The effect of numerical integration in mixed finite element approximation in the simulation of miscible displacement

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Abstract

We consider the effect of numerical integration in finite element procedures applied to a nonlinear system of two coupled partial differential equations describing the miscible displacement of one incompressible fluid by another in a reservoir Ω ⊆ R2 of unit thickness and local elevation z(x), x ∈ Ω, with the Darcy velocity of the fluid mixture given by

\[ u = \frac{k(x)}{\mu(c)} (\nabla p - \gamma_0(c) \nabla z), \]

(1)
can be described by differential system that can be put in the slightly more general form [6]

\[
\begin{aligned}
\nabla \cdot u &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x,c) \left( \frac{\partial p}{\partial x_i} - \gamma_i(x,c) \right), \quad q, x, x \in \Omega, t \in [0, T], (a) \\
\phi \frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (D \nabla c) &= (\hat{c} - c) q = g(x,t,c), x \in \Omega, t \in [0, T], (b) \\
u \cdot \vartheta &= 0, \quad x \in \partial \Omega, t \in [0, T], (c) \\
\sum_{i,j} D_{ij}(\phi, u) \frac{\partial c}{\partial x_i} \delta_{ij} &= 0, \quad x \in \partial \Omega, \quad t \in [0, T], (d) \\
c(x,0) &= c_0(x), \quad x \in \Omega. \quad (e)
\end{aligned}
\]

(2)

In the above, p is the pressure and the initial pressure, modulo an additive constant, can be determined from (2a) and (2c); c is the concentration, c0 the initial concentration such that 0 ≤ c0(x) ≤ 1, and the term \( \hat{c} \) must be specified where \( q > 0 \) and \( \hat{c} = c \) where \( q < 0 \); \( q = g(x,t) \) is the imposed external flow, positive for injection and negative for projection, \( \vartheta \) is the exterior normal to \( \partial \Omega \); for compatibility (q, 1) = ∫Ω q(x,t)dx = 0, \( t \in [0, T] \); \( a = a(c) = a(x,c) = \frac{k(x)}{\mu(c)} \).

where \( k(x) \) is the permeability of the medium, \( \mu(c) \) the viscosity of the fluid; \( \gamma_i \) is the density of the fluid and \( \gamma(x,c) = \gamma_i(c) \nabla z(x) \). The diffusion coefficient \( D = D(\phi, u) = 2 \times 2 \) matrix given by

\[ D = \phi(x) \left[ d_x I + | u | (d_1 E(u) + d_2 E^{-1}(u)) \right], \]

(3)

where \( \phi \) is the porosity of the medium and the matrix E is the projection along the direction of flow given by \( E(u) = (a_i a_i/|u|^2) \).

1. Introduction

The miscible displacement of one incompressible fluid by another in a reservoir \( \Omega \subset \mathbb{R}^2 \) of unit thickness and local elevation \( z(x), x \in \Omega \), with the Darcy velocity of the fluid mixture given by

\[ u = \frac{k(x)}{\mu(c)} (\nabla p - \gamma_0(c) \nabla z), \]

(1)
can be described by differential system that can be put in the slightly more general form [6]

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\begin{aligned}
\nabla \cdot u &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(x,c) \left( \frac{\partial p}{\partial x_i} - \gamma_i(x,c) \right), \quad q, x, x \in \Omega, t \in [0, T], (a) \\
\phi \frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (D \nabla c) &= (\hat{c} - c) q = g(x,t,c), x \in \Omega, t \in [0, T], (b) \\
u \cdot \vartheta &= 0, \quad x \in \partial \Omega, t \in [0, T], (c) \\
\sum_{i,j} D_{ij}(\phi, u) \frac{\partial c}{\partial x_i} \delta_{ij} &= 0, \quad x \in \partial \Omega, \quad t \in [0, T], (d) \\
c(x,0) &= c_0(x), \quad x \in \Omega. \quad (e)
\end{aligned}
\]

(2)

In the above, p is the pressure and the initial pressure, modulo an additive constant, can be determined from (2a) and (2c); c is the concentration, c0 the initial concentration such that 0 ≤ c0(x) ≤ 1, and the term \( \hat{c} \) must be specified where \( q > 0 \) and \( \hat{c} = c \) where \( q < 0 \); \( q = g(x,t) \) is the imposed external flow, positive for injection and negative for projection, \( \vartheta \) is the exterior normal to \( \partial \Omega \); for compatibility (q, 1) = ∫Ω q(x,t)dx = 0, \( t \in [0, T] \);\( a = a(c) = a(x,c) = \frac{k(x)}{\mu(c)} \).

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\[ D = \phi(x) \left[ d_x I + | u | (d_1 E(u) + d_2 E^{-1}(u)) \right], \]

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where \( \phi \) is the porosity of the medium and the matrix E is the projection along the direction of flow given by \( E(u) = (a_i a_i/|u|^2) \).

