Oscillation Criteria of Third Order Nonlinear Neutral Differential Equations

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Abstract
In this paper we will study the criteria for oscillation of the equation

$$\left[ a(t)(z''(t))^{\gamma} \right]' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0$$

and establish new oscillation criteria some examples of the obtained results are given. Our technique is Riccati’s method.

Keywords: Oscillation, Third order, Nonlinear equation, Neutral type.

1 Introduction
Consider the couple of third-order neutral differential equation of the form

$$\left[ a(t)(z''(t))^{\gamma} \right]' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0, \quad (1)$$

where $z(t) = x(t) \pm \sum_{i=1}^{n} p_i(t)x(\sigma_i(t))$, under the assumptions.
(I_1) \( a(t) \in C([t_0, \infty), (0, \infty)), \gamma > 0, \int_{t_0}^{\infty} a(s)^{-1} ds = \infty; \)

(II) \( p_i(t) \in C([t_0, \infty), R), -\mu \leq p_i(t) \leq 1 \quad \text{for all} \ i = 1, 2, 3, \ldots, n; \mu \in (0, 1); \)

(III) \( \sigma_i(t) \in C([t_0, \infty), R), \sigma_i(t) \leq t; \lim_{t \to \infty} \sigma_i(t) = \infty \quad \text{for all} \ i = 1, 2, 3, \ldots, n; \)

(IV) \( \tau_j(t) \in C([t_0, \infty), R), \tau_j(t) \leq t; \lim_{t \to \infty} \tau_j(t) = \infty \quad \text{for all} \ j = 1, 2, 3, \ldots, m; \)

(V) \( f_j(t, x(\tau_j(t))) \in C([t_0, \infty) \times R, R) \) and there exists \( q_j(t) \in C([t_0, \infty), (0, \infty)) \) such that \( f_j(t, x(\tau_j(t))) \text{ sign } x \geq q_j(t) | x^{\gamma}(\tau_j(t)) |^\gamma. \)

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see [1-15]. In the last decades, there are many studies that have been made on the oscillatory behavior of solutions of second and first differential equations. For the third-order equations we have greatly fewer results than first and second-order equations; For instance, B. Baculikova, J. Dzurina [4] examined the oscillation of the third-order nonlinear differential equations

\[
[a(t)(x''(t))^{\gamma} + q(t)f(x(\tau(t))) = 0,
\]

where \( a(t), q(t) \) are positive and \( \gamma \) is a quotient of odd positive integers, B. Baculikova, J. Dzurina [3] studied the oscillation of the third-order nonlinear neutral differential equations

\[
[a(t)\{[x(t) \pm p(t)x(\delta(t))]''\}^{\gamma} + q(t)x^{\gamma}(\tau(t)) = 0,
\]

where \( a(t), q(t) \) and \( p(t) \) are positive and \( \gamma \) is a quotient of odd positive integers, B. Baculikova, E. M. Elabbasy, S. H. Saker, J. Dzurina [2] considered the third-order equation

\[
[b(t)((a(t)x'(t))')^{\gamma} + q(t)x^{\gamma}(\tau(t)) = 0,
\]

where \( b(t), a(t) \) and \( q(t) \) are positive and \( \gamma \) is a quotient of odd positive integers.

In this paper we will improve and extend some of main results of [6]. we will use the technique in [1] to obtain criteria for oscillation of third-order neutral delay differential equations.

For simplicity we define

\[
A(t) = \int_{t_0}^{\infty} a(s)^{-1} ds; \quad \hat{Q}(t) = \int_t^{\infty} \sum_{j=1}^m q_j(s)[1 - \sum_{i=1}^n p_i(t)(\tau_j(s))]^{\gamma} ds;
\]

\[
R(t) = \gamma \tau_j(t)A[\tau_j(t)]; \quad \hat{Q}(t) = \int_t^{\infty} \sum_{j=1}^m q_j(s) ds
\]


2 Main Results

2.1 Oscillation criteria for \(0 \leq p_i(t) \leq 1\) for all \(i = 1, 2, 3, ..., n\).

In this part we will study the criteria for

\[ a(t)[(x(t) + \sum_{i=1}^{n} p_i(t)x(\sigma_i(t)))''\gamma] + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0. \quad (2) \]

The following lemma, we will be needed in the sequel.

