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Oscillation Criteria of Third Order Nonlinear Neutral Differential Equations

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Abstract

In this paper we will study the criteria for oscillation of the equation

$$\left[a(t)(z''(t))^{\gamma}\right]' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0$$

and establish new oscillation criteria some examples of the obtained results are given. Our technique is Riccati's method.

Keywords: *Oscillation, Third order, Nonlinear equation, Neutral type.*

1 Introduction

Consider the couple of third-order neutral differential equation of the form

$$[a(t)(z''(t))^{\gamma}]' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0, \tag{1}$$

where $z(t) = x(t) \pm \sum_{i=1}^{n} p_i(t)x(\sigma_i(t))$, under the assumptions.

(I_1) $a(t) \in C([t_0,\infty),(0,\infty))$, $\gamma > 0$, $\int_{t_0}^{\infty} a(s)^{\frac{-1}{\gamma}} ds = \infty$;

(I_2) $p_i(t) \in C([t_0,\infty),R), -\mu \leq p_i(t) \leq 1 \quad for\ all\ i = 1,2,3,...,n; \mu \in (0,1)$;

(I_3) $\sigma_i(t) \in C([t_0,\infty),R), \sigma_i(t) \leq t; lim_{t\to\infty}\sigma_i(t) = \infty\ \ for\ \ all\ \ i = 1,2,3,...,n$;

(I_4) $\tau_j(t) \in C([t_0,\infty),R), \tau_j(t) \leq t; lim_{t\to\infty}\tau_j(t) = \infty\ \ for\ \ all\ \ j = 1,2,3,...,m$;

(I_5) $f_j(t, x(\tau_j(t))) \in C([t_0,\infty)\times R, R)$ and there exists $q_j(t) \in C([t_0,\infty),(0,\infty))$ such that $f_j(t, x(\tau_j(t)))\ \ sign\ x \geq q_j(t) \mid x^{\gamma}(\tau_j(t)) \mid^{\gamma}$.

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines;see[1-15].
In the last decades, there are many studies that have been made on the oscillatory behavior of solutions of second and first differential equations. For the third-order equations we have greatly fewer results than first and second-order equations;
For instance, B.Baculikova, J.Dzurina [4],examined the oscillation of the third-order nonlinear differential equations

$$[a(t)(x''(t))^{\gamma}]' + q(t)f(x(\tau(t))) = 0,$$

where $a(t)$, $q(t)$ are positive and γ is a quotient of odd positive integers,
B.Baculikova, J.Dzurina [3] studied the oscillation of the third-order nonlinear neutral differential equations

$$[a(t)([x(t) \pm p(t)x(\delta(t))]'')^{\gamma}]' + q(t)x^{\gamma}(\tau(t)) = 0,$$

where a(t),q(t) and p(t)are positive and γ is a quotient of odd positive integers,
B.Baculikova, E.M.Elabbasy, S.H.Saker, J.Dzurina [2] considered the third-order equation

$$[b(t)((a(t)x'(t)')^{\gamma}]' + q(t)x^{\gamma}(\tau(t)) = 0,$$

where $b(t)$, $a(t)$ and $q(t)$are positive and γ is a quotient of odd positive integers.

In this paper we will improve and extend some of main results of [6]. we will use the technique in [1] to obtain criteria for oscillation of third-order neutral delay differential equations.
For simplicity we define
$A(t) = \int_{t_0}^{\infty} a(s)^{\frac{-1}{\gamma}} ds$; $\quad \widehat{Q}(t) = \int_{t}^{\infty} \sum_{j=1}^{m} q_j(s)[1 - \sum_{i=1}^{n} p_i(t)(\tau_j(s))]^{\gamma} ds$;
$R(t) = \gamma\tau_j'(t)A[\tau_j(t)]$; $\qquad Q(t) = \int_{t}^{\infty} \sum_{j=1}^{m} q_j(s) ds$

2 Main Results

2.1 Oscillation criteria for $0 \leq p_i(t) \leq 1$ for all$i = 1, 2, 3, ..., n$.

In this part we will study the criteria for

$$[a(t)[(x(t)+\sum_{i=1}^{n} p_i(t)x(\sigma_i(t)))'']^{\gamma}]' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0. \tag{2}$$

The following lemma , we will be needed in the sequel .