When numerical integration is not used the problem (2) has been studied by Jim Douglas Jr., Richard E. Ewing and Mary Fanett So-Hsiang Chou and Li Qian [4], Li Qian and Wang Daoyu [5] and others. When numerical integration is not used the problem (2) has been studied by Jim Douglas Jr., Richard E. Ewing and Mary Fanett So-Hsiang Chou and Li Qian [4], Li Qian and Wang Daoyu [5] and others. When numerical integration is not used the problem (2) has been studied by Jim Douglas Jr., Richard E. Ewing and Mary Fanett So-Hsiang Chou and Li Qian [4], Li Qian and Wang Daoyu [5] and others.
We shall consider $W^{m,1}(\Omega), H^m(\Omega) = W^{m,2}(\Omega), L^2(\Omega) = H^0(\Omega) = W^{0,2}(\Omega)$ and $L^2(\Omega) = W^{0,1}(\Omega)$ for any integer $m \geq 0$ and any number $s$ such that $1 \leq s < \infty$, as the usual Sobolev and Lebesgue spaces on $\Omega$ respectively. The associated norms are denoted as follows: $\| \cdot \|_{m,x} = \| \cdot \|_{W^{m,1}(\Omega)}$, $\| \cdot \|_m = \| \cdot \|_{H^m(\Omega)}$ or $\| \cdot \|_{H^0(\Omega)}$ as appropriate, $\| \cdot \|_{L^2(\Omega)}$ or $\| \cdot \|_{L^2(\Omega)}$ as appropriate, $\| \cdot \|_{L^2(\Omega)}$.

Let $X$ be any of $L^2$ or Sobolev spaces; for a function $f(x,t)$ defined on $\Omega \times [0,T]$ we set $\| f \|_{L^2(\Omega)} = \int \int |f(x,t)|^2 dt$, $\| f \|_{L^2(\Omega)} = \sup_{0 \leq t \leq T} |f(x,t)|$.

Set $H(div;\Omega) = \{ v, v \in L^2(\Omega)^2, \nabla \cdot v \in L^2(\Omega) \}$, provided with the norm

$$\| v \|_{H(div;\Omega)} = \left( \sum_{i=1}^2 \| v_i \|_{0,\Omega}^2 + \| \nabla \cdot v \|_{0,\Omega}^2 \right)^{\frac{1}{2}}$$

and let

$$V = \{ v, v \in H(div;\Omega), \nabla \cdot v = 0 \text{ on } \partial\Omega \}$$

and $W = L^2(\Omega)^2/\{ \phi \equiv \text{constant on } \Omega \}$.

**Assumptions (A)**

(i) The external flow is smoothly distributed, and the coefficients and domain are sufficiently regular as to allow a smooth solution of the differential problem.

(ii) The functions $a_i(x,c),(c \in [0,1])$ are bounded above and below by positive constants and there exist a uniform positive constant $M$ such that $|q(x,t)| + |V_c(x)| + |f(x,c)| \leq M$; and the matrix $D$ should be uniformly positive-definite:

$$\sum_{i,j=1}^2 D_{ij}(\phi,u)\xi_i\xi_j \geq D_0 \| \xi \|^2, \xi \in \mathbb{R}^2,$

with $D_0$ being independent of $x$ and $u$.

(iii) $g$ is Lipschitz continuous and the various bounds that are used for the coefficients and their derivatives hold only in a neighborhood of the solution of the differential problem.

Let $h = (h_c,h_p)$, where $h_c$ and $h_p$ are positive, and different in general. Let $M_h = M_h \subset H^1(\Omega)$ be a standard finite element space of index at least $l$ associated with a quasi-regular polygonalization $T_h$ of $\Omega$ and having the following approximation and inverse hypotheses:

$$\inf_{z \in M_h} \| z - z_h \| \leq h^{l+1} M_h \| z \|_{H^{l+1}(\Omega)}, (a)$$

$$\| z_h \|_{M_h} \leq h^{-m} M_h^{1-m} \| z \|_{l+1}, 1 \leq m \leq l+1, z_h \in M_h. (b)$$

Suppose that $\Omega$ is a polygonal domain. Let $V_h \times W_h$ be one of the Raviart-Thomas spaces of index at least $k$ associated with a quasi-regular triangulation or quadrilateralization $T_h$ of $\Omega$ such that the elements have diameters much smaller than $h$.

Set $V_h = \{ v \in V_h : v \cdot \theta = 0 \text{ on } \partial\Omega \}$ and $W_h = W_h/\{ \phi \equiv \text{constant on } \Omega \}$.

The approximation properties of $V_h \times W_h$ are given by the following relations:

$$\inf_{v \in V_h} \| v - v_h \|_{L^2(\Omega)} \leq M \| v \|_{H^{l+1}(\Omega)} + \| \nabla v \|_{H^{l+1}(\Omega)} h_p^{k+1}. (a)$$

$$\inf_{w \in W_h} \| w - w_h \| \leq M \| w \|_{H^{l+1}(\Omega)} h_p^{k+1}. (b)$$

whenever the norms on the right-hand side are finite.