Lemma 2.1. Let \(x(t)\) be a positive solution of Eq(2), then there are only the following two cases for \(z(t)\):

1. \(z(t) > 0, z'(t) > 0, z''(t) > 0\);
2. \(z(t) > 0, z'(t) < 0, z''(t) > 0\);

for \(t \geq t_1\), where \(t_1\) is sufficiently large.

Proof. Assume that \(x(t)\) be a positive solution of Eq(2) on \([t_0, \infty)\) we see that \(z(t) > 0\), \(x(t) > 0\)

From (I_0), Eq(2) we have

\[ [a(t)(z''(t))\gamma]' = -\sum_{j=1}^{m} f_j(t, x(\tau_j(t))) \]

\[ \leq -\sum_{j=1}^{m} x^{\gamma}(\tau_j(t)) \]

\[ < 0, \]

Thus \([a(t)(z''(t))\gamma]\] is nonincreasing and of one sign. There for \(z''(t)\) is also of one sign and so we have two cases: \(z''(t) < 0\) or \(z''(t) > 0\) for all \(t \geq t_1\). If we admit that \(z''(t) < 0\) then there exist a constant \(M > 0\) such that

\[ a(t)(z''(t))\gamma \leq -M < 0, \]

integrating on \([t_1, t]\) we obtain

\[ z'(t) \leq z'(t_1) - M^{\frac{1}{\gamma}} \int_{t_1}^{t} a^{-\frac{1}{\gamma}}(s)ds, \]

as \(t \to \infty\), we get \(z'(t) \to -\infty\) thus \(z'(t) < 0\) eventually. But \(z''(t) < 0\) and \(z' < 0\) eventually imply \(z(t) < 0\) for \(t \leq t_1\). This contradiction, and this proves that \(z''(t) > 0\).

\[ \square \]
Theorem 2.2. Suppose that Eq(2) is nonoscillatory, then there exist a positive function $V(t)$ on $[T, \infty)$ such that

1. 
$$
\hat{Q}(t) < \infty, \quad \int_{t}^{\infty} R(s)\hat{Q}^{\frac{n}{n-1}}(s)ds < \infty \quad (3)
$$

2. 
$$
V(t) \geq \hat{Q}(t) + \int_{t}^{\infty} R(s)V^{\frac{n}{n-1}}(s)ds \quad \text{for all } t \geq T \geq t_0 \quad (4)
$$

3. 
$$
\limsup_{t \to \infty} V(t) \left[ \int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds \right]^{\gamma} \leq 1 \quad (5)
$$

Proof. Let $x(t)$ be nonoscillatory solution of Eq(2). Assume that $x(t) > 0$, $x(\sigma_i(t)) > 0$ for all $i = 1, 2, ..., n$; $x(\tau_j(t)) > 0$ for all $j = 1, 2, ..., m$ then $z(t) > 0$, $z(t) > x(t)$. from(I5)

$$
[a(t)(z''(t))^{\gamma}']' = -\sum_{j=1}^{m} f_j(t, x(\tau_j(t))) \leq -\sum_{j=1}^{m} q_j(t)x^{\gamma}(\tau_j(t)), \quad (6)
$$

since;

$$
x(t) = z(t) - \sum_{i=1}^{n} p_i(t)x(\sigma_i(t)),$$

$$
x(\tau_j(t)) = z(\tau_j(t)) - \sum_{i=1}^{n} p_i(\tau_j(t))x(\sigma_i(\tau_j(t))),$$

$$
\geq z(\tau_j(t)) - \sum_{i=1}^{n} p_i(\tau_j(t))z(\sigma_i(\tau_j(t))),$$

$$
\geq [1 - \sum_{i=1}^{n} p_i(\tau_j(t))]z(\tau_j(t)), \quad (7)
$$

from(6), (7)

$$
\frac{[a(t)(z''(t))^{\gamma}']}{z^{\gamma}(\tau_j(t))} \leq -\sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\tau_j(t))]^{\gamma}.
$$