Lemma 2.1. *Let x(t) be a positive solution of Eq(2), then there are only the following two cases for $z(t)$:*

1. $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0$;
2. $z(t) > 0,\ z'(t) < 0,\ z''(t) > 0$;

for $t \geq t_1$, where t_1 is sufficiently large.

Proof. Assume that $x(t)$ be a positive solution of Eq(2) on $[t_0, \infty)$ we see that $z(t) > 0$, $x(t) > 0$
From (I_5), Eq(2) we have

$$\begin{aligned}[a(t)(z''(t))^{\gamma}]' &= -\sum_{j=1}^{m} f_j(t, x(\tau_j(t))) \\ &\leq -\sum_{j=1}^{m} x^{\gamma}(\tau_j(t)) \\ &< 0,\end{aligned}$$

Thus $[a(t)(z''(t))^{\gamma}]$ is nonincreasing and of one sign. There for $z''(t)$ is also of one sign and so we have two cases: $z''(t) < 0$ or $z''(t) > 0$ for all $t \geq t_1$. If we admit that $z''(t) < 0$.then there exist a constant $M > 0$ such that

$$a(t)(z''(t))^{\gamma} \leq -M < 0,$$

integrating on $[t_1, t]$, we obtain

$$z'(t) \leq z'(t_1) - M^{\frac{1}{\gamma}} \int_{t_1}^{t} a^{\frac{-1}{\gamma}}(s)ds,$$

as $t \to \infty$, we get $z'(t) \to -\infty$ .thus $z'(t) < 0$ eventually. But $z''(t) < 0$ and $z' < 0$ eventually imply $z(t) < 0$ for $t \leq t_1$. This contradiction, and this proves that $z''(t) > 0$. □

Theorem 2.2. *Suppose that Eq(2) is nonoscillatory, then there exist a positive function $V(t)$ on $[T,\infty)$.such that*

1.
$$\widehat{Q}(t) < \infty \quad , \quad \int_t^\infty R(s)\widehat{Q}^{\frac{\gamma+1}{\gamma}}(s)ds < \infty \tag{3}$$

2.
$$V(t) \geq \widehat{Q}(t) + \int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds \quad for \quad all \quad t \geq T \geq t_0 \tag{4}$$

3.
$$\limsup_{t\to\infty} V(t)\left[\int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds\right]^\gamma \leq 1 \tag{5}$$

Proof. Let $x(t)$ be nonoscillatory solution of Eq(2). Assume that $x(t) > 0$, $x(\sigma_i(t)) > 0$ for all $i = 1, 2, ..., n$; $x(\tau_j(t)) > 0$ for all $j = 1, 2, ..., m$ then $z(t) > 0$, $z(t) > x(t)$. from(I_5)

$$\begin{aligned}
[a(t)(z''(t))^\gamma]' &= -\sum_{j=1}^m f_j(t, x(\tau_j(t))) \\
&\leq -\sum_{j=1}^m q_j(t)x^\gamma(\tau_j(t)),
\end{aligned} \tag{6}$$

since;

$$\begin{aligned}
x(t) &= z(t) - \sum_{i=1}^n p_i(t)x(\sigma_i(t)), \\
x(\tau_j(t)) &= z(\tau_j(t)) - \sum_{i=1}^n p_i(\tau_j(t))x(\sigma_i(\tau_j(t))), \\
&\geq z(\tau_j(t)) - \sum_{i=1}^n p_i(\tau_j(t))z(\sigma_i(\tau_j(t))), \\
&\geq [1 - \sum_{i=1}^n p_i(\tau_j(t))]z(\tau_j(t)),
\end{aligned} \tag{7}$$

from(6), (7)

$$\frac{[a(t)(z''(t))^\gamma]'}{z^\gamma(\tau_j(t))} \leq -\sum_{j=1}^m q_j(t)[1 - \sum_{i=1}^n p_i(\tau_j(t))]^\gamma.$$