The weak form of (2) is defined by finding the map $\{c,u,p\} : [0,T] \rightarrow H^1 \times V \times W$ such that

$$\int_{\Omega} (\phi \cdot ) + (u \cdot V_c, z) + (D(u) V_c, V_c) = (g,c), \quad z \in H^1(\Omega), 0 < t \leq T, (a)$$

$$A(c,u,v) + B(v,p) = (f,c), \quad v \in V, \quad 0 < t \leq T, (b)$$

$$B(u,\phi) = -(q,\phi), \quad \phi \in W, \quad 0 \leq t \leq T, (c)$$

$$c(0) = c(0), D(u) = D(f,u); \quad c_t = \frac{\partial c}{\partial t}, \quad u \cdot V_c \in L^2(\Omega), (6)$$

$$A(\theta,\alpha,\beta) = \sum_{i=1}^2 \frac{1}{a_i(\theta)} \alpha_i \beta_i, \quad \beta_i \in \Omega \sum_{i=1}^2 \frac{1}{a_i(\theta)} \alpha_i \beta_i dx, \quad \alpha, \beta \in V, \quad \theta \in L^2(\Omega),$$

$$B(\alpha,\phi) = -\langle \nabla \cdot \alpha, \phi \rangle = -\int_{\Omega} \nabla \cdot \alpha \phi \text{dx}, \quad \phi \in W.$$}

Following [1], we now give a general description of the corresponding formulation of (6) when numerical integration is present. In what follows let $f$ be $c$ or $p$ as appropriate, and $s$ be $l$ or $k$ as appropriate.

Let $T_{h}^j$ be a quasi-regular polygonalisation of the set $\Omega$ with elements $(K_j,P_j,C_j)$, with diameters $\leq h_j$.

The following assumptions shall be made

(i) The family $(K_j,P_j,C_j), K_j \in T_{h}^j$ for all $h_j$ is a regular affine family with a single reference finite element $(K_0,P_0,C_0)$.

(ii) $\hat{P}_j = P_j(K_j)$, the set of polynomials of degree less than or equal to $s$.

(iii) The family of triangulations or quadrilateralizations $T_{h}^j$ satisfies an inverse hypothesis.

(iv) Each polygonalization $T_{h}^j$ is associated with a finite-dimensional subspace $M_h^0$ or $V_h$ or $W_h$ of trial functions which is contained in $H^1(\Omega) \cap C^0(\Omega)$.

We now introduce a quadrature scheme over the reference set $K_j$. A typical integral $\int_{K_j} \hat{\phi} d\hat{x}$ is approximated by

$$\sum_{I_j=1}^{L_j} \omega_{I_j} \hat{\phi}(b_{I_j}),$$

where the points $b_{I_j} \in K_j$ and the numbers $\omega_{I_j} > 0, 1 \leq I_j \leq L_j$ are respectively the nodes and the weights of the quadrature.

Let $F_{K_j} : \hat{x} \mapsto x \equiv F_{K_j}(\hat{x}) \equiv B_{K_j} \hat{x} + b_{K_j}$ be the inverse affine mapping from $K_j$ onto $K_j$ with the Jacobian of $F_{K_j}, \text{det}(B_{K_j}) > 0$.

Any two functions $\phi$ and $\hat{\phi}$ on $K_j$ and $\hat{K}_j$ are related as $\phi(\hat{x}) = \hat{\phi}(x)$ for all $x \in F_{K_j}(\hat{x}), \hat{x} \in K_j$.

The induced quadrature scheme over $K_j$ is

$$\int_{K_j} \phi(x) dx \approx \text{det}(B_{K_j}) \int_{\hat{K}_j} \hat{\phi}(\hat{x}) d\hat{x} \approx \sum_{I_j=1}^{L_j} \omega_{I_j} \phi(b_{I_j}),$$

with $\omega_{I_j} \equiv \text{det}(B_{K_j}) \omega_{I_j}$, and $b_{I_j} \equiv F_{K_j}(b_{I_j}), 1 \leq I_j \leq L_j$.

Accordingly, we introduce the quadrature error functionals

$$E_{K_j}(\phi) \equiv \int_{K_j} \phi(x) dx - \sum_{I_j=1}^{L_j} \omega_{I_j} \phi(b_{I_j}).$$
\[ \hat{E}(\hat{\phi}) \equiv \int_{\Omega} \hat{\phi}(x) dx - \sum_{i=1}^{L_\phi} \hat{\alpha}_i \hat{\phi}(\hat{b}_i), \]  
(8)

which are related by

\[ E_{K_\phi}(\phi) = \det(B_{K_\phi})\hat{E}(\hat{\phi}). \]  
(9)

The quadrature scheme is exact for the space of functions \( \hat{\phi} \), if \( \hat{E}(\hat{\phi}) = 0, \forall \hat{\phi}. \)

If the approximations for the concentration, the velocity, and the pressure are denoted by \( C(U), P \), respectively, then using these quadrature formulas