Now, we define function $V(t)$

$$
V(t) = \frac{[a(t)(z''(t))^{\gamma}]}{z^{\gamma}(\tau_j(t))},
$$
\[
V'(t) = \frac{[a(t)(z''(t))^\gamma]'}{z^\gamma(\tau_j(t))} - \gamma \frac{[a(t)(z''(t))^\gamma]z'((\tau_j(t))(\tau'_j(t))}{z^{\gamma+1}(\tau_j(t))} \\
\leq - \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\tau_j(t))]\gamma - \gamma V(t)\frac{z'((\tau_j(t))(\tau'_j(t))}{z(\tau_j(t))},
\]

since

\[
z'(t) \geq \int_{t_0}^{t} z''(s)ds = \int_{t_0}^{t} a^{-1}(s)[a(s)(z''(s))^\gamma]^{\frac{1}{\gamma}}ds,
\]

from (I_1) and (I_5), we get

\[
z'(t) \geq [a(t)(z''(t))^\gamma]\frac{1}{\gamma} \int_{t_0}^{t} a^{-1}(s)ds,
\]

\[
z'(\tau_j(t)) \geq [a(\tau_j(t))(z''(\tau_j(t)))^\gamma]\frac{1}{\gamma} \int_{t_0}^{\tau_j(t)} a^{-1}(s)ds,
\]

\[
z'(\tau_j(t)) \geq [a(\tau_j(t))(z''(\tau_j(t)))^\gamma]\frac{1}{\gamma} A[\tau_j(t)],
\]

from (2), (I_1), (I_4) and (I_5), we get

\[
[a(t)(z''(t))^\gamma] \leq [a(\tau_j(t))(z''(\tau_j(t)))^\gamma],
\]

from (9), (10)

\[
z'(\tau_j(t)) \geq [a(t)(z''(t))^\gamma]\frac{1}{\gamma} A[\tau_j(t)],
\]

from (8) and (11), we have

\[
V'(t) \leq - \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\tau_j(t))]\gamma - \gamma V(t)\frac{[a(t)(z''(t))^\gamma]\frac{1}{\gamma} A[\tau_j(t)](\tau'_j(t))}{z(\tau_j(t))},
\]

\[
\leq - \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\tau_j(t))]\gamma - \gamma V^{\frac{\gamma+1}{\gamma}}(t)A[\tau_j(t)](\tau'_j(t)),
\]

\[
V'(t) + \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\tau_j(t))]\gamma + R(t)V^{\frac{\gamma+1}{\gamma}}(t),
\]

integrating on \([t, t']\)

\[
V(t') - V(t) + \int_{t}^{t'} \sum_{j=1}^{m} q_j(s)[1 - \sum_{i=1}^{n} p_i(\tau_j(s))]\gamma ds + \int_{t}^{t'} R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds \leq 0,
\]

assume that \(\hat{Q}(t) < \infty\) for all \(t \geq t_2\), otherwise from the above inequality

\[
V(t') \leq V(t) - \int_{t}^{t'} \sum_{j=1}^{m} q_j(s)[1 - \sum_{i=1}^{n} p_i(\tau_j(s))]\gamma ds,
\]
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this lead to \( V(t') \to -\infty \) as \( t' \to \infty \). Which contradiction \((V(t)\) positive ), Similarly we can show that

\[
\int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds < 0. \tag{15}
\]

To prove (4) from Theorem 2.2 from (12) we define \( \lim_{t \to \infty} V(t) = V^* \), from (15) letting \( t' \to \infty \) in (14)

\[
V(t) \geq \hat{Q}(t) + \int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds \text{ for all } t \geq T \geq t_0.
\]

To prove (5) from (13) and \( \sum_{j=1}^m q_j(t)[1 - \sum_{i=1}^n p_i(\tau_j(t))]^\gamma > 0 \), then

\[
V'(t) + R(t)V^{\frac{\gamma+1}{\gamma}}(t) < 0, \tag{16}
\]

since \( V(t) > 0 \) and \( R(t) > 0 \), then from (16), \( V'(t) < 0 \), and

\[
-\frac{V'(t)}{\gamma V^{\frac{\gamma+1}{\gamma}}(t)} > A[\tau_j(t)](\tau_j'(t)) \left( V^{\frac{1}{\gamma}}(t) \right)' > A[\tau_j(t)](\tau_j'(t)).
\]

