

Finite time stability of linear fractional order dynamical system with variable delays

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Abstract

In this paper, some sufficient condition ensuring finite time stability are derived for a class of linear fractional order dynamical system with variable delay using generalized Gronwall inequality as well as classical Bellman-Gronwall inequality.

Keywords: Dynamical System; Fractional Calculus; Finite Time Stability; Time Varying Delays.

1. Introduction

Time delay systems are important class of dynamical systems. Time delays are very often encountered in different technical systems such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, control systems etc. [19], [20]. These delays are due to transportation of materials, energy or information, [54], [14]. The existence of time delays may cause undesirable system transient response and frequently the source of instability. Consequently, the question of stability of these class of systems is of theoretical and practical importance, see [2], [37], [56].

Numerous remarkable results relating to the stability of time delay systems have been published with particular emphasis on the application of Frequency domain techniques [25], Lyapunov methods, or idea of matrix measure [33], [41], small-gain-base methods [17].

In practice one is not only interested in system stability (in the sense of Lyapunov), but also in bounds of system trajectories. A system could be stable but still completely useless if it possesses undesirable transient performance. Thus, it may be useful to consider the stability of such systems with respect to certain subset of the state space which are defined a priori in a given problem. Besides that, it is of particular significance to consider the behavior of dynamical system only over a finite time interval. For this purpose, the concept of finite time stability and practical stability has been used. A system is said to be finite time stable if, given a bound on the initial condition, its state does not exceed a certain threshold within a specified time interval. To verify the finite time stability of systems, several authors have developed different techniques to investigate stability criteria, see [1], [7], [8], [23], [24], [26], [36], [37], [42], [49] and references therein.

Fractional calculus was first introduced 300 years ago and dates back to the works of Leibnitz, Liouville, Riemann, Grunwald, and Letnikov [28], [48]. Fractional calculus is a generalization of the traditional (integer order) calculus in which the order of the derivatives and integrals can be any real or complex number. In recent years, the study of fractional order derivative has attracted increasing interest due to an important role it plays not only in mathemat-

ics, but also in physics, engineering, control systems, dynamical systems and in particular in the modeling of many natural phenomena [24], [50], [28], [53]. There are two aspects that essentially differentiates fractional order models and integer order, which makes it more realistic to characterize real world physical systems by fractional order state equation. First for the integer order derivative indicates a variation of a certain attributes at a particular time for a physical or mechanical process, while fractional order is concerned with the whole-time domain. Second, integer order derivatives describe the local properties of a certain position for physical process, while fractional order derivative is related to the whole space, [3], [18].

However, in many real world physical systems, fractional calculus is more feasible than integer calculus to model the behaviors of such system. For example, fractional electrical networks [39], fractional order Schrodinger equation [43], fractional oscillator equation [46], fractional Lotka-Volterra equation [18], robotics [15]. In particular, stability analysis is one of the most fundamental and important issues for systems.

It is well known that the analysis of the stability of fractional systems is more complex than that of classical differential equation, since fractional derivatives are nonlocal and have more weakly singular kernels. Algebraic criteria for stability analysis of classical (integer order) system, such as Hurwitz and Routh criteria or Jury's, cannot be used for fractional order system, because fractional order systems do not have characteristics polynomials but pseudo polynomial with rational power multivalued function. Thus, there remain only geometric methods (Nyquist) which can be used for the stability check for bounded input bounded output.

Different techniques have been proposed in the investigation of the stability for various fractional dynamical system, such as analytical approach [4], [27], fixed point theorem [9], [10], [51], the Lyapunov method [34], [35], linear matrix inequality [47], Gronwall inequality [12], [30].

Recently there have been advances in control theory of fractional order dynamical systems for different kinds of stability. [38] considered the structural stability result of fractional differential equations with applications to control processing from both algebraic and analytical point of views. In their paper [11], they analyzed the stability of linear fractional differential system with multiple

time delays. Based on the characteristic equation defined by using Laplace transform, several stability criteria are derived.

