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Research paper

A class of new exact solutions of the system of PDE for the plane motion of viscous incompressible fluids in the presence of body force

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Abstract

The purpose of this paper is to indicate a class of exact solutions of the system of partial differential equations governing the steady, plane motion of incompressible fluid of variable viscosity with body force term to the right-hand side of Navier-Stokes equations. The class consists of the stream function ψ characterized by the equation $\theta = f(r) + v(\psi)$ in polar coordinates r and θ where f and v are continuously differentiable functions and the function $v(\psi)$ is such that $v''(\psi) = c\,v'^2(\psi)$ where a non-zero constant is c and overhead prime represents derivative with respect to ψ . When $c>0$ or $c<0$ we show exact solutions for given one component of the body force for both the cases when the function $f(r)$ is arbitrary and when it is not. For the arbitrary function case, $f(r)$ appears in the coefficient of a linear second order ordinary differential equation showing a large numbers of solutions of this equation. This in turn establishes an infinite set of exact solutions to the problem concerned however; we show three examples of such exact solutions. The alternate case fixes $f(r)$ and provides viscosity as derivative of temperature function for $c>0$ and $c<0$. Anyhow, we find an infinite set of streamlines, the velocity components, viscosity function, generalized energy function and temperature distribution.

Keywords: *Some Exact Solutions in the Presence of Body Force; Exact Solutions to the Flow Equations of Incompressible Fluids; Exact Solutions of Variable Viscosity Fluids; Navier-Stokes Equations with Body Force.*

1. Introduction

The basic system of partial differential equations (PDE) for the motion of a viscous fluid consists of the equation of continuity, Navier-Stokes equations (NSE) and energy equation. As the Navier-Stokes equations have base on Newton's law, therefore it allows us to add body forces term to right-hand side of it in addition to surface force. The examples of the body forces are constant gravity force, coriolis force, centrifugal force etc. In the presence of body force the basic dimensionless form of system of PDE's for the steady motion of incompressible fluid of variable viscosity in tensor notation are

Continuity

$$\frac{\partial v_i}{\partial x_i} = 0 \qquad (1)$$

NSE

$$\left(v_j \frac{\partial v_i}{\partial x_j}\right) = F_i - \frac{\partial p}{\partial x_i} + \frac{1}{R_e}\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\} \qquad (2)$$

Energy

$$\left(v_j \frac{\partial T}{\partial x_j}\right) = \frac{1}{R_e P_r}\frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i}\right) + \frac{\mu E_c}{R_e}\frac{\partial v_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (3)$$

where $\mathbf{F} = (F_1(x_i), F_2(x_i), F_3(x_i))$ is the body force per unit mass, $\mathbf{v} = (v_1(x_i), v_2(x_i), v_3(x_i))$ the fluid velocity, $p = p(x_i)$ is pressure, $\rho = \rho(x_i)$ the fluid density, the coefficients of viscosity $\mu > 0$, the space coordinates x_i and $i, j \in \{1, 2, 3\}$. The dimensionless quantities R_e , P_r and E_c are the Reynolds number, the Prandtl number and the Eckert number respectively. The non-dimensional parameters used in equations (2-3) with constant thermal conductivity k are mentioned in [1-2].

For the plane Cartesian space case we take $i, j \in \{1, 2\}$, $x_1 = x$, $x_2 = y$, $v_1 = u(x, y)$, $v_2 = v(x, y)$, $F_1 = F_1(x, y)$, $F_2 = F_2(x, y)$ in equations (1-3) and obtain

$$u_x + v_y = 0 \qquad (4)$$

$$u\,u_x + v\,u_y = F_1 - p_x + \frac{1}{R_e}[(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \qquad (5)$$

$$u\,v_x + v\,v_y = F_2 - p_y + \frac{1}{R_e}[(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \qquad (6)$$

$$u\,T_x + v\,T_y = \frac{1}{R_e P_r}(T_{xx} + T_{yy}) + \frac{E_c}{R_e}[2\mu(u_x{}^2 + v_y{}^2) + \mu(u_y + v_x)^2] \qquad (7)$$

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The solution of the equation (4) demands the existence of a stream function $\psi(x,y)$ such that

$$u=\frac{\partial\psi}{\partial y} \quad v=-\frac{\partial\psi}{\partial x} \tag{8}$$

However the nonlinear terms in equations (5-7) offers a great difficulty for exact solutions. In order to handle nonlinearities some coordinates transformation techniques and dimension analysis methods have proved helpful for some exact solutions of NSE with surface force. The readers interested in these methods/techniques may refer to [3]-[16] and the references therein. For a class of some exact solutions of NSE with body force we refer [2] and the references therein.