Integrating the above inequality on \( [\tau_j(t), \tau_j'(t)] \), yields

\[
V^{\frac{1}{\gamma}}(t) > \int_{\tau_j(t)}^{\tau_j'(t)} A[\tau_j(s)](\tau_j'(s))ds \geq \frac{1}{V} \left[ \int_{\tau_j(t)}^{\tau_j'(t)} A[\tau_j(s)](\tau_j'(s))ds \right]^\gamma.
\]

This complete proof of Theorem 2.2

\[ \square \]

**Corollary 2.3.** Assume that

\[
\liminf_{t \to \infty} \frac{1}{\hat{Q}(t)} \int_t^\infty \hat{Q}^{\frac{\gamma+1}{\gamma}}(s)R(s)ds > \frac{\gamma}{(\gamma + 1)\frac{\gamma+1}{\gamma}}. \tag{17}
\]

Then Eq.(2) is oscillatory.

**Proof.** Suppose the contrary that Eq(2) is nonoscillatory from theorem (4) we find

\[
V(t) \geq \hat{Q}(t) + \int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds,
\]
and from the assumption of the corollary there exist $\beta > \frac{\gamma}{(\gamma+1)}$ such that

$$\liminf_{t \to \infty} \frac{1}{Q(t)} \int_t^\infty \hat{Q}^{\frac{\gamma+1}{\gamma}}(s)R(s)ds > \beta,$$

from (4)

$$\frac{V(t)}{Q(t)} \geq 1 + \frac{1}{Q(t)} \int_t^\infty R(s)\hat{Q}^{\frac{\gamma+1}{\gamma}}(s)ds$$

$$\geq 1 + \frac{1}{Q(t)} \int_t^\infty R(s)\hat{Q}^{\frac{\gamma+1}{\gamma}}(s)\left(\frac{V(s)}{Q(s)}\right)^{\frac{\gamma+1}{\gamma}} ds.$$

Let $\lambda = \inf_{t \geq T} \left(\frac{V(t)}{Q(t)}\right)$, then $\lambda \geq 1$ and $\lambda \geq 1 + \beta \lambda^{\frac{\gamma+1}{\gamma}}$ by simple calculation we get $\lambda - \beta \lambda^{\frac{\gamma+1}{\gamma}} \leq \frac{1}{(\gamma+1)}\beta$, a contradiction, $\beta = \frac{\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}}$ this contradiction lead to Eq(2) is oscillatory. \hfill \square

Following [1], let $\{y_n(t)\}_{n=0}^\infty$ be a sequence of continuous functions defined as follows (if they exist):

$$y_0(t) = \hat{Q}(t)$$

for all $t \geq t_0$ and

$$y_n(t) = \int_t^\infty R(s)y_{n-1}^{\frac{\gamma+1}{\gamma}}(s)ds + \hat{Q}(t).$$

\textbf{Lemma 2.4.} [1]. If Eq(2) is nonoscillatory, then $y_n(t) \leq V(t)$ where $V(t)$ be as defined in Theorem 2.2 and there exists a positive function $y(t)$ on $[t, \infty)$, such that $\lim_{t \to \infty} y_n(t) = y$ for $t \geq T \geq t_0$. In addition we have

$$y(t) = \int_t^\infty R(s)y^{\frac{\gamma+1}{\gamma}}(s)ds + \hat{Q}(t).$$

\textit{Proof}. Suppose that Eq(2) is nonoscillatory. From (Theorem 2.2) $y_0(t) \leq y_1(t)$, by induction $y_n(t) \leq y_{n+1}(t)$ for $n = 0, 1, 2, ...$

By the other hand from (3), we have $V(t) \geq \hat{Q}(t) = y_0(t)$. Inductively, we get $V(t) \geq y_n(t)$ for $n = 0, 1, 2, ...$

Thus, the sequence $\{y_n(t)\}_{n=0}^\infty$ converges to $y(t)$ on $[T, \infty)$ by Lebesgue’s monotone convergence theorem, and letting $n \to \infty$ in Eq (18) we get (19). \hfill \square