Now, we define function $V(t)$

$$V(t) = \frac{[a(t)(z''(t))^\gamma]}{z^\gamma(\tau_j(t))},$$

$$V'(t) = \frac{[a(t)(z''(t))^{\gamma}]'}{z^{\gamma}(\tau_j(t))} - \gamma\frac{[a(t)(z''(t))^{\gamma}]z'(\tau_j(t))(\tau_j'(t))}{z^{\gamma+1}(\tau_j(t))}$$
$$\leq -\sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\tau_j(t))]^{\gamma} - \gamma V(t)\frac{z'(\tau_j(t))(\tau_j'(t))}{z(\tau_j(t))}, \quad (8)$$

since

$$z'(t) \geq \int_{t_0}^{t} z''(s)ds = \int_{t_0}^{t} a^{\frac{-1}{\gamma}}(s)[a(s)(z''(s))^{\gamma}]^{\frac{1}{\gamma}}ds,$$

from (I_1) and (I_5), we get

$$z'(t) \geq [a(t)(z''(t))^{\gamma}]^{\frac{1}{\gamma}} \int_{t_0}^{t} a^{\frac{-1}{\gamma}}(s)ds,$$
$$z'(\tau_j(t)) \geq [a(\tau_j(t))(z''(\tau_j(t)))^{\gamma}]^{\frac{1}{\gamma}} \int_{t_0}^{\tau_j(t)} a^{\frac{-1}{\gamma}}(s)ds,$$
$$z'(\tau_j(t)) \geq [a(\tau_j(t))(z''(\tau_j(t)))^{\gamma}]^{\frac{1}{\gamma}} A[\tau_j(t)], \quad (9)$$

from (2), (I_1), (I_4) and (I_5), we get

$$[a(t)(z''(t))^{\gamma}] \leq [a(\tau_j(t))(z''(\tau_j(t)))^{\gamma}], \quad (10)$$

from (9), (10)

$$z'(\tau_j(t)) \geq [a(t)(z''(t))^{\gamma}]^{\frac{1}{\gamma}} A[\tau_j(t)], \quad (11)$$

from (8) and (11), we have

$$V'(t) \leq -\sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\tau_j(t))]^{\gamma} - \gamma V(t)\frac{[a(t)(z''(t))^{\gamma}]^{\frac{1}{\gamma}} A[\tau_j(t)](\tau_j'(t))}{z(\tau_j(t))},$$
$$\leq -\sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\tau_j(t))]^{\gamma} - \gamma V^{\frac{\gamma+1}{\gamma}}(t)A[\tau_j(t)](\tau_j'(t)), \quad (12)$$

$$V'(t) + \sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\tau_j(t))]^{\gamma} + R(t)V^{\frac{\gamma+1}{\gamma}}(t), \quad (13)$$

integrating on $[t, t']$

$$V(t')-V(t)+\int_{t}^{t'} \sum_{j=1}^{m} q_j(s)[1-\sum_{i=1}^{n} p_i(\tau_j(s))]^{\gamma}ds+\int_{t}^{t'} R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds \leq 0, \quad (14)$$

assume that $\widehat{Q}(t) < \infty$ for all $t \geq t_2$, otherwise from the above inequality

$$V(t') \leq V(t) - \int_{t}^{t'} \sum_{j=1}^{m} q_j(s)[1-\sum_{i=1}^{n} p_i(\tau_j(s))]^{\gamma}ds,$$

this lead to $V(t') \to -\infty$ as $t' \to \infty$. Which contradiction ($V(t)$ positive), Similarly we can show that

$$\int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds < 0. \tag{15}$$

To prove (4) from Theorem 2.2
from (12) we define $\lim_{t\to\infty} V(t) = V^*$, from (15)letting $t' \to \infty$ in (14)

$$V(t) \geq \widehat{Q}(t) + \int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds \quad for \quad all \quad t \geq T \geq t_0.$$

To prove (5)
from(13) and $\sum_{j=1}^m q_j(t)[1 - \sum_{i=1}^n p_i(\tau_j(t))]^\gamma > 0$, then

$$V'(t) + R(t)V^{\frac{\gamma+1}{\gamma}}(t) < 0, \tag{16}$$

since $V(t) > 0$ and $R(t) > 0$, then from (16), $V'(t) < 0$, and

$$-\frac{V'(t)}{\gamma V^{\frac{\gamma+1}{\gamma}}(t)} > A[\tau_j(t)](\tau_j'(t))$$
$$\left(V^{\frac{-1}{\gamma}(t)}\right)' > A[\tau_j(t)](\tau_j'(t)).$$