In [31], they extend some basic results of finite and practical stability of linear, continuous fractional order invariant time delay systems with delay in the state by proposing a stability test procedure using Bellman-Gronwall theorem. [12] obtained sufficient condition for the finite time stability of a class of fractional singular time delay systems by giving the Mittag-Leffler estimate of the solution for an equivalent system. Also, finite time stability analysis of fractional order time delay systems is studied in [30, 32]. Our aim in this paper is to develop sufficient condition for the finite time stability of a class of fractional order systems with time varying delays.

2. Preliminaries

Consider the equation

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + A_1(t)x(t - \tau(t)) + B(t)u(t) \\ x(t) &= \psi_x(t), \quad \forall t \in [-\tau_M, 0] \end{aligned} \quad (2.1)$$

Where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input control vector, $A_0(t), A_1(t) \in R^{n \times n}$, $B(t) \in R^{n \times m}$ are matrices of bounded variation, $\tau(t)$ is time varying delays, $\psi_x \in C([- \tau, 0], R^n)$ is an admissible initial state and $C([- \tau_M, 0], R^n)$ is a Banach space of continuous function mapping the interval $[- \tau_M, 0]$ into R^n which converges uniformly, and the norm defined by

$$\|\varphi\| = \sup_{-\tau_{\max} \leq \theta \leq 0} |\varphi(\theta)|$$

The system behavior is defined over the time interval $I = [0, T]$, where T is a positive number.

For the time invariant sets S_δ , used as bounds of the system trajectories are assumed to be bounded, open and connected. Let S_δ be a given set of all allowable states of the system for $\forall t \in I$. Let S_ε be the set of all initial states of the system such that $S_\delta \subseteq S_\varepsilon$ and S_γ denote the set of all allowable control actions. The sets S_δ and S_ε are connected and a priori known. Generally, the set

$$S_\rho = \{x(t) : \|x(t)\|_\rho < \rho\}, \quad \rho \in [\varepsilon, \delta]$$

The time varying delay is assumed to satisfy the following condition

$$0 \leq \tau(t) \leq \tau_M \quad (2.2)$$

Where τ_M represents the maximum delays. Before proceeding further, we will introduce the following definitions and theorems which will be used in the next section.

Matrix measures have been extensively studied in [13, 21] and it is used to estimate upper bounds of matrix exponential. The following theorem relates an upper bound of a matrix exponential to its matrix measures.

Theorem 2.1: For any matrix $A \in R^{n \times n}$ the estimate

$$\|\exp(A(t))\| \leq \exp(\mu(A)(t)) \quad (2.3)$$

Holds [12], [22].

Theorem 2.2: The matrix norm or Lozinskii logarithm norm of a $n \times n$ matrix A is

$$\mu(A) = \lim_{h \rightarrow 0} \frac{\|I + hA\| - 1}{h} \quad (2.4)$$

Where $\|\cdot\|$ is any matrix norm compatible with some vector norm $\|x\|$. The matrix measure defined in theorem 2.2 has three variants depending on the norm utilized in the definition.

It is assumed that the usual smoothness condition is satisfied by system (2.1) so that there will be no difficulty with the question of existence, uniqueness and continuity of solutions with respect to initial data.

Before stating our results, we introduce the concept of finite-time stability for time-delay system (2.1). This concept can be formalized through the following definition.

Definition 2.1: Time delayed control system is finite time stable with respect to $\{S_\varepsilon, S_\delta, T, \|\cdot\|, \mu(A_0) \neq 0\}$, $\delta < \varepsilon$ if and only if;

$$\psi_x(t) \in S_\delta, \quad \forall t \in [-\tau, 0]$$

And

$$u(t) \in S_\gamma, \quad \forall t \in T$$

Implies

$$x(t; t_0, x_0) \in S_\varepsilon, \quad \forall t \in [0, T]$$

See [6], [31].