The objective of this communication is to obtain a class of exact solutions of the problem of the steady plane motion of incompressible fluid of variable viscosity in the presence of body force with a new characterization of streamlines. To achieve the aim we transform the fundamental flows equations from Cartesian space (x,y) into a curvilinear coordinates (ϕ,ψ) with Martin's definition. Martin [17] defined the curvilinear coordinate lines $\psi=$ const. as streamlines and left the curvilinear coordinate lines $\phi=$ const. arbitrary. We will refer it as Martin's system $-(\phi,\psi)$. As the coordinate ϕ is arbitrary in Martin's system, therefore, we take $\phi=r(x,y)$ to achieve our plan and we characterize the streamlines of the class of flows under consideration by

$$\theta-f(r)=\text{const.} \tag{9}$$

Where $f(r)$ is a continuously differentiable function, r, θ the polar coordinates. As $\psi=$ const. are the streamlines therefore, for the class of flows under consideration of this communication we take

$$\theta=f(r)+v(\psi) \tag{10}$$

With v as a continuously differentiable function of ψ such that $v''(\psi)=c\,v'^2(\psi)$ where a non-zero constant is c and overhead prime represents derivative with respect to ψ.

We organize this paper as follow: In section (2), we transform the fundamental non-dimensional flow equations from Cartesian space (x,y) into Martin's system (ϕ,ψ). In section (3), we find exact solution taking $\phi=r(x,y)$. In last section, we present conclusion.

2. Fundamental flow equations in Martin's system

Following [1-3], we reduce the basic system Eqs. (5-7) to a convenient form by introducing the vorticity function w and the total energy function L defined by

$$w=v_x-u_y \tag{11}$$

$$L=p+\frac{1}{2}(u^2+v^2)-\frac{2\mu u_x}{R_e} \tag{12}$$

And find

$$-v\,w=F_1-L_x+\frac{1}{R_e}A_y \tag{13}$$

$$u\,w=F_2-L_y-\frac{1}{R_e}B_y+\frac{1}{R_e}A_x \tag{14}$$

$$u\,T_x+v\,T_y=\frac{1}{R_e P_r}(T_{xx}+T_{yy})+\frac{E_c}{R_e}\frac{1}{4\mu}\left(B^2+4A^2\right) \tag{15}$$

Where

$$A=\mu(u_y+v_x) \qquad B=4\mu u_x \tag{16}$$

Now we transform Eqs. (13-15) into Martin's system (ϕ,ψ) through transformation

$$x=x(\phi,\psi)\ ,\ y=y(\phi,\psi) \tag{17}$$

Such that the Jacobian $J=\dfrac{\partial(x,y)}{\partial(\phi,\psi)}\neq 0$ of the transformation is finite. At a common point $P(x,y)$ let α be the angle between the tangents to the streamlines lines $\psi=const.$ and the curves $\phi=const.$ then we have shown in [1-2] that Eqs. (13-15) in Martin's system are following

$$-R_e w\,J\,E=R_e J\sqrt{E}\left[-F\left(F_1\cos\alpha+F_2\sin\alpha\right)+J\left(F_1\sin\alpha-F_2\cos\alpha\right)\right]$$
$$+R_e J\,E\,L_\psi+A_\phi\left((F^2-J^2)\cos 2\alpha-2FJ\sin 2\alpha\right)$$
$$+EA_\psi\left(J\sin 2\alpha-F\cos 2\alpha\right)\big)-B_\phi\left(\frac{1}{2}(F^2-J^2)\sin 2\alpha+FJ\cos 2\alpha\right)$$
$$+E\,B_\psi\left(\frac{1}{2}F\sin 2\alpha+J\cos^2\alpha\right), \tag{18}$$