Integrating the above inequality on $[t_0, \tau_j(t)]$, yields

$$V^{\frac{-1}{\gamma}}(t) > \int_{t_0}^{\tau_j(t)} A[\tau_j(s)](\tau_j'(s))ds$$
$$\frac{1}{V} > \left[\int_{t_0}^{\tau_j(t)} A[\tau_j(s)](\tau_j'(s))ds\right]^\gamma.$$

This complete proof of Theorem 2.2 □

Corollary 2.3. *Assume that*

$$\liminf_{t\to\infty} \frac{1}{\widehat{Q}(t)} \int_t^\infty \widehat{Q}^{\frac{\gamma+1}{\gamma}}(s)R(s)ds > \frac{\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}} . \tag{17}$$

Then Eq.(2) is oscillatory.

Proof. Suppose the contrary that Eq(2) is nonoscillatory from theorem (4) we find

$$V(t) \geq \widehat{Q}(t) + \int_t^\infty R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds,$$

and from the assumption of the corollary .there exist $\beta > \frac{\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}}$ such that

$$\liminf_{t\to\infty} \frac{1}{\widehat{Q}(t)} \int_t^\infty \widehat{Q}^{\frac{\gamma+1}{\gamma}}(s) R(s) ds > \beta,$$

from(4)

$$\frac{V(t)}{\widehat{Q}(t)} \geq 1 + \frac{1}{\widehat{Q}(t)} \int_t^\infty R(s) V^{\frac{\gamma+1}{\gamma}}(s) ds$$

$$\begin{aligned} \frac{V(t)}{\widehat{Q}(t)} &\geq 1 + \frac{1}{\widehat{Q}(t)} \int_t^\infty R(s) V^{\frac{\gamma+1}{\gamma}}(s) ds \\ &\geq 1 + \frac{1}{\widehat{Q}(t)} \int_t^\infty R(s) \widehat{Q}^{\frac{\gamma+1}{\gamma}}(s) \left(\frac{V(s)}{\widehat{Q}(s)} \right)^{\frac{\gamma+1}{\gamma}} ds. \end{aligned}$$

Let $\lambda = \inf_{t\geq T} \left(\frac{V(t)}{\widehat{Q}(t)} \right)$, then $\lambda \geq 1$ and $\lambda \geq 1 + \beta \lambda^{\frac{\gamma+1}{\gamma}}$ by simple calculation we get $\lambda - \beta \lambda^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^\gamma} \frac{1}{\beta^\gamma}$ a contradiction, $\beta = \frac{\gamma^\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}}$ this contradiction lead to Eq(2) is oscillatory. □

Following [1],Let $\{y_n(t)\}_{n=0}^\infty$ be a sequence of continuous functions defined as follows (if they exist):
$y_0(t) = \widehat{Q}(t)$ for all $t \geq t_0$ and

$$y_n(t) = \int_t^\infty R(s) y_{n-1}^{\frac{\gamma+1}{\gamma}}(s) ds + \widehat{Q}(t). \tag{18}$$

Lemma 2.4. *[1]. If Eq(2) is nonoscillatory, then $y_n(t) \leq V(t)$ where $V(t)$ be as defined in Theorem 2.2 and there exists a positive function $y(t)$ on $[t, \infty)$, such that $\lim_{t\to\infty} y_n(t) = y$ for $t \geq T \geq t_0$.In addition we have*

$$y(t) = \int_t^\infty R(s) y^{\frac{\gamma+1}{\gamma}}(s) ds + \widehat{Q}(t). \tag{19}$$

Proof. suppose that Eq(2) is nonoscillatory . from (Theorem 2.2) $y_0(t) \leq y_1(t)$, by induction $y_n(t) \leq y_{n+1}(t)$ for $n = 0, 1, 2, ...$
By the other hand from (3), we have $V(t) \geq \widehat{Q}(t) = y_0(t)$.
Inductively, we get $V(t) \geq y_n(t)$ for $n = 0, 1, 2, ...$
Thus, the sequence $\{y_n(t)\}_{n=0}^\infty$ converges to $y(t)$ on $[T, \infty)$ .by Lebesgue's monotone convergence theorem, and letting $n \to \infty$ in Eq (18)we get (19). □

Corollary 2.5. *Let $y_n(t)$ defined by (18).if there exists some $y_n(t)$ such that*

$$\limsup_{t\to\infty} y_n(t) \left[\int_{t_0}^{\tau_j(t)} \tau_j'(s) A[\tau_j(s)] ds \right]^\gamma > 1 \quad for \quad all \quad n = 0, 1, 2, ... \quad . \tag{20}$$

Then Eq(2) is oscillatory.