Definition 2.2: For any real matrix $A = (a_{ij})_{n \times n}$, its matrix measure is defined as

$$\mu_\rho(A) = \lim_{\varepsilon \rightarrow 0^+} \frac{\|I - \varepsilon A\| - 1}{\varepsilon} \quad (2.5)$$

Where $\|\cdot\|_\rho$ denotes the matrix norm in $R^{n \times n}$, I is the identity matrix and $\rho = [1, 2, \infty]$ norm. the matrix norms are defined as follows

$$\|A\|_1 = \max_j \left\{ \sum_{i=1}^n |a_{ij}| \right\}, \quad \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

And

$$\|A\|_\infty = \max_i \left\{ \sum_{j=1}^n |a_{ij}| \right\}$$

Lemma 2.1: For the definition of matrix measure, for any $A, B \in R^{n \times n}$, $\rho = 1, 2, \infty$, we have

- 1) $-\|A\|_\rho \leq \mu_\rho(A) \leq \|A\|_\rho$
- 2) $\mu_\rho(\alpha A) = \alpha \mu_\rho(A), \quad \forall \alpha > 0$
- 3) $\mu_\rho(A + B) \leq \mu_\rho(A) + \mu_\rho(B)$

Definition 2.3: System given by eq. (2.1) satisfying the initial condition $x(t) = \psi_x(t)$, $-\tau_M \leq t \leq 0$ is finite time stable w.r.t.

$\{\delta, \varepsilon, \gamma, t_0, J\}$, $\delta < \varepsilon$ if and only if $\|\psi_x\|_c < \delta$ and $\|u(t)\| < \gamma, \quad \forall t \in J$ imply $\|x(t)\| < \varepsilon, \quad \forall t \in J$.

3. Fundamentals of fractional calculus

At first, the differential and integral operators can be generalized into one fundamental D_t^α operator which is known as fractional calculus [5], [28], [44], [45].

$${}_a D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0 \\ 1 & \Re(\alpha) = 0 \\ \int_a^t (t-\tau)^{-\alpha} d\tau & \Re(\alpha) < 0 \end{cases}$$

There are many ways to define fractional derivatives and integrals. The definition generally used in recent studies are, Grunwald-Letnikov, Riemann-Liouville and Caputo definitions.

Definition 3.1: The Grunwald-Letnikov (GL) fractional derivative of order $\alpha, \alpha > 0$ and fractional integral of order $\alpha, \alpha > 0$ of a continuous function $f(t)$ defined on the interval $[a, b]$ are defined by

$${}_a^{GL} D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor (t-a)/h \rfloor} (-1)^k \binom{\alpha}{k} f(t - kh), \quad (3.1)$$

$$\binom{\alpha}{k} = \frac{\Gamma(-\alpha) \Gamma(k+1)}{\Gamma(-\alpha) \Gamma(k+1)}$$

And the fractional integral as

$${}_a^{GL} D_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \quad (3.2)$$

Where a and t are limits of the operator, $\lfloor (\cdot) \rfloor$ denotes the integer part of $(\cdot)$ and $\Gamma(\cdot)$ is the Euler's gamma function that generalizes factorial for non-integer arguments:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Gamma(z+1) = \Gamma(z), \quad z = x + iy$$

One basic property of the gamma function is that it satisfies the functional equation

$$\Gamma(z) = z \Gamma(z), \Rightarrow \Gamma(n+1) = n(n+1)! = n!$$

Definition 3.2: Let $[a, b]$ be a finite interval, $-\infty < a < b < \infty$, $[a, b] \subset \mathbb{R}$ and $f(t)$ be a continuous function defined on $[a, b]$, the Riemann-Liouville fractional derivative of order α is given by

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds \quad (3.3)$$

For $(n-1 < \alpha < n)$ and $\Gamma(\cdot)$ is Euler's gamma function.