$$0=R_e J\sqrt{E}\left[F_1\cos\alpha+F_2\sin\alpha\right]-R_e J\,L_\phi+E\,A_\psi\cos 2\alpha$$
$$-A_\phi\left[F\cos 2\alpha-J\sin 2\alpha\right]+$$
$$B_\phi\left(\frac{1}{2}F\sin 2\alpha-J\sin^2\alpha\right)-\frac{E\,B_\psi}{2}\sin 2\alpha\ , \tag{19}$$

$$\frac{1}{J R_e P_r}\left[\left(\frac{G T_\phi-F T_\psi}{J}\right)_\phi+\left(\frac{E T_\psi-F T_\phi}{J}\right)_\psi\right]$$
$$=-\frac{E_c}{R_e}\frac{1}{4\mu}\left(B^2+4A^2\right)+\frac{T_\phi}{J} \tag{20}$$

Where the coefficients of first fundamental form are

$$E=x_\phi^2+y_\phi^2\ ,\ F=x_\phi x_\psi+y_\phi y_\psi\ ,\ G=(x_\psi)^2+(y_\psi)^2 \tag{21}$$

And

$$J=\pm\sqrt{EG-F^2} \tag{22}$$

$$B(\phi,\psi)=\frac{4\mu}{EJ^3}\left[E_\phi(F\sin\alpha+J\cos\alpha)^2-2E(F\sin\alpha+J\cos\alpha)\right.$$
$$\left.(F_\phi\sin\alpha+J_\phi\cos\alpha)+E^2(J_\psi\sin 2\alpha+G_\phi\sin^2\alpha)\right], \tag{23}$$

$$A(\phi,\psi)=\mu\left[-\frac{(F\cos\alpha-J\sin\alpha)}{4E^2J^5}\ \{E_\phi(2E\,J^3\cos\alpha+F\sqrt{E}\sin\alpha)\right.$$
$$-4E^2J^2J_\phi\cos\alpha-2E\sqrt{E}\,F_\phi\sin\alpha+E\sqrt{E}\,E_\psi\sin\alpha\,\}$$
$$+\frac{\cos\alpha}{2J^3}\{E_\psi(F\sin\alpha+J\cos\alpha)-2EJ_\psi\cos\alpha-E\,G_\phi\sin\alpha\,\}$$

$$+ \frac{(F\sin\alpha + J\cos\alpha)}{2EJ^3}\{(J E_\phi - 2EJ_\phi)\sin\alpha$$

$$+ \cos\alpha\ [-FE_\phi + 2E F_\phi - E E_\psi]\}$$

$$- \frac{\sin\alpha}{2J^3}\{(E_\psi(J\sin\alpha - F\cos\alpha) - 2EJ_\psi\sin\alpha + EG_\phi\cos\alpha\}], \qquad (24)$$

$$w = \frac{(F\sin\alpha + J\cos\alpha)}{2EJ^3}\{(J E_\phi - 2EJ_\phi)\sin\alpha$$

$$+ \cos\alpha\ [-FE_\phi + 2E F_\phi - E E_\psi]\}$$

$$- \frac{\sin\alpha}{2J^3}\{E_\psi(J\sin\alpha - F\cos\alpha) - 2EJ_\psi\sin\alpha + EG_\phi\cos\alpha\}]$$

$$+ \frac{(F\cos\alpha - J\sin\alpha)}{4E^2J^5}\{E_\phi(2E J^3\cos\alpha + F\sqrt{E}\sin\alpha)$$

$$- 4E^2J^2J_\phi\cos\alpha - 2E\sqrt{E}F_\phi\sin\alpha + E\sqrt{E}E_\psi\sin\alpha\}$$

$$-[\frac{\cos\alpha}{2J^3}\{E_\psi(F\sin\alpha + J\cos\alpha) - 2EJ_\psi\cos\alpha - E G_\phi\sin\alpha\}], \qquad (25)$$

Thus, the basic transformed equations into Martin's system are Eqs. (18-25).