\textbf{Corollary 2.5.} Let $y_n(t)$ defined by (18). If there exists some $y_n(t)$ such that

$$\limsup_{t \to \infty} y_n(t) \left[\int_{\tau_j(t)}^{\tau_j(t)} \tau_j(s)A[\tau_j(s)]ds\right]^\gamma > 1 \text{ for all } n = 0, 1, 2, ...$$

Then Eq(2) is oscillatory.
Proof. Suppose that Eq(2) is nonoscillatory, then from Theorem 2.2

$$\limsup_{t \to \infty} V(t) \left[ \int_{t_0}^{\tau_j(t)} \tau_j'(s) A[\tau_j(s)]ds \right]^{\gamma} \leq 1,$$

by contrary and from Lemma 2.4 we get

$$\limsup_{t \to \infty} y_n(t) \left[ \int_{t_0}^{\tau_j(t)} \tau_j'(s) A[\tau_j(s)]ds \right]^{\gamma} > 1.$$

This lead to Eq(2) is oscillatory.

Corollary 2.6. Assume that

$$\limsup_{t \to \infty} \left[ \int_{t_0}^{\tau_j(t)} \tau_j'(s) A[\tau_j(s)]ds \right]^{\gamma} \left[ \int_t^{\infty} R(s) \hat{Q}^{\gamma+1}(s)ds + \hat{Q}(t) \right] > 1.$$ 

Then Eq(2) is oscillatory.

2.2 Oscillation criteria for $- \mu \leq \sum_{i=1}^{n} p_i(t) \leq 0$.

In this section we will study the criteria for

$$[a(t)[(x(t) - \sum_{i=1}^{n} p_i(t)x(\sigma_i(t))]'']'' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0, \quad (21)$$

Theorem 2.7. Assume that every solution of Eq(21) is neither oscillatory nor tends to zero.then there exists a positive function $V(t)$ on $[T, \infty)$ such that

1. $Q(t) < \infty$, $\int_t^{\infty} R(s) Q^{\gamma+1}(s)ds < \infty.$ \hfill (22)

2. $V(t) \geq Q(t) + \int_t^{\infty} R(s) V^{\gamma+1}(s)ds$ for all $t \geq T \geq t_0.$ \hfill (23)

3. $\limsup_{t \to \infty} V(t) \left[ \int_{t_0}^{\tau_j(t)} \tau_j'(s) A[\tau_j(s)]ds \right]^{\gamma} \leq 1.$ \hfill (24)
Proof. Let $x(t)$ be a solution which is neither oscillatory nor tends to zero such that $x(t) > 0$; $x(\sigma_i(t)) > 0$; $x(\tau_j(t)) > 0$ from (6) we have

$$[a(t)(z''(t)))'] = - \sum_{j=1}^{m} f_j(t, x(\tau_j(t)))$$

$$\leq - \sum_{j=1}^{m} q_j(t)x^\gamma(\tau_j(t)),$$

we have two possible cases for $z(t)$

I $z(t) > 0$

II $z(t) < 0$

(I) $z(t) > 0$, then proof will be as proof of theorem 2.2 but $\hat{Q}(t)$ replaced by $Q(t)$ (II) $z(t) < 0$ eventually for all $t \geq t_2 \geq t_1 \geq t_0$, then we have two cases for $x(t)$

(a) $x(t)$ is unbounded

(b) $x(t)$ is bounded

(a) $x(t)$ is unbounded

Assume that $x(t)$ is unbounded, then

$$x(t) = z(t) - \sum_{i=1}^{n} p_i(t)x(\sigma_i(t)) < - \sum_{i=1}^{n} p_i(t)x(\sigma_i(t)) < \sum_{i=1}^{n} x(\sigma_i(t)). \quad (25)$$

Since $x(t)$ is unbounded, then we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ satisfying $\lim_{n \to \infty} T_n = \infty$ from which $\lim_{n \to \infty} x(T_n) = \infty$ and $\max_x x(t) = x(T_n)$ by choosing $N$ large such that $\sigma_i(T_N) > T_1$ for all $T_N > t_2$. Thus $\max_x x(t) = x(T_N)$. this contradict with (25).