Closely related to Riemann-Liouville Fractional order derivative is the fractional integral defined by

$${}_a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad \alpha < 0 \quad (3.4)$$

In [3] it has been proposed that the integer order (classical) derivative of function x , as are commonly used in initial value problem with integer order equations be incorporated.

Definition 3.3 [5]: The Caputo fractional derivative of order $\alpha < 0$ of a function $f: (0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}_0^C D_t^\alpha [f(t)] = \frac{d^\alpha f}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \quad (3.5)$$

$$n-1 < \alpha < n, \quad f^{(n)}(s) = \frac{d^n f}{ds^n}$$

Some properties of Riemann-Liouville and Caputo derivatives are recalled here, [40], [55].

Property 3.1

When $0 < \alpha < 1$, we have

$${}_0^C D_t^\alpha x(t) = {}_0 D_t^\alpha x(t) - \frac{x(t_0)}{\Gamma(1-\alpha)} (t-t_0)^{-\alpha}$$

In particular, if ${}_0^C D_t^\alpha x(t) = {}_0 D_t^\alpha x(t)$

Property 3.2

For $\nu > 1$, we have

$${}_0 D_t^\nu (t-t_0)^\nu = \frac{\Gamma(1+\nu)}{(1+\nu-\alpha)} (t-t_0)^{\nu-\alpha}$$

In particular, if $0 < \alpha < 1$ and $x(t) = (t-t_0)^\nu$ then, from property 3.1, we have

$${}_0^C D_t^\alpha (t-t_0)^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} (t-t_0)^{\nu-\alpha}$$

Property 3.3

From the definition of Caputo derivation eq. (3.5) when $0 < \alpha \leq 1$ we have

$$I_0^\alpha {}_0^C D_t^\alpha x(t) = x(t) - x(t_0)$$

Where

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > t_0$$

Property 3.4

Fractional order differentiation is a linear operator:

$$D^\alpha (\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t)$$

Also, the chain rule has the form

$$\frac{d^\beta f(g(t))}{dt^\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k} \left(\frac{d^{\beta-k}}{dt^{\beta-k}} I \right) \frac{d^k}{dt^k} f(g(t))$$

Where $k \in \mathbb{N}$ and $\binom{\beta}{k}_r$ are the coefficients of the generalized binomial

$$\binom{\beta}{k}_r = \frac{\Gamma(1+\beta)}{\Gamma(1+k) \Gamma(1-k+\beta)}$$

There are also two functions that play an important role in the study of stability of FDE's

Definition 3.4: The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$$

Where $\operatorname{Re}(z) > 0$, $z \in \mathbb{C}$. the two parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$$

Where $\operatorname{Re}(z) > 0$, and $\beta \in \mathbb{C}$, $z \in \mathbb{C}$

Definition 3.5: The α -exponential function is defined by

$$e^{\lambda z} = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^{\alpha})$$

Where $z \in \mathbb{C} \setminus 0$, $\operatorname{Re}(\alpha) > 0$ and $\lambda \in \mathbb{C}$. $E_{\alpha,\alpha}(\cdot)$ is the two parameter Mittag-Leffler function. Mittag-Leffler function is frequently used in the solution of fractional order system and is a generalization of the exponential function.

4. Main result

Here, a class of linear dynamical system with time varying delays in the state of the form

$$\begin{aligned} D_t^{\alpha} &= \frac{d^{\alpha}}{dt^{\alpha}} \\ &= A_0 x(t) + A_1 x(t - \tau(t)) + Bu(t) \end{aligned} \quad (4.1)$$

With initial condition $x(t) = \psi_x(t)$, where the time varying delays satisfy eq. (2.2) and D_t^{α} denotes Caputo fractional derivative of order α , $0 < \alpha < 1$ is considered.