3. Exact solutions

As the coordinate ϕ is arbitrary in Martin's system (ϕ,ψ), therefore, we take

$$\phi = r(x,y) \qquad (26)$$

Where

$$x = r\,\mathrm{Cos}\theta,\ y = r\,\mathrm{Sin}\theta \qquad (27)$$

Utilizing equations (26-27) and writing the fundamental equations equation (18-25) in terms of independent variables r and ψ, we get

$$q = \frac{\sqrt{E}}{J} \qquad (28)$$

$$-R_e w = -R_e J F_2 + R_e L_\psi - JA_r + \sqrt{E-1}A_\psi + B_\psi \qquad (29)$$

$$0 = R_e\left(F_1 + F_2\sqrt{E-1}\right) - R_e L_r + \frac{A_\psi(2-E)}{J}$$

$$+ A_r\sqrt{E-1} - \frac{\sqrt{E-1}B_\psi}{J} \qquad (30)$$

$$JT_{rr} - 2\sqrt{E-1}\ T_{vr}v' + \frac{E}{J}T_{vv}(v')^2 + \left(J_r - \frac{E_\psi}{2\sqrt{E-1}} - R_e P_r\right)T_r$$

$$+\left(\frac{E_\psi}{J} - \frac{E_r}{2\sqrt{E-1}} - \frac{EJ_\psi}{J^2} + \frac{E}{J}\left(\frac{v''}{v'}\right)\right)T_v v' = -\frac{J E_c P_r}{4\mu}\left(B^2 + 4A^2\right) \qquad (31)$$

$$w = \left[\frac{f'(r)}{r} + f''(r)\right]\left(\frac{1}{v'(\psi)}\right) + \left[\frac{1}{r^2} + \{f'(r)\}^2\right]\left(\frac{v''(\psi)}{\{v'(\psi)\}^3}\right) \qquad (32)$$

Where

$$A(r,\psi) = \frac{\mu}{J}\left[\frac{-2J_r\sqrt{E-1}}{J} + \frac{E_r}{2\sqrt{E-1}} + \frac{-(2-E)J_\psi}{J^2}\right] \qquad (33)$$

$$B(r,\psi) = 4\mu\,\frac{1}{J^3}\,[-J\ J_r + \sqrt{E-1}\ J_\psi] \qquad (34)$$

$$E = 1 + r^2\,[f'(r)]^2 \qquad (35)$$

$$F = J\sqrt{E-1} \qquad (36)$$

$$G = r^2\,v'(\psi)^2 \qquad (37)$$

$$J = r\,v'(\psi) \qquad (38)$$

$$\cos\alpha = \frac{1}{\sqrt{E}} \qquad (39)$$

We are using q is the magnitude of velocity vector $\mathbf{q} = (u,v)$ for the plane motion.
In order to determine the solution of the flow equations (28-39), we follow [1-3] and construct the following equation using the natural integrability condition $L_{r\psi} = L_{\psi r}$ on equations (29) and (30).

$$r v' A_{rr} - 2rf'A_{r\psi} - \frac{\left[1 - r^2(f')^2\right]}{r\,v'}A_{\psi\psi} + v'A_r - A_\psi\left(f' + rf''\right)$$

$$-\left\{B_r - \frac{f'B_\psi}{v'}\right\}_\psi = R_e w_r + R_e\left(F_1 + F_2 r(f')\right)_\psi - R_e\left(r\,v'F_2\right)_r \qquad (40)$$

This equation involves the functions $f(r)$, $v(\psi)$, the body force components $F_1(r,\psi)$, $F_2(r,\psi)$ and the viscosity μ. Once a solution of this equation (40) is determined, the function L and temperature distribution T are determined from equations (29-30) and (31), respectively, the pressure from equation (12) and velocity components from (8).
In order to attempt for solution of equation (40) the first inspiration that we take is from [2] which guides to focus on the vorticity function w and write it as a product function of independent variables r and ψ through

$$v'' = c\,v'^2 \qquad (41)$$

Where c is constant. The case for $c = 0$ has already discussed in [2]. The case when $c > 0$ or $c < 0$ the solution of equation (41) is

$$v = \frac{1}{c}\ln\left[\frac{-1}{c(k_1\psi + k_2)}\right] \qquad (42)$$

Where k_1 and k_2 are constant of integration. Utilizing equation (42) in equations (32-34), we have