(b) $x(t)$ is bounded

Suppose that $x(t)$ is bounded, we show that $x(t) \to 0$ as $t \to \infty$ since

$$\lim_{t \to \infty} \sup_t(z(t)) \leq 0,$$

then we have
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\[
\limsup_{t \to \infty} (x(t) + \sum_{i=1}^{n} p_i(t)x(\sigma_i(t))) \leq 0
\]

\[
\limsup_{t \to \infty} x(t) + \limsup_{t \to \infty} \sum_{i=1}^{n} p_i(t)x(\sigma_i(t)) \leq 0
\]

\[(1 - \mu) \limsup_{t \to \infty} x(t) \leq 0.
\]

This shows that \(x(t) \to 0\) as \(t \to \infty\).

\[\square\]

**Corollary 2.8.** Assume that

\[
\limsup_{t \to \infty} \left[ \int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds \right]^{\gamma} \left[ \int_{t}^{\infty} R(s)Q^{\frac{\gamma+1}{\gamma}}(s)ds + Q(t) \right] > 1.
\] (26)

Then every solution of Eq(21) is either oscillatory or tends to zero.

**Corollary 2.9.** Assume that

\[
\liminf_{t \to \infty} \frac{1}{Q(t)} \int_{t}^{\infty} Q^{\frac{\gamma+1}{\gamma}}(s)R(s)ds > \frac{\gamma}{(\gamma + 1)^{\frac{\gamma+1}{\gamma}}}.
\] (27)

Then every solution of Eq(21) is either oscillatory or tends to zero.

**Remark 2.10.** We can get oscillation criteria of the equation

\[\left[a(t)(z''(t))^{\gamma}\right]' + f(t, x(\tau(t))) = 0,\]

where \(z(t) = x(t) \pm p(t)x(\sigma(t))\), if we put \(j = i = 1\).

### 3 Examples

**Example 1.** Consider

\[
[t(z''(t))^3]' + \sum_{j=1}^{2} f_j(t, x(\tau_j(t))) = 0, \quad t \geq 0
\] (28)

\[z(t) = x(t) + \sum_{i=1}^{2} p_i(t)x(\sigma_i(t)), \quad p_1(t) = \frac{1}{2}, \quad p_2(t) = \frac{1}{4}, \quad q_1(t) = \frac{2a}{e^r}, \quad q_2(t) = \frac{4a}{e^r}, \quad a > 0, \quad \tau_1(t) = \frac{t}{2}, \quad \tau_2(t) = \frac{t}{3}.
\]

It is clear that

\[a(t) = t, \quad \int_{t}^{\infty} a^{-\frac{1}{\gamma}}(s)ds = \int_{t}^{\infty} s^{-\frac{1}{\gamma}}ds = \infty, \quad 0 \leq p_1(t) \leq 1; \quad q_1(t), q_2(t) \text{ positive.}
\]

since
\[ \hat{Q}(t) = \int_t^\infty \sum_{j=1}^m q_j(s)[1 - \sum_{i=1}^n p_i(t)(\tau_j(s))\gamma]ds \]

\[ \hat{Q}(t) = \int_t^\infty [q_1(s)(1 - (p_1(s) + p_2(s)))^3 + [q_2(s)(1 - (p_1(s) + p_2(s)))^3]ds \]

\[ = \left(\frac{1}{4}\right)^3 \int_t^\infty \frac{6a}{s^6}ds \]

\[ = \left(\frac{1}{4}\right)^3 \left(\frac{6a}{5}\right) \left(\frac{1}{t^5}\right) . \]

\[ R(t) = \gamma \tau'_j(t)A[\tau_j(t)] \]

\[ = 3[\tau'_1(t)A[\tau_1(t)] + \tau'_2(t)A[\tau_2(t)] \]

\[ = \left(\frac{9}{2}\right) \left[\left(\frac{1}{2}\right)^{5/3} + \left(\frac{1}{3}\right)^{5/3}\right] t^{2/3} . \]

from (17)

\[ \liminf_{t\to\infty} \frac{1}{Q(t)} \int_t^\infty \hat{Q}(s)R(s)ds = \left(\frac{9}{10}\right) \left(\frac{1}{4}\right) \left(\frac{6a}{5}\right) \left[\left(\frac{1}{2}\right)^{5/3} + \left(\frac{1}{3}\right)^{5/3}\right] . \]

since \( \frac{\gamma}{(\gamma + 1)\gamma} = \frac{3}{(4)^{\gamma + 1/2}} \)

Then the solution of (28) is oscillatory when \( a > 71.89238009 \).

**Example 2.** Consider

\[ [t(z''(t))^3] + \sum_{j=1}^2 f_j(t, x(\tau_j(t))) = 0, \quad t \geq 0 \quad (29) \]

\[ z(t) = x(t) + \sum_{i=1}^2 p_i(t)x(\sigma_i(t)), \quad p_1(t) = \frac{-1}{2}, \quad p_2(t) = \frac{-1}{4}, \quad q_1(t) = \frac{2a}{e}, \quad q_2(t) = \frac{4a}{e}, \quad a > 0, \quad \tau_1(t) = \frac{t}{2}, \quad \tau_2(t) = \frac{t}{4}. \]

It is clear that

\[ a(t) = t, \quad \int_t^\infty a^{-\gamma}(s)ds = \int_t^\infty s^{-\gamma}ds = \infty, \]

\[ 0 \leq p_1(t), \quad p_2(t) \leq 1 ; \quad q_1(t), q_2(t) \text{ positive.} \]

since

\[ Q(t) = \int_t^\infty \sum_{j=1}^m q_j(s)ds = \left(\frac{6a}{5}\right) \left(\frac{1}{t^5}\right) . \]
\[ R(t) = \left( \frac{9}{2} \right) \left[ \left( \frac{1}{2} \right)^{5/3} + \left( \frac{1}{3} \right)^{5/3} \right] t^{2/3}. \]

from (27)

\[
\liminf_{t \to \infty} \frac{1}{Q(t)} \int_t^\infty Q^{\gamma+1}(s)R(s)ds = \left( \frac{9}{10} \right) \left( \frac{6a}{5} \right)^{1/3} \left[ \left( \frac{1}{2} \right)^{5/3} + \left( \frac{1}{3} \right)^{5/3} \right].
\]

since \( \frac{\gamma}{(\gamma+1)\gamma} = \frac{3}{(4)^{5/3}} \)

Then the solution of (29) is oscillatory or tends to zero when \( a > 1.123318444 \).

**Example 3.** [13] Consider

\[
((x''(t))^3)' + \frac{a}{t^7}x^3(\lambda t) = 0, \quad a > 0, \quad 0 < \lambda < 1, \quad t \geq 1. \tag{30}
\]

Note that \( p(t) = 0, \quad q(t) = \frac{a}{t^7}, \quad \gamma = 3, \quad \tau(t) = \lambda t, \quad a(t) = 1, \quad \int_t^\infty a^{-1}\hat{Q}(s)ds = \infty \)

since

\[
\hat{Q}(t) = \int_t^\infty \sum_{j=1}^m q_j(s)[1 - \sum_{i=1}^n p_i(t)(\tau_j(s))]^{\gamma}ds
\]

\[
= \int_t^\infty q(s)[1 - p(t)(\tau(s))]^{\gamma}ds
\]

\[
= \frac{a}{6t^6}.
\]

\[ R(t) = \gamma \tau_j'(t)A[\tau_j(t)] \]

\[ = \gamma \tau'(t)A[\tau(t)] \]

\[ = (3\lambda)(\lambda t - t_0). \]

from (17)

\[
\liminf_{t \to \infty} \frac{1}{Q(t)} \int_t^\infty \hat{Q}^{\gamma+1}(s)R(s)ds = \frac{3\lambda^2}{6} \left( \frac{a}{6} \right)^{1/3}
\]

since \( \frac{\gamma}{(\gamma+1)\gamma} = \frac{3}{(4)^{3/4}} \)

Then Eq(3.3.1) is oscillatory if \( a\lambda^6 > \frac{3^4}{4^3} \).

This result is consistent with the result in Example 168 of [13].
References