Lemma 4.1: (Bellman-Gronwall inequality [16], [52])

Suppose $x(t)$ and $a(t)$ are nonnegative and locally integrable on $0 \leq t < T$, $T < \infty$, and $g(t)$ is nonnegative continuous function defined on $0 \leq t < T$, $g(t) \leq M$, M a constant, $\alpha > 0$ with

$$x(t) = a(t) + g(t) \int_0^t (t-s)^{\alpha-1} x(s) ds$$

On this interval, then:

$$\begin{aligned} x(t) &\leq a(t) \\ &+ \int_0^t \left(\frac{(g(s)\Gamma(\alpha))}{\Gamma(\alpha)} (t-s)^{\alpha-1} \right) ds, \quad 0 \leq t < T \end{aligned}$$

Theorem 4.1: The dynamical system eq. (4.1) satisfying the initial condition $x(t) = \psi_x(t)$, $-\tau_M \leq t \leq 0$ is finite time stable w.r.t. $\{\delta, \varepsilon, \alpha_0, J\}$, $\delta < \varepsilon$ if the following condition is satisfied

$$\begin{aligned} &\left(1 + \frac{\mu_x(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} \right) e^{\frac{\alpha_0(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}} \\ &+ \frac{\gamma^* [b(t-t_0) + \tau_M^*]}{\Gamma(\alpha+1)} \leq \varepsilon / \delta \end{aligned}$$

Where $\gamma^* = \alpha_0 \alpha_0 / \delta$, $\Gamma(\cdot)$ Euler's gamma function.

Proof

In accordance with the property of fractional order $0 < \alpha < 1$, the solution can be obtained in the form of an equivalent Volterra integral equation:

$$\begin{aligned} x(t) &= x(t_0) \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \begin{pmatrix} A_0 x(s) \\ + A_1 x(s - \tau(s)) \\ + Bu(s) \end{pmatrix} ds \end{aligned} \quad (4.2)$$

To obtain an estimate of the solution we apply the norm $\|(\cdot)\|$ to eq. (4.2) and using appropriate property of the norm, the following applies:

$$\begin{aligned} \|x(t)\| &\leq \|x(t_0)\| \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\| \begin{pmatrix} A_0 x(s) \\ + A_1 x(s - \tau(s)) \\ + Bu(s) \end{pmatrix} \right\| ds \end{aligned} \quad (4.3)$$

Also, applying the norm $\|(\cdot)\|$ to eq. (4.1), it holds:

$$\begin{aligned} \left\| \frac{d^{\alpha} x(t)}{dt^{\alpha}} \right\| &\leq \|A_0\| \|x(t)\| + \|A_1\| \|x(t - \tau(t))\| + \|B\| \|u(t)\| \\ &\leq (\sigma_{\max}(A_0)) \|x(t)\| \\ &+ (\sigma_{\max}(A_1)) \|x(t - \tau(t))\| + \|B\| \|u(t)\| \end{aligned} \quad (4.4)$$

Where $\|A\|$ denotes the induced norm of matrix A, considering

$$\begin{aligned} &\|x(t - \tau(t))\| \\ &\leq \sup \{ \|x(t^*)\| : t^* \in [t - \tau_{\max}, t] \} \end{aligned} \quad (4.5)$$

Applying the inequality eq. (4.5), eq. (4.4) can be written as

$$\begin{aligned} \left\| \frac{d^{\alpha} x(t)}{dt^{\alpha}} \right\| &\leq (\sigma_{\max}(A_0)) \|x(t)\| \\ &+ (\sigma_{\max}(A_1)) \sup_{t^* \in [t - \tau_M, t]} \|x(t^*)\| + \|B\| \|u(t)\| \\ &\leq \sigma_{\Sigma} \sup_{t^* \in [t - \tau_M, t]} \|x(t^*)\| + b \|u(t)\| \end{aligned} \quad (4.6)$$

So,

$$\begin{aligned} &\left\| \frac{d^{\alpha} x(t)}{dt^{\alpha}} \right\| \\ &\leq \sigma_{\Sigma} \left(\sup_{t^* \in [t - \tau_M, t]} \|x(t^*)\| + \|\psi_x\|_{\mathbb{C}} \right) \\ &\quad + b \|u(t)\| \end{aligned} \quad (4.7)$$

