X) \to P(X)$ be any p-operation, then the following are equivalent:*

1. *X is λ_{pc}-T_1.*
2. *$\lambda_{pc}Cl(\{x\}) = \{x\}$ for all $x \in X$.*
3. *X is λ_{pc}-R_0 and λ_{pc}-T_0.*

Proof. (1) $\Rightarrow$ (2) : Let $y \notin \{x\}$, then there exists $U \in PO_{\lambda pc}(X)$ such that $y \in U$, $x \notin U$, so $U \cap \{x\} = \phi$. Hence $y \notin \lambda_{pc}Cl(\{x\})$ implies $\lambda_{pc}Cl(\{x\}) \subseteq \{x\}$ also $\{x\} \subseteq \lambda_{pc}Cl(\{x\})$ always, hence $\lambda_{pc}Cl(\{x\}) = \{x\}$ for all $x \in X$.
(2) $\Rightarrow$ (3) : Let $x,\ y \in X$ with $x \neq y$. Then $\{x\}$ and $\{y\}$ are λ_{pc}-closed and hence $X \backslash \{x\}$ is a λ_{pc}-open set containing y but not x. This shows that X is λ_{pc}-T_0. Again, $x,\ y \in X$ with $x \neq y$, then $\lambda_{pc}Cl(\{x\}) \neq \lambda_{pc}Cl(\{y\})$. Also, $\lambda_{pc}Cl(\{x\}) \cap \lambda_{pc}Cl(\{y\}) = \phi$. Thus, by Theorem 4.20, X is λ_{pc}-R_0.
(3) $\Rightarrow$ (1) : Let $x,\ y \in X$ with $x \neq y$. there exists $U \in PO_{\lambda pc}(X)$ such that $x \in U$ and $y \notin U$ then, $\lambda_{pc}Cl(\{x\}) \subseteq U$ (as X is λ_{pc}-R_0) and so $y \notin \lambda_{pc}Cl(\{x\})$. Hence $x \in U \in PO_{\lambda pc}(X)$, $y \notin U$ and $y \in X \backslash \lambda_{pc}Cl(\{x\}) \in PO_{\lambda pc}(X)$, $x \notin X \backslash \lambda_{pc}Cl(\{x\})$. Therefore, X is a λ_{pc}-T_1 space.

Definition 4.22 *Let (X, τ) be a topological space $\lambda : PO(X) \to P(X)$ be an p-operation. The space X is said to be λ_{pc}-R_1 if for $x,\ y \in X$ with $\lambda_{pc}Cl(\{x\}) \neq \lambda_{pc}Cl(\{y\})$, there exist disjoint λ_{pc}-open sets U and V such that $\lambda_{pc}Cl(\{x\}) \subseteq U$ and $\lambda_{pc}Cl(\{y\}) \subseteq V$.*

Theorem 4.23 *If $\lambda : PO(X) \to P(X)$ is an p-operation and X is λ_{pc}-R_1, then X is λ_{pc}-R_0.*

Proof. Let $U \in PO_{\lambda pc}(X)$ and $x \in U$. If $y \notin U$, then $\lambda_{pc}Cl(\{x\}) \neq \lambda_{pc}Cl(\{y\})$ (as $x \notin \lambda_{pc}Cl(\{y\})$). Hence there exists $V \in PO_{\lambda pc}(X)$ such that $\lambda_{pc}Cl(\{y\}) \subseteq V$ and $x \notin V$. This gives that $y \notin \lambda_{pc}Cl(\{x\})$, so $\lambda_{pc}Cl(\{x\}) \subseteq U$. Hence, X is a λ_{pc}-R_0 space.

By the following examples, we show the converse of above theorem is not true in general, and also we show λ_{pc}-R_0 and pre-R_0 are independent.