$$w = \left(\frac{1}{v'}\right)\left[\frac{M'}{r} + \frac{c\,(1+M^2)}{r^2}\right], \qquad (43)$$

$$A = \frac{\mu}{r^2v'}\,[rM' - 2M - c\left(1 - M^2\right)] \qquad (44)$$

And

$$B = \frac{4\mu}{r^2v'}[-1 + c\,M] \qquad (45)$$

Where

$$M(r) = r f'(r) \tag{46}$$

Other observation about the compatibility equation (40) is that it involves functions A, B which depends upon the viscosity function μ, the function $f(r)$ and derivative of $f(r)$ therefore it is extremely difficult to solve it analytically. The second guidance that we take from [2] is that the equation resulting from compatibility condition provide solution on eliminating μ from the functions A and B. Therefore we eliminate μ from equation (44) and (45) by introducing function $Y(r)$ through

$$A = Y(r)\ B \tag{47}$$

Where

$$Y(r) = \frac{rM' - 2M - c(1-M^2)}{4(-1+cM)} \neq 0 \tag{48}$$

Inserting equation (47) in equation (40), we have

$$rY\ B_{rr} -$$

$$(1+2MY)B_{\nu r} + B_{\nu\nu}\left(-\frac{(1-M^2)Y}{r} + \frac{M}{r}\right) + B_\nu\left[-2MY' - YM'\right]$$

$$+\ B_r(2rY' + Y) + B(rY'' + Y')$$

$$= R_e\left(\frac{1}{\nu'^2}\right)\left[\frac{M'}{r} + \frac{c(1+M^2)}{r^2}\right]' + R_e(F_1 + MF_2)_\nu - R_e(rF_2)_r \tag{49}$$

Since the solution of equation (49) is to lead us for the function L from equations (29–30) and temperature distribution T from equation (31) whereas the use of (35) and (38) in the equation (31) provides a term with factor $\left(1 - \frac{R_e P_r}{\nu'}\right)$ therefore it guides us to search for the function B as solution of equation (49) of the type

$$B(r,\psi) = \left(1 - \frac{R_e P_r}{\nu'}\right) R(r) \tag{50}$$

Where $R(r)$ is an unknown function to be determined. Utilizing equation (50) in equation (49), we get

$$\left(1 - \frac{R_e P_r}{\nu'}\right)\left[rY\ R'' + R'(2rY' + Y) + R(rY'' + Y')\right]$$

$$+ \frac{c R_e P_r}{\nu'}\left[-(1+2MY)R' - R\left(-\frac{c(1-M^2)Y}{r} + \frac{cM}{r} + 2MY' + YM'\right)\right]$$

$$= R_e\left(\frac{1}{\nu'^2}\right)\left[\frac{M'}{r} + \frac{c(1+M^2)}{r^2}\right]' + R_e(F_1)_\nu + R_e M(F_2)_\nu - R_e(rF_2)_r \tag{51}$$

Here equation (51) is to provide the function $R(r)$, but it involves the components of unknown body force $F_1(r,\psi)$ and $F_2(r,\psi)$ therefore its solution depends upon the form of F_1 and F_2. We can select many possible forms of F_1 and F_2 leading to the solution of equation (49) for $R(r)$, however we find that not all arbitrarily selected forms lead to the solution of the momentum equations (29-30) for the function L and the energy equation (31) for T. Our search for the appropriate form of F_1 and F_2 revealed that the solution of the momentum equations (29-30) and energy equation (31) is obtainable when the function $F_2(r,\psi) = F_2(r)$ is a solution of the following differential equation

$$R_e(arF_2)_r = -[r(Y\ R)']' \tag{52}$$

Or

$$R_e F_2 = -(Y\ R)' + \frac{h_1}{r} \tag{53}$$

Where h_1 is constant.

On substituting equation (53) in equation (51) utilizing equation (41) and solving for F_1, we obtain

$$R_e F_1 = \left(\frac{R_e P_r}{cK_1}\right)\left[r(Y\ R)'\right]' e^{-c\nu}$$

$$+ \frac{R_e P_r e^{-c\nu}}{K_1}\left[(1+2MY)R' + R\left(-\frac{c(1-M^2)Y}{r} + \frac{cM}{r} + 2MY' + YM'\right)\right]$$

$$+\left(\frac{R_e e^{-2c\nu}}{2cK_1^2}\right)\left[\frac{M'}{r} + \frac{c(1+M^2)}{r^2}\right]' + H(r) \tag{54}$$

On substituting equations (53–54), in equations (29–30) and solving for the function L, we find

$$R_e L = \nu h_1 + \left(\frac{R_e e^{-2c\nu}}{2cK_1^2}\right)\left[\frac{M'}{r} + \frac{c(1+M^2)}{r^2}\right] + \frac{[r(Y\ R)']}{c}\left(\frac{R_e P_r e^{-c\nu}}{K_1}\right)$$

$$+(MY+1)R\left(\frac{R_e P_r e^{-c\nu}}{K_1}\right) +$$

$$\int\left(\frac{M h_2}{r} + 2M(YR)' + H(r)\right)dr + h_2 \tag{55}$$

Where h_2 is constant.

We can obtain viscosity from either of equations (32) or (33)

$$\mu = -\frac{\nu' r^2}{4(-1+cM)}\left(1 - \frac{R_e P_r}{\nu'}\right)R(r) \tag{56}$$

The energy equation (31) on using (35), (38), (47), (50) and (56) becomes

$$(r\nu')T_{rr} - 2M\ T_{\nu r}\nu' + \frac{(1+M^2)}{r}T_{\nu\nu}(\nu') + (\nu' - R_e P_r)T_r - M'\ T_\nu \nu'$$

$$= -\frac{E_c P_r(-1+cM)}{r}(1+4Y^2)\left(1 - \frac{R_e P_r}{\nu'}\right)R(r) \tag{57}$$

Right-hand side of equation (57) suggests searching for solution of the type

$$T(r,\nu) = \frac{K(r)}{\nu'} \tag{58}$$

Where $K(r)$ is unknown function to be determined. Utilizing equation (58) in equation (57) we find

$$rK'' + 2cMK' + \frac{(1+M^2)}{r}(c^2K) + \left(1 - \frac{R_e P_r}{\nu'}\right)K' + cKM'$$

$$= -\frac{E_c P_r(-1+cM)}{r}(1+4Y^2)\left(1 - \frac{R_e P_r}{\nu'}\right)R(r) \tag{59}$$

Comparing the coefficient of $\left(1-\frac{R_e P_r}{\nu'}\right)$ on both side of equation (59), we get

$$R(r) = -\frac{r K'}{E_c P_r(-1+cM)\left(1+4Y^2\right)} \tag{60}$$

And

$$r^2 K'' + 2c\, r\, M\;K' + K\left[c^2(1+M^2)+crM'\right] = 0 \tag{61}$$

The value of the unknown function $R(r)$ is to be determined from equation (60) but it involves another unknown function $K(r)$ satisfying equation (61). The coefficients of this equation depends on the function $M(r) = r f'(r)$ and for one arbitrary choice of $f(r)$ the solution of equation (61) could be found using computer algebra system (CAS) software in terms of special functions. Finding $K(r)$ from equation (61) we find $R(r)$ from equation (60), T from equation (58), viscosity μ from equation (56), pressure p from (12) using (55) and velocity components from (8) for the components of body force F_1 and F_2 from (53-54). This indicates an infinite set of exact solutions however; we present here three examples of such exact solutions.

Example 1:
The equation (61) reduces to a Cauchy-Euler equation for

$$M = m_1 = Const. \tag{62}$$

Or

$$f(r) = m_1 \ln r + m_2 \tag{63}$$

Where m_2 is constant of integration. Taking equation (62) in equation (61), we get

$$r^2 K'' + 2m\, r\;K' + (c^2+m^2)K = 0 \tag{64}$$

Where

$$m = c\, m_1 = Const. \tag{65}$$

Whose solution is

$$K(r) = k_3 r^{\lambda_1} + k_4 r^{\lambda_2} \tag{66}$$

Where

$$\lambda_1 = \frac{-(2m-1)+\sqrt{1-4(m+c^2)}}{2} \tag{67}$$

$$\lambda_2 = \frac{-(2m-1)-\sqrt{1-4(m+c^2)}}{2} \tag{68}$$

Inserting equation (66-68) in equation (58), we get

$$T(r,\nu) = \frac{\left(k_3 r^{\lambda_1} + k_4 r^{\lambda_2}\right)}{k_1 e^{c\nu}} \tag{69}$$

Thus for $K(r)$ from equation (66) we find $R(r)$ from equation (60) , pressure p using (55), viscosity μ from equation (56) and T from equation (69) , viscosity μ from equation (56), pressure p from (12) using (55) and velocity components from (8) for the components of body force F_1 and F_2 from (53-54).

Example 2:
In order to reduce the order of equation (61) we set

$$c(1+M^2) + rM' = 0 \tag{70}$$

Or

$$M(r) = \mathrm{Tan}(m_3 - c\ln r) \tag{71}$$

Substituting (71) in equation (46) and solving the resulting differential equation, we find

$$f(r) = \frac{1}{c}\ln\cos(m_3 - c\ln r) + m_4 \tag{72}$$

Taking equation (72) in equation (61), we get

$$r^2 K'' + 2c\, r\,\mathrm{Tan}(m_3 - c\ln r)K' = 0 \tag{73}$$

Whose solution is

$$K(r) = k_5\int \sec^2(m_3 - c\ln r)dr + k_6 \tag{74}$$

Inserting equation (74) in equation (58), we get

$$T(r,\nu) = \frac{k_5\int sce^2(m_3 - c\ln r)dr + k_6}{k_1 e^{c\nu}} \tag{75}$$

Thus finding $K(r)$ from equation (74) we find T from equation (75), $R(r)$ from equation (60), viscosity μ from equation (56), pressure p from (12) using (55) for the components of body force F_1 and F_2 from (53-54).

Example 3:
Here we let solution of equation (61) is

$$K(r) = U(r)V(r) \tag{76}$$

Taking (76) in equation (61) and removing the first derivative term [18], we find

$$U(r) = e^{-cf(r)} \tag{77}$$

And

$$r^2 V'' + V\left(c^2 + crf'\right) = 0 \tag{78}$$

A solution of (78) is

$$V(r) = k_7\,\mathrm{Cos}\, r + k_8\,\mathrm{Sin}\, r \tag{79}$$

When

$$f(r) = \frac{r^2}{2c} - c\ln r \tag{80}$$

Therefore inserting (77) and (79) in (76), we get

$$K(r) = r^{c^2} e^{-\frac{r^2}{2}}\left(k_7\,\mathrm{Cos}\, r + k_8\,\mathrm{Sin}\, r\right) \tag{81}$$

And inserting equation (81) in equation (58), we find

$$T(r,\nu) = \frac{r^{c^2} e^{-\frac{r^2}{2}}\left(k_7\,\mathrm{Cos}\, r + k_8\,\mathrm{Sin}\, r\right)}{k_1 e^{c\nu}} \tag{82}$$

Where k_i, $i \in \{1,2,3,4,5,6,7,8\}$ and m_i, $i \in \{1,2,3,4\}$ are constants. Thus finding $K(r)$ from equation (81) we find T from equation (82), $R(r)$ from equation (60), viscosity μ from equation (56), pressure p from (12) using (55) for the components of body force F_1 and F_2 from (53-54). The streamline patterns can be drawn to observe the effect of various parameters for $c>0$ and $c<0$.
For the case $Y(r)=0$, the equation (48) implies

$$rM' - 2M - c(1-M^2) = 0 \tag{83}$$

Since here, either $c>0$ or $c<0$ therefore we provide solution of equation (83) for $c=+1$ and $c=-1$ as an example and likewise one can find for $c>0$ or $c<0$. When $c=+1$ and $c=-1$, the equation (83) on utilizing equation (46) provides

$$f(r) = c_2 + \ln r + \ln\left[\cosh\left(\sqrt{2}(c_1 - \ln r)\right)\right] \tag{84}$$

And

$$f(r) = c_2 - \ln r - \ln\left[\cosh\left(\sqrt{2}(c_1 + \ln r)\right)\right] \tag{85}$$

Respectively. Inserting equation (83) in equation (44), we find

$$A = 0 \tag{86}$$

Using (86) in (40), we get

$$-\left\{B_r - \frac{f'B_\psi}{\nu'}\right\}_\psi = R_e w_r + R_e\left(F_1 + F_2 r(f')\right)_\psi - R_e\left(r\,\nu' F_2\right)_r \tag{87}$$

Here equation (87) is to provide the function B, but it involves the components of unknown body force $F_1(r,\psi)$, $F_2(r,\psi)$ and vorticity function w therefore its solution will depend upon the form of these functions. We can select many possible forms of these functions leading to the solution of equation (87) for B. However, we find that arbitrarily selected forms of these functions do not lead to the solution of the momentum equations (29-30) for the function L and the energy equation (31) for T. One successful search for the form of the functions $F_2(r,\psi)$ is to take $F_2(r,\psi) = F_2(r)$ satisfying

$$R_e\left(r\,\nu' F_2\right)_r = R_e w_r \tag{88}$$

Or

$$R_e\left(r\,\nu' F_2\right) = R_e w + G(\psi) \tag{89}$$

Where $G(\psi)$ is function of integration and vorticity function w is arbitrary. Utilizing (89) in (87), we get

$$R_e\left(F_1 + F_2 r(f')\right)_\psi = -\left\{B_r - \frac{f'B_\psi}{\nu'}\right\}_\psi \tag{90}$$

Or

$$R_e F_1 = -R_e F_2 r(f') - \left\{B_r - \frac{f'B_\psi}{\nu'}\right\} + H_1(r) \tag{91}$$

Where $H_1(r)$ is a function of integration. On substituting equations (89) and (91), in equations (29–30) and solving for the function L, we have

$$R_e L = -B - \int G(\psi)d\psi + \int H_1(r)dr + h_3 \tag{92}$$

Where h_3 is constant of integration.
Now the energy equation (30) on using equation (86), becomes

$$\begin{aligned} &r\,T_{rr} - 2M\,T_{\nu r} + \frac{(1+M^2)}{r}T_{\nu\nu} \\ &+\left(1 - \frac{R_e P_r}{\nu'}\right)T_r - M'T_\nu = -\frac{r\,E_c P_r}{4\mu}\left(B^2\right) \end{aligned} \tag{93}$$

Substituting equation (45) in equation (93), we find a relation for viscosity in terms of the derivative of the temperature function T

$$\mu = \left(\frac{-r^3(\nu')^2}{4E_c P_r (cM-1)^2}\right)\left[\begin{aligned} &r\,T_{rr} - 2M\,T_{\nu r} \\ &+\frac{(1+M^2)}{r}T_{\nu\nu} \\ &+\left(1 - \frac{R_e P_r}{\nu'}\right)T_r - M'T_\nu \end{aligned}\right] \tag{94}$$

Where the function $M(r) = r f'(r)$ is according to the equations (84) and (85).
The streamline patterns can be drawn to observe the effect of various parameters for $c>0$ and $c<0$.

4. Results and discussion

We have found some new exact solutions of the system of partial differential equations governing the steady, plane motion of incompressible fluid of variable viscosity with body force term to the right-hand side of Navier-Stokes equations of a class. The class consists of the stream function ψ characterized by $\theta = f(r) + \frac{1}{c}\ln\left[\frac{-1}{c(k_1\psi + k_2)}\right]$ in polar coordinates r and θ where a continuously differentiable function is f and the constant $c>0$ or $c<0$. For both values of the constant c, we found solutions for given one component of the body force for arbitrary $f(r)$ which indicates an infinite set of exact solutions to the problem concerned however, we mentioned three example exact solutions. When the case fixes the function $f(r)$ we find viscosity as derivative of function of temperature for $c>0$ or $c<0$ and mention such relation for $c=+1$ and $c=-1$ as an example. In both the cases when $f(r)$ is arbitrary and when it is not, we indicated an infinite set of streamlines, the velocity components, viscosity function, generalized energy function and temperature distribution. For $c>0$ and $c<0$ the streamline patterns can be drawn using CAS to observe the effect of various parameters.

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