Proof. Suppose that Eq(2) is nonoscillatory, then from Theorem 2.2

$$\limsup_{t\to\infty} V(t)\left[\int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds\right]^{\gamma} \leq 1,$$

by contrary and from Lemma 2.4 we get

$$\limsup_{t\to\infty} y_n(t)\left[\int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds\right]^{\gamma} > 1.$$

This lead to Eq(2) is oscillatory. □

Corollary 2.6. *Assume that*

$$\limsup_{t\to\infty}\left[\int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds\right]^{\gamma}\left[\int_{t}^{\infty} R(s)\widehat{Q}^{\frac{\gamma+1}{\gamma}}(s)ds + \widehat{Q}(t)\right] > 1.$$

Then Eq(2) is oscillatory.

2.2 Oscillation criteria for $-\mu \leq \sum_{i=1}^{n} p_i(t) \leq 0$.

In this section we will study the criteria for

$$[a(t)[(x(t)-\sum_{i=1}^{n} p_i(t)x(\sigma_i(t)))'']^{\gamma}]' + \sum_{j=1}^{m} f_j(t, x(\tau_j(t))) = 0, \tag{21}$$

Theorem 2.7. *Assume that every solution of Eq(21) is neither osillatory nor tends to zero .then there exists a positive function $V(t)$ on $[T,\infty)$ such that*

1.
$$Q(t) < \infty \quad , \quad \int_{t}^{\infty} R(s)Q^{\frac{\gamma+1}{\gamma}}(s)ds < \infty. \tag{22}$$

2.
$$V(t) \geq Q(t) + \int_{t}^{\infty} R(s)V^{\frac{\gamma+1}{\gamma}}(s)ds \quad for \quad all \quad t \geq T \geq t_0. \tag{23}$$

3.
$$\limsup_{t\to\infty} V(t)[\int_{t_0}^{\tau_j(t)} \tau_j'(s)A[\tau_j(s)]ds]^{\gamma} \leq 1. \tag{24}$$

Proof. Let $x(t)$ be a solution which is neither oscillatory nor tends to zero such that $x(t) > 0$; $x(\sigma_i(t)) > 0$; $x(\tau_j(t)) > 0$
from (6) we have

$$\begin{aligned}[a(t)(z''(t))^{\gamma}]' &= -\sum_{j=1}^{m} f_j(t, x(\tau_j(t))) \\ &\leq -\sum_{j=1}^{m} q_j(t)x^{\gamma}(\tau_j(t)),\end{aligned}$$

we have two possible cases for $z(t)$

I $z(t) > 0$,

II $z(t) < 0$,

(I) $z(t) > 0$, then proof will be as proof of theorem 2.2 but $\widehat{Q}(t)$ replaced by $Q(t)$ (II)$z(t) < 0$ eventually for all $t \geq t_2 \geq t_1 \geq t_0$, then we have two cases for $x(t)$

(a) $x(t)$ is unbounded

(b) $x(t)$ is bounded

(a) $x(t)$ is unbounded
Assume that $x(t)$ is unbounded, then

$$x(t) = z(t) - \sum_{i=1}^{n} p_i(t)x(\sigma_i(t)) < -\sum_{i=1}^{n} p_i(t)x(\sigma_i(t)) < \sum_{i=1}^{n} x(\sigma_i(t)). \tag{25}$$

Since $x(t)$ is unbounded, then we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} T_n = \infty$ from which $\lim_{n\to\infty} x(T_n) = \infty$ and $\max_x x(t) = x(T_n)$ by choosing N large such that $\sigma_i(T_N) > T_1$ for all $T_N > t_2$. Thus $\max x(t) = x(T_N)$. this contradict with(25).

