International Journal of Basic and Applied Sciences, 1 (3) (2012) 270-282

www.sciencepubco.com/index.php/IJBAS

Generalized Synchronization of Bidirectionally Coupled Chaotic Systems Via Linear Transformations

Mohammad Ali Khan

Department of Mathematics, Garhbeta Ramsundar Vidyabhaban,
West Bengal, India
E-mail: mdmaths@gmail.com

Abstract

Several important properties of chaos synchronization of bidirectionally coupled systems remain still unexplored. This paper designes a new synchronization scheme for generalized bidirectionally coupled chaotic systems via linear transformations. The proposed synchronization scheme of bidirectionally coupled chaotic systems are discussed taking coupled unified chaotic systems. In this method, we can predict the driven systems behavior in advance, knowing the driving systems behavior. Simulation results are presented to show the effectiveness of the proposed scheme.

Keywords: *Bidirectionally coupled, Generalized synchronization (GS), Unified chaotic system, Linear transformation.*

1 Introduction

Since the pioneer work by Pecora and Carroll (1990) [1], chaos synchronization has received much attention because of its fundamental importance in non-linear dynamics and potential applications to laser dynamics, electronic circuits, chemical and biological systems and secure communication. One can use synchronized chaotic behavior for designing secure communication devices in the following way. One can have two remote systems behaving chaotically, but synchronized with each other through only one driving signal. A sender can add a given message to the drive, thus musking the information from any third party who wants to intercept it. The receiver can extract the message by using the synchronization scheme. Many chaos synchronization and control methods have been developed, such as backstepping design method [2], impulsive control method [3], invariant manifold method [4], control strategy method [5], active control method [6], synchronization in unidirectionally coupled system [7] and bidirectionally coupled system [8].

A pair of dynamical systems

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y) \end{aligned} \tag{1}$$

are said to be unidirectionally coupled if

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y) + h(x,y) \end{aligned} \tag{2}$$

where h(x,y) is a nontrivial function of x and y. Physically, this means that in part of the phase space, the behavior of one system has no influence on the behavior of the other. If coupling is not unidirectional then it must be bidirectional. Systems are called bidirectionally coupled if

$$\begin{aligned} \dot{x} &= f(x) + k(x,y) \\ \dot{y} &= g(y) + h(x,y) \end{aligned} \tag{3}$$

Where h(x,y) and k(x,y) are nontrivial functions of x and y.
Two dynamical systems are called synchronized if the distance between the corresponding states of the systems converges to zero as time goes to infinity. This type of synchronization is known as identical synchronization [1]. However in the coupled chaotic systems identical synchronization is a fairly restrictive concept and often difficult to achieve except under ideal conditions. Applications of GS may be more practical than those of identical synchronization because parameter mismatches and distortions always exist in the physical world. Recently, a more elaborate form of synchronization called generalized synchronization (GS) was proposed by Kocarev and Parlitz (1996) [9]. They formulated a condition for the occurence of GS for the unidirectionally coupled systems of following type

$$\begin{aligned} \dot{x} &= f(x) && \text{driving system} \\ \dot{y} &= g(y,u) = g(y,h(x)) && \text{driven system} \end{aligned} \tag{4}$$