Example 4.24 *Let $X = \{a, b\}$ and $\tau = \{\phi, \{a\}, X\}$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty and $\lambda(A) = X$ otherwise. Clearly X is λ_{pc}-R_0, but it is neither pre-R_0 nor pre-R_1.*

Example 4.25 *Let $X = \{a, b\}$, and $\tau = P(X)$. We define an p-operation $\lambda : PO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A = \{a\}$ and $\lambda(A) = X$ otherwise. Clearly X is pre-R_0 and pre-R_1, but it is not λ_{pc}-R_0.*

Theorem 4.26 *Let (X, τ) be a topological space $\lambda : PO(X) \to P(X)$ be an p-operation. Then the following are equivalent:*

1. *X is λ_{pc}-T_2.*
2. *X is λ_{pc}-R_1 and λ_{pc}-T_1.*
3. *X is λ_{pc}-R_1 and λ_{pc}-T_0.*

Proof. (1) $\Rightarrow$ (2) : Let X be λ_{pc}-T_2, then X is clearly λ_{pc}-T_1. Now if $x,\ y \in X$ with $\lambda_{pc}Cl(\{x\}) \neq \lambda_{pc}Cl(\{y\})$ then $x \neq y$, so there exist $U,\ V \in PO_{\lambda pc}(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Hence by Theorem 4.21, $\lambda_{pc}Cl(\{x\}) = \{x\} \subseteq U$ and $\lambda_{pc}Cl(\{y\}) = \{y\} \subseteq V$ and $U \cap V = \phi$. Therefore, X is λ_{pc}-R_1.
(2) $\Rightarrow$ (3) :It is obvious.
(3) $\Rightarrow$ (1) : Let X be λ_{pc}-R_1 and λ_{pc}-T_0, then by Theorem 4.23, X is λ_{pc}-R_0 and λ_{pc}-T_0. Hence, by Theorem 4.21, X is λ_{pc}-T_1 . If $x, y \in X$ with $x \neq y$, then $\lambda_{pc}Cl(\{x\}) = \{x\} \neq \{y\} = \lambda_{pc}Cl(\{y\})$. Since X is λ_{pc}-R_1, so there exist $U, V \in PO_{\lambda pc}(X)$ such that $\lambda_{pc}Cl(\{x\}) = \{x\} \subseteq U, \lambda_{pc}Cl(\{y\}) = \{y\} \subseteq V$ and $U \cap V = \phi$. Hence, X is λ_{pc}-T_2.

References

[1] B. Ahmad and S. Hussain, Properties of γ -operations in topological spaces, *Aligarh Bull. Math.*, 22(1):45-51, 2003.

[2] H. Maki, J. Umehara, and T. Nori, Every Topological Space is Pre–$T_{1/2}$, *Mem. Fac. Sci. Kochi Univ. Ser. A(Mathes.)*, 17(1996), 33-42.

[3] S. G. Crossley and S. K. Hildebrand, Semi-closure, *Texas J. Sci.*, 22:99-112, 1971.

[4] H. Corson and E. Michael, Metrizability of certain countable union, *Illinois J. Math.*, 8(1964), 351-360.

[5] M. Caldas, S. Jafari, and T. Nori, Characterizations of pre-R_0, and pre-R_1 topological spaces, *Topology Proceedings*, 25(2000), 17-30.

[6] S. Kasahara, Operation-compact spaces, *Math. Japonica*, 24:97-105, 1979.

[7] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On pre-continuous and week precontinuous mappings, *Proc. Math. Phys. Soc.*, [Egypt], 53(1982), 47-53.

[8] N. Levine, Generalized closed sets in topology, *Rend. Circ. Math. Palermo*, 19:89-96, 1970.

[9] A. Chattopadhyay, pre-T_0 and pre-T_1 Topological Spaces, *J. Indian Acad. Math.*, 17(2),(1995), 156-159.

[10] H. Ogata, Operation on topological spaces and associated topology, *Math. Japonica*, 36(1):175 -184, 1991.