(b) $x(t)$ is bounded
Suppose that $x(t)$ is bounded, we show that $x(t) \to 0$ as $t \to \infty$
since

$$\limsup_{t\to\infty}(z(t)) \leq 0,$$

then we have

$$\limsup_{t\to\infty}(x(t)+\sum_{i=1}^{n}p_i(t)x(\sigma_i(t)))\leq 0$$
$$\limsup_{t\to\infty}x(t)+\limsup_{t\to\infty}x(t)\sum_{i=1}^{n}p_i(t)x(\sigma_i(t))\leq 0$$
$$(1-\mu)\limsup_{t\to\infty}x(t)\leq 0.$$

This shows that $x(t)\to 0$ as $t\to\infty$. □

Corollary 2.8. *Assume that*

$$\limsup_{t\to\infty}\left[\int_{t_0}^{\tau_j(t)}\tau_j'(s)A[\tau_j(s)]ds\right]^{\gamma}\left[\int_t^{\infty}R(s)Q^{\frac{\gamma+1}{\gamma}}(s)ds+Q(t)\right]>1. \tag{26}$$

Then every solution of Eq(21) is either oscillatory or tends to zero.

Corollary 2.9. *Assume that*

$$\liminf_{t\to\infty}\frac{1}{Q(t)}\int_t^{\infty}Q^{\frac{\gamma+1}{\gamma}}(s)R(s)ds>\frac{\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}}\ . \tag{27}$$

Then every solution of Eq(21) is either oscillatory or tends to zero.

Remark 2.10. *We can get oscillation criteria of the equation*

$$[a(t)(z''(t))^{\gamma}]'+f(t,x(\tau(t)))=0,$$

where $z(t)=x(t)\pm p(t)x(\sigma(t))$*, if we put* $j=i=1$.

3 Examples

Example 1. *Consider*

$$[t(z''(t))^3]'+\sum_{j=1}^{2}f_j(t,x(\tau_j(t)))=0,\qquad t\geq 0 \tag{28}$$

$z(t)=x(t)+\sum_{i=1}^{2}p_i(t)x(\sigma_i(t))$, $p_1(t)=\frac{1}{2}$, $p_2(t)=\frac{1}{4}$, $q_1(t)=\frac{2a}{t^6}$, $q_2(t)=\frac{4a}{t^6}$, $a>0$, $\tau_1(t)=\frac{t}{2}$, $\tau_2(t)=\frac{t}{3}$.
It is clear that
$a(t)=t,\quad \int_t^{\infty}a^{\frac{-1}{\gamma}}(s)ds=\int_t^{\infty}s^{\frac{-1}{3}}ds=\infty,$
$0\leq p_1(t)\ ,\ p_2(t)\leq 1\ ;\quad q_1(t),q_2(t)$ *positive.*
since

$$\widehat{Q}(t) = \int_t^\infty \sum_{j=1}^m q_j(s)[1 - \sum_{i=1}^n p_i(t)(\tau_j(s))]^\gamma ds$$

$$\begin{aligned}\widehat{Q}(t) &= \int_t^\infty [q_1(s)(1-(p_1(s)+p_2(s)))]^3 + [q_2(s)(1-(p_1(s)+p_2(s)))]^3 ds \\ &= \left(\frac{1}{4}\right)^3 \int_t^\infty \frac{6a}{s^6} ds \\ &= \left(\frac{1}{4}\right)^3 \left(\frac{6a}{5}\right)\left(\frac{1}{t^5}\right).\end{aligned}$$

$$\begin{aligned}R(t) &= \gamma\tau_j'(t)A[\tau_j(t)] \\ &= 3[\tau_1'(t)A[\tau_1(t)] + \tau_2'(t)A[\tau_2(t)]] \\ &= \left(\frac{9}{2}\right)\left[\left(\frac{1}{2}\right)^{5/3} + \left(\frac{1}{3}\right)^{5/3}\right] t^{2/3}.\end{aligned}$$

from (17)

$$\liminf_{t\to\infty} \frac{1}{\widehat{Q}(t)} \int_t^\infty \widehat{Q}^{\frac{\gamma+1}{\gamma}}(s)R(s)ds = \left(\frac{9}{10}\right)\left(\frac{1}{4}\right)\left(\frac{6a}{5}\right)^{1/3}\left[\left(\frac{1}{2}\right)^{5/3} + \left(\frac{1}{3}\right)^{5/3}\right].$$

since $\frac{\gamma}{(\gamma+1)^\gamma} = \frac{3}{(4)^{4/3}}$
Then the solution of (28)is oscillatory when $a > 71.89238009$.