where $x \in R^n, y \in R^m$ and $u(t) = (u_1(t), u_2(t), ...u_k(t))$ with $u_j = h_j(x(t, x_0))$. Here the variable u_j are introduced to include explicitly the case that a function $u = h(x)$ of x is used for driving the response system. According to Kocarev and Parlitz (1996) [9] the system (4) possess the property of GS between x and y if there exists a transformation $H: R^n \to R^m$, a manifold $M = \{(x,y): y = H(x)\}$, and a subset $B = B_x \times B_y \subseteq R^n \times R^m$ with $M \subseteq B$ such that all trajectories of (4) with initial conditions in the basin B approach M as time t goes to infinity. If H equals to the identity transformation, this definition of generalized synchronization coincides with the usual definition of synchronization e.g., identical synchronization. generalized synchronization (GS), which is defined by a time-independent nonlinear functional relation $y = \Phi(x)$ between the states x and y of two systems. Experimental detection and characterization of GS from observed data is a challenging problem, especially in biology; e.g., for study on nonlinear interdependence observed in binding of different features in cognitive process and epilepsies in the brain. In unidirectionally coupled system, a way to detect GS is to make an identical copy Y' of the response system Y driven by the common signal from the driver system X, then investigate whether orbits of both Y and Y' coincide after transient. In 1995 Rulkov et. al. [10] discussed generalized synchronization of chaos in unidirectionally coupled chaotic systems. Hramov et.al. [11] proposed GS by a modified system approach in 2005. They investigated the physical reasons leading to GS appearance in unidirectionally coupled chaotic systems. Hramov et.al.(2005) [12] explained the peculiarity of the GS onset in the unidirectionally coupled Rossler oscillators. Yang and Chua (1999) [13] proposed a method for obtaining GS of two coupled chaotic systems via linear transformations. They also considered unidirectional coupled chaotic systems. In 2007 Poria [14] discussed generalized chaos synchronization of two Lorenz dynamical systems via linear transformation considering unidirectional coupling. There are very few results about synchronization of bidirectionally coupled chaotic systems. But most of the natural systems are bidirectionally coupled. Therefore the study of bidirectionally coupled systems are necessary. Nobody have discussed the GS of two bidirectionally coupled chaotic systems via linear transformation. In this paper we introduce the theory of GS of two bidirectionally coupled chaotic systems via linear transformations. We discuss the theory considering two bidirectionally coupled unified chaotic systems. Finally simulation results are presented and discussed.

2 Generalized Synchronization of Bidirectional Coupled Systems

A dynamical system can be decomposed into two parts

$$\dot{x} = Ax + \Psi(x) \tag{5}$$

where A is an $n \times n$ constant matrix and $\Psi : R^n \to R^n$. We assume that the driving system transmit the signal $\Psi(x)$ to the driven system and consider the following bidirectionally coupled systems:

$$\begin{aligned} \dot{x} &= Ax + \Psi(x) + \Phi(y) && \text{driving system} \\ \dot{y} &= Ay + \Lambda\Psi(x) + \Lambda\Phi(y) && \text{driven system} \end{aligned} \tag{6}$$

where A, Λ are $n \times n$ matrices and $\Psi, \Phi : R^n \to R^n$.

Theorem 2.1

If the matrix Λ commutes with A, then the two coupled chaotic dynamical systems in (6) are in a state of generalized synchronization via the following linear transformation

$$y(\infty) = H(x) = \Lambda x$$

if and only if all eigenvalues of the matrix A have negetive real parts.

Proof:

Let $z = y - \Lambda x$, then the stability of the GS between the two dynamical systems in (6) via the GS transformation $y = H(x) = \Lambda x$ is equivalent to that of the origin of the following system:

$$\begin{aligned}\dot{z} &= [Ay + \Lambda\Psi(x) + \Lambda\Phi(y)] - [\Lambda Ax + \Lambda\Psi(x) + \Lambda\Phi(y)]\\ &= Ay - \Lambda Ax\\ &= A(y - \Lambda x) \text{ since } \Lambda \text{ commutes with } A\\ &= Az. \end{aligned} \tag{7}$$

Therefore z=0 is asymptotically stable if and only if all eigen values of the matrix A have negative real parts.

The matrices Λ which commute with an $n \times n$ matrix A must be an $n \times n$ matrix which satisfies the following equation:

$$A\Lambda = \Lambda A \tag{8}$$

Clearly the above equaton has infinite number of solutions for Λ. Therefore we can construct several methods of linear GS between two chaotic systems. Three simple solutions of equations (8) are (i) scalar multiplication of any identity matrix of same order commutes with A, (ii) any square matrix commutes with itself and (iii) if A is invertible then A^{-1} commutes with A.

3 Generalized Synchronization of Bidirectionally Coupled Unified Chaotic Systems

In this section we discuss the theory considering two bidirectionally coupled unified chaotic systems via linear transformations. A unified chaotic system is presented by Lu and Wu in 2004 [15]. The unified chaotic system can be described by the following system of differential equations.

$$\dot{x} = (25a + 10)(y - x)$$

$$\dot{y} = (28-35a)x - xz + (29a-1)y \tag{9}$$
$$\dot{z} = xy - \frac{8+a}{3}z$$

where $a \in [0,1]$ For a =0, 0.8, 1 the system (9) represents the Lorenz chaotic system, Lu chaotic system and Chen chaotic system respectively. Practically, unified chaotic system is chaotic for any $a \in [0,1]$ According to equation (6) the master and slave systems with bidirectional coupling are constructed as

Master system :

$$\begin{aligned}
\dot{x}_1 &= (25a+10)(x_2+y_2-x_1)\\
\dot{x}_2 &= (28-35a)x_1 - x_2 - x_1x_3 - y_1y_3 + 29a(x_2+y_2)\\
\dot{x}_3 &= x_1x_2 + y_1y_2 - \frac{8+a}{3}x_3
\end{aligned} \tag{10}$$

Slave system:

$$\begin{aligned}
\dot{y}_1 &= -(25a+10)y_1 + \lambda_{11}(25a+10)(x_2+y_2) + \lambda_{12}\{29a(x_2+y_2) - x_1x_3 - y_1y_3\}\\
&\quad +\lambda_{13}(x_1x_2+y_1y_2)\\
\dot{y}_2 &= (28-35a)y_1 - y_2 + \lambda_{21}(25a+10)(x_2+y_2) + \lambda_{22}\{29a(x_2+y_2) - x_1x_3 - y_1y_3\}\\
&\quad +\lambda_{23}(x_1x_2+y_1y_2)\\
\dot{y}_3 &= -\frac{8+a}{3}y_3 + \lambda_{31}(25a+10)(x_2+y_2) + \lambda_{32}\{29a(x_2+y_2) - x_1x_3 - y_1y_3\}\\
&\quad +\lambda_{33}(x_1x_2+y_1y_2)
\end{aligned} \tag{11}$$

where
$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}, \quad A = \begin{pmatrix} -(25a+10) & 0 & 0 \\ (28-35a) & -1 & 0 \\ 0 & 0 & -\frac{8+a}{3} \end{pmatrix} \tag{12}$$

4 Results and Discussions

Numerical simulations have been performed to show the usefulness of the newly proposed synchronization method. Fourth-order Runge-Kutta method is used for solving bidirectionally coupled unified chaotic system with step size .001. The initial conditions for the master and slave systems are taken as
$(x_1(0), x_2(0), x_3(0)) = (1, 1, 1)$ and $(y_1(0), y_2(0), y_3(0)) = (-2,-2, -2)$.

Case 1. The parameter of unified chaotic system is chosen as $a=0$. Figures 1(a)-1(c), 2(a)-2(c) and 3(a)-3(c) shown the simulation results for bidirectionally coupled unified chaotic systems for $a=0$, i.e., in the Lorenz systems.

Simulation 1.

In this simulation, we take

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{13}$$

where $\lambda \neq 0$. Clearly $\Lambda A = A\Lambda$. Therefore all conditions of the theorem are satisfied. Here the driving unified chaotic system is (10) and the driven unified chaotic system is given by

$$\begin{aligned} \dot{y}_1 &= (25a+10)(\lambda(x_2+y_2)-y_1) \\ \dot{y}_2 &= (28-35a)y_1 - y_2 + \lambda(-x_1x_3 - y_1y_3 + 29a(x_2+y_2)) \\ \dot{y}_3 &= \lambda(x_1x_2+y_1y_2) - \frac{8+a}{3}y_3 \end{aligned} \tag{14}$$

The simulation results of synchronization are shown in Figure 1(a)-1(c). We choose $\lambda = 2$. Then the state variables of the driving system and driven systems are connected by the linear transformation

$$\begin{aligned} y_1 &= 2x_1 \\ y_2 &= 2x_2 \\ y_3 &= 2x_3 \end{aligned} \tag{15}$$

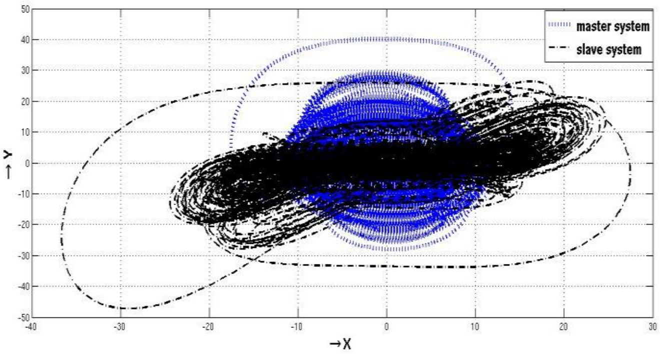

Figure 1(a). Phase diagram of (x_1, x_2) and (y_1, y_2) for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 1.

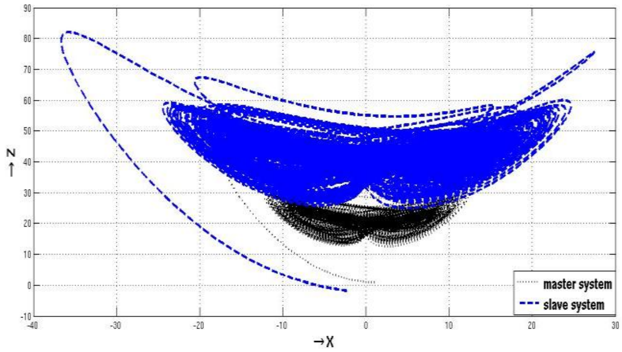

Figure 1(b). Phase diagram of (x_1, x_3) with (y_1, y_3) for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 1.

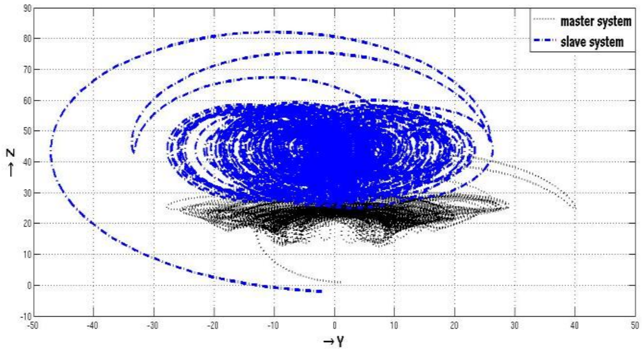

Figure 1(c). Phase diagram of (x_2, x_3) with (y_2, y_3) for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 1.

Simulation 2.

In this simulation we choose

$$\Lambda = A = \begin{pmatrix} -(25a+10) & 0 & 0 \\ (28-35a) & -1 & 0 \\ 0 & 0 & -\frac{8+a}{3} \end{pmatrix}$$

Clearly, the matrix Λ commutes with A. In this case, the driven unified chaotic system is given by

$$\begin{aligned} \dot{y}_1 &= -(25a+10)y_1 - (25a+10)^2(x_2+y_2) \\ \dot{y}_2 &= (28-35a)y_1 - y_2 + (28-35a)(25a+10)(x_2+y_2) + x_1x_3 + y_1y_3 - 29a(x_2+y_2) \\ \dot{y}_3 &= -\frac{8+a}{3}(y_3 + x_1x_2 + y_1y_2) \end{aligned} \tag{16}$$

If the GS between the driven and driving systems is achieved, then the following relations should be satisfied.

$$
\begin{aligned}
y_1 &= -(25a+10)x_1 \\
y_2 &= (28-35a)x_1 - x_2 \\
y_3 &= -\frac{8+a}{3}x_3
\end{aligned} \tag{17}
$$

The simulation results of synchronization are shown in Figures 2(a)-2(c).

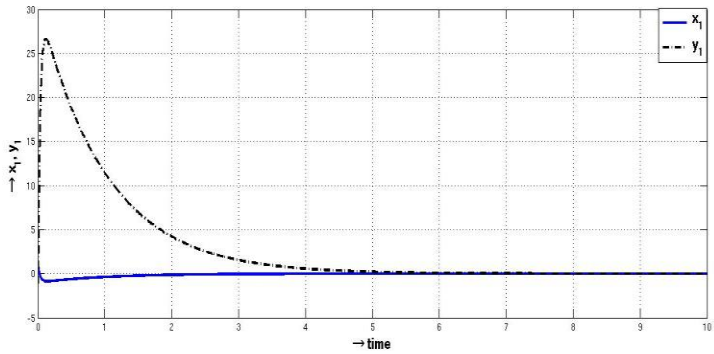

Figure 2(a). t-x_1 with t-y_1 for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 2.

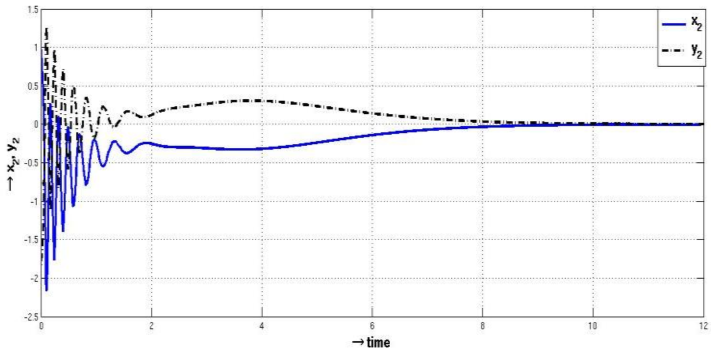

Figure 2(b). t-x_2 with t-y_2 for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 2.

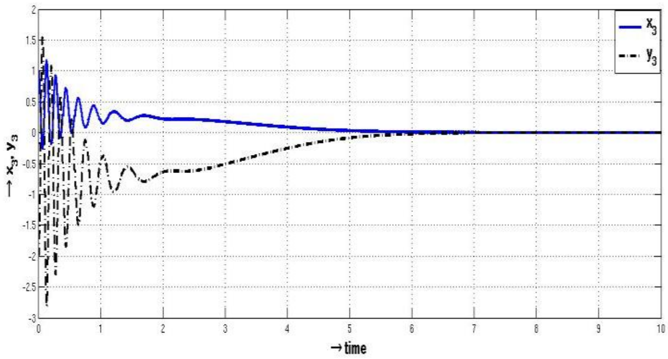

Figure 2(c). t-x_3 with t-y_3 for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 2.

Simulation 3.

In this case, we take

$$\Lambda = A^{-1} = \begin{pmatrix} -\dfrac{1}{(25a+10)} & 0 & 0 \\ -\dfrac{28-35a}{25a+10} & -1 & 0 \\ 0 & 0 & -\dfrac{3}{8+a} \end{pmatrix}$$

Obviously, Λ commutes with A. Therefore driven unified chaotic system is given by

$$\begin{aligned} \dot{y}_1 &= -(25a+10)y_1 - (x_2 + y_2) \\ \dot{y}_2 &= (28-35a)y_1 - y_2 - (28-35a)(x_2+y_2) + x_1x_3 + y_1y_3 - 29a(x_2+y_2) \\ \dot{y}_3 &= -\frac{8+a}{3}y_3 - \frac{3}{8+a}(x_1x_2 + y_1y_2) \end{aligned} \tag{18}$$

For the GS between the driven and driving systems the following relations should be satisfied, then the following relations should be satisfied.

$$\begin{aligned} y_1 &= -\frac{x_1}{25a+10} \\ y_2 &= -\frac{28-35a}{25a+10}x_1 - x_2 \\ y_3 &= -\frac{3}{8+a}x_3 \end{aligned} \tag{19}$$

The simulation results of synchronization are shown in Figures 3(a)-3(c).

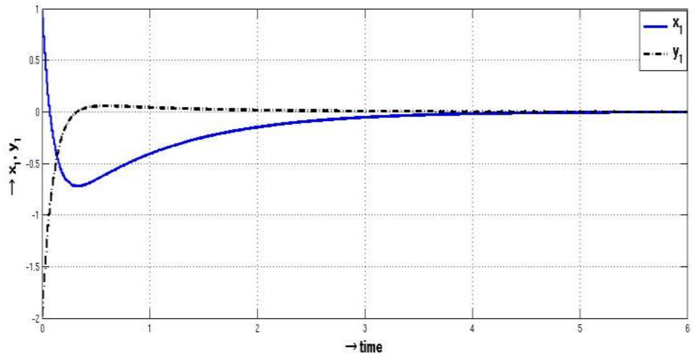

Figure 3(a). t-x_1 with t-y_1 for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 3.

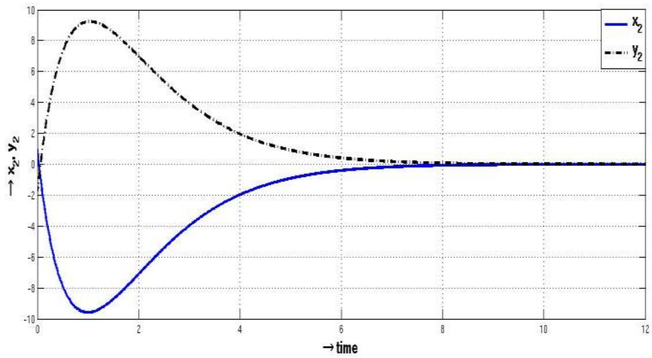

Figure 3(b). t-x_2 with t-y_2 for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 3.

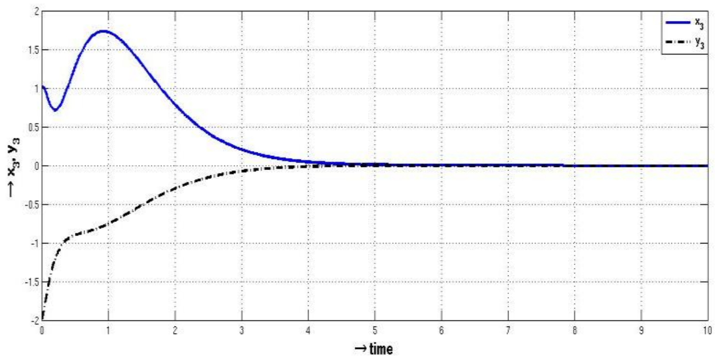

Figure 3(c). t-x_3 with t-y_3 for bidirectionally coupled unified chaotic systems for a=0 i.e., in the Lorenz systems for simulation 3.

Case 2. The parameter of unified chaotic system is chosen as $a = 0$ .8.
Figures 4(a)-4(c), Figure 5 and Figure 6 shown the simulation results for bidirectionally coupled unified chaotic systems for $a = 0.8$, i.e., in the Lu systems.

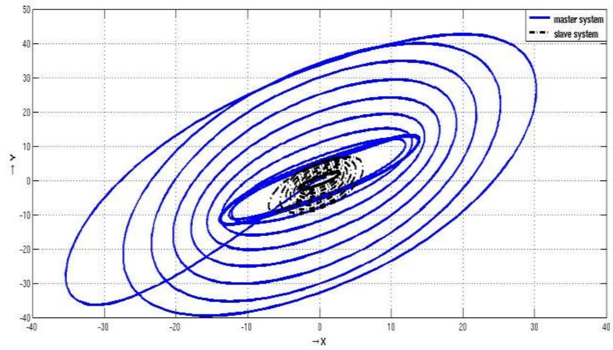

Figure 4(a). Phase diagram of (x_1, x_2) with (y_1-y_2) for bidirectionally coupled unified chaotic systems for a=0.8 i.e., in the Lu systems for simulation 1.

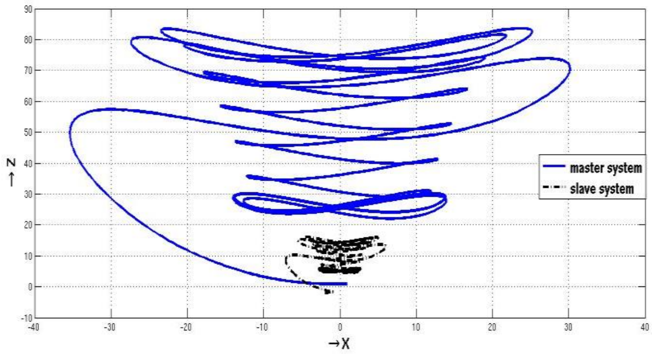

Figure 4(b). Phase diagram of (x_1, x_3) with (y_1, y_3) for bidirectionally coupled unified chaotic systems for a=0.8 i.e., in the Lu systems for simulation 1.

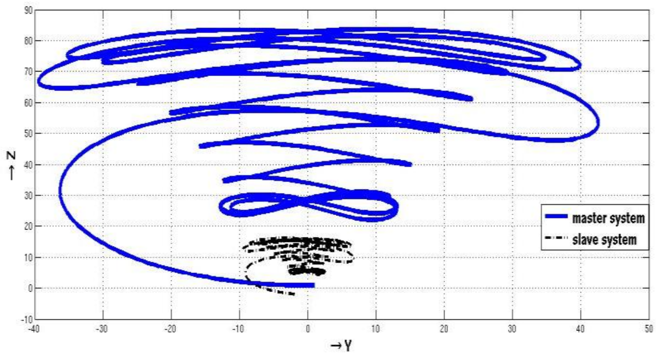

Figure 4(c). Phase diagram (x_2, x_3) with (y_2, y_3) for bidirectionally coupled unified chaotic systems for a=0.8 i.e., in the Lu systems for simulation 1.

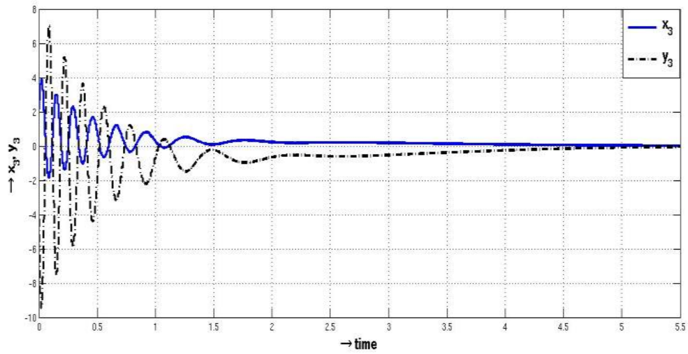

Figure 5. t-x_3 with t-y_3 for bidirectionally coupled unified chaotic systems for a=0.8 i.e., in the Lu systems for simulation 2.

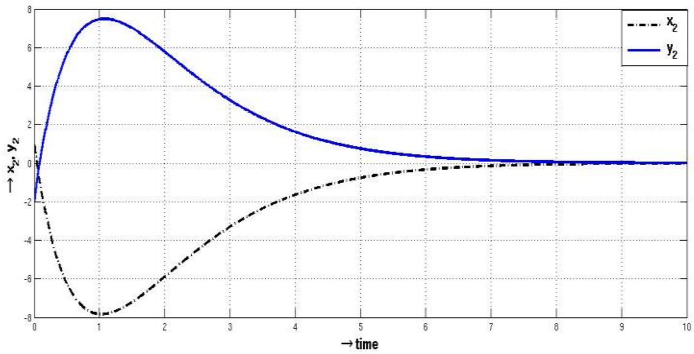

Figure 6. t-x_2 with t-y_2 for bidirectionally coupled unified chaotic systems for a=0.8 i.e., in the Lu systems for simulation 3.

5 Conclusion

In this paper we have proposed a generalized synchronization scheme for bidirectionally coupled chaotic systems via linear transformations. We derive the condition for generalized synchronization of bidirectionally coupled chaotic systems. We have also predicted the relationship between master system and slave system after generalized synchronization. We have done numerical simulation for unified chaotic system. The numerical simulation results shows that the proposed method is very effective. Our proposed generalized synchronization scheme may be very useful for secure communication purpose.

References

[1] L.M. Pecora and T.L. Carroll , Synchronization in chaotic systems, Phys. Rev. Lett, 64 (1990) 821-824.
[2] Y. Yongguang and Z. Suochun, Controlling uncertain Lu system using backstepping design, Chaos Solitons and Fractals, 15 (2003) 897-902.

[3] T. Yang, L-B. Yang and C-M. Yang , Impulsive control of Lorenz system, Physica D, 110 (1997) 18-24.
[4] K. Josic, Synchronization of chaotic systems and invariant manifolds, Nonlinearity, 13 (2000) 1321.
[5] M.A.Khan, A.K.Mondal and S.Poria, Three control strategies for unified chaotic system, Int. J. of Applied Mechanics and Engineering. 16 (2011) 597-608.
[6] MC Ho and YC Hung, Synchronization of two different systems by using generalized active control, Physics Letters A, 301 (2002) 424-428.
[7] M.A. Khan and A.K.Mandal, Generalized chaos synchronization of coupled Rossler systems, Bull. Cal. Math. Soc., 101 (2009), 197-204.
[8] A Tarai(Poria), S Poria and P. Chatterjee, Synchronization of bidirectionally coupled chaotic Chen's system with delay, Chaos Solitons and Fractals, 41 (2009) 643-647.
[9] L. Kocarev and U. Parlitz, Generalized synchronization, predictability and equivalence of unidirectionally coupled dynamical systems, Physical Review Letters, 76 (1996), 1816-1819.
[10] N. F. Rulkov, M. M. Suschik and L. S. Tsimring, Generalized synchronization of chaos in directionally coupled chaotic systems, Physical Review E, 51 (1995), 980-994.
[11] A. E. Hramov and A. A. Koronovskii, Generalized synchronization: a modified system approach, Physical Review E, 71 (2005), 067201.
[12] A.E. Hramov, A. A. Koronovskii and O. I. Moskalenko, Generalized synchronization onset, Europhysics Letters, 72 (2005), 901-907.
[13] T. Yang and L. O. Chua, Generalized synchronization of chaos via linear transformations, Int. J. Bifurcation and chaos, 9 (1999), 215-219.
[14] S. Poria , The linear generalized chaos synchronization and predictability, Int. J. of Applied Mechanics and Engineering, 12 (2007), 879-885.
[15] Lu and Wu, A unified chaotic system with continuous periodic switch, Chaos, Solitons and Fractals, 20 (2004), 245-251.