Where

$$\sigma_{\Sigma} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1), \quad b = \sigma_{\max}(B)$$

Combining eq. (4.7) with eq. (4.3) yields

$$\begin{aligned} \|x(t)\| &\leq \|x(t_0)\| \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ \sigma_{\Sigma} \left(\sup_{t^* \in [t - \tau_M, t]} \|x(t^*)\| \right) \right. \\ &\quad \left. + b \|u(t)\| \right\} ds \end{aligned}$$

$$\begin{aligned}
& \leq \| \psi_x \|_c \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ \sigma_z \left(\sup_{t^* \in [t-\tau_M, t]} \|x(t^*)\| \right) \right. \\
& \quad \left. + \| \psi_x \|_c \right. \\
& \quad \left. + b \|u(t)\| \right\} ds \\
& \leq \| \psi_x \|_c \left(1 + \frac{\sigma_z (t-t_0)^\alpha}{\Gamma(\alpha+1)} \right) + \frac{\alpha_{u_0}}{\Gamma(\alpha+1)} (b(t-t_0) + \tau_M^*) \\
& + \frac{\sigma_z}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\| \sup_{t^* \in [t-\tau_M, t]} \|x(t^*)\| \right\| ds
\end{aligned} \tag{4.8}$$

Let

$$\begin{aligned}
a(t) &= \| \psi_x \|_c \left(1 + \frac{\sigma_z (t-t_0)^\alpha}{\Gamma(\alpha+1)} \right) \\
& + \frac{\alpha_u \alpha_0}{\Gamma(\alpha+1)} [b(t-t_0) + \tau_M^*] \\
g(t) &= \frac{\sigma_z}{\Gamma(\alpha)}
\end{aligned} \tag{4.9}$$

By eq. (4.8), we have

$$\begin{aligned}
& \|x(t)\| \leq a(t) \\
& + g(t) \int_{t_0}^t (t-s)^{\alpha-1} \left\| \sup_{t^* \in [t-\tau_M, t]} \|x(t^*)\| \right\| ds
\end{aligned} \tag{4.10}$$

Obviously, the right-hand side of eq. (4.10) is a nondecreasing continuous function defined on $[0, T]$, hence

$$\begin{aligned}
& \sup_{t^* \in [t-\tau_M, t]} \|x(t^*)\| \leq a(t) \\
& + g(t) \int_{t_0}^t (t-s)^{\alpha-1} \left\| \sup_{t^* \in [t-\tau_M, t]} \|x(t^*)\| \right\| ds
\end{aligned} \tag{4.11}$$

Applying the generalized Gronwall inequality, lemma 4.1, leads to

$$\begin{aligned}
& \|x(t)\| \leq \sup_{t^* \in [t-\tau_M, t]} \|x(t^*)\| \leq a(t) e^{\int_{t_0}^t (t-s)^{\alpha-1} g(s) ds} \\
& = a(t) e^{\frac{\sigma_z \alpha_0 (t-t_0)^\alpha}{\Gamma(\alpha+1)}}
\end{aligned}$$

And the relation

$$\begin{aligned}
& \|x(t)\| \leq \delta \left(1 + \frac{\sigma_z (t-t_0)^\alpha}{\Gamma(\alpha+1)} \right) e^{\frac{\sigma_z (t-t_0)^\alpha}{\Gamma(\alpha+1)}} \\
& + \frac{\alpha_u \alpha_0}{\Gamma(\alpha+1)} [b(t-t_0) + \tau_M^*]
\end{aligned}$$

Hence by the basic conditions of theorem (4.1), eq. (4.1) yields

$$\|x(t)\| < \varepsilon \quad \forall t \in J$$

This completes the proof.

5. Conclusion

In this article, stability results of a class of linear fractional order systems with time varying delays in the state was considered. Stability criteria for this class of system was derived by applying Bellman-Gronwall theorem where sufficient conditions of finite time stability are obtained.

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