Example 2. *Consider*

$$[t(z''(t))^3]' + \sum_{j=1}^2 f_j(t, x(\tau_j(t))) = 0, \qquad t \geq 0 \tag{29}$$

$z(t) = x(t) + \sum_{i=1}^2 p_i(t)x(\sigma_i(t))$, $p_1(t) = \frac{-1}{2}$, $p_2(t) = \frac{-1}{4}$, $q_1(t) = \frac{2a}{t^6}$, $q_2(t) = \frac{4a}{t^6}$, $a > 0$, $\tau_1(t) = \frac{t}{2}$, $\tau_2(t) = \frac{t}{3}$.
It is clear that
$a(t) = t$, $\int_t^\infty a^{\frac{-1}{\gamma}}(s)ds = \int_t^\infty s^{\frac{-1}{3}}ds = \infty$,
$0 \leq p_1(t)$, $p_2(t) \leq 1$; $q_1(t), q_2(t)$ *positive.*

since

$$Q(t) = \int_t^\infty \sum_{j=1}^m q_j(s)ds = \left(\frac{6a}{5}\right)\left(\frac{1}{t^5}\right)$$

$$R(t) = \left(\frac{9}{2}\right)\left[\left(\frac{1}{2}\right)^{5/3} + \left(\frac{1}{3}\right)^{5/3}\right] t^{2/3}.$$

from (27)

$$\liminf_{t\to\infty} \frac{1}{Q(t)} \int_t^\infty Q^{\frac{\gamma+1}{\gamma}}(s)R(s)ds = \left(\frac{9}{10}\right)\left(\frac{6a}{5}\right)^{1/3}\left[\left(\frac{1}{2}\right)^{5/3} + \left(\frac{1}{3}\right)^{5/3}\right].$$

since $\frac{\gamma}{(\gamma+1)^\gamma} = \frac{3}{(4)^{4/3}}$
Then the solution of (29)is oscillatory or tends to zero when $a > 1.123318444$.

Example 3. [13] *Consider*

$$((x''(t))^3)' + \frac{a}{t^7}x^3(\lambda t) = 0 \qquad ,a > 0, \quad 0 < \lambda < 1, \quad t \geq 1. \tag{30}$$

Note that $p(t) = 0$, $q(t) = \frac{a}{t^7}$, $\gamma = 3$, $\tau(t) = \lambda t$, $a(t) = 1$, $\int_t^\infty a^{\frac{-1}{\gamma}}(s)ds = \infty$ *since*

$$\begin{aligned} \widehat{Q}(t) &= \int_t^\infty \sum_{j=1}^m q_j(s)[1 - \sum_{i=1}^n p_i(t)(\tau_j(s))]^\gamma ds \\ &= \int_t^\infty q(s)[1 - p(t)(\tau(s))]^\gamma ds \\ &= \frac{a}{6t^6}. \end{aligned}$$

$$\begin{aligned} R(t) &= \gamma\tau_j'(t)A[\tau_j(t)] \\ &= \gamma\tau'(t)A[\tau(t)] \\ &= (3\lambda)(\lambda t - t_0). \end{aligned}$$

from (17)

$$\liminf_{t\to\infty} \frac{1}{\widehat{Q}(t)} \int_t^\infty \widehat{Q}^{\frac{\gamma+1}{\gamma}}(s)R(s)ds = \frac{3\lambda^2}{6}\left(\frac{a}{6}\right)^{1/3}$$

since $\frac{\gamma}{(\gamma+1)^\gamma} = \frac{3}{(4)^{4/3}}$
Then Eq(3.3.1)is oscillatory if $a\lambda^6 > \frac{3^4}{4^2}$
This result is consistent with the result in Example 168 of [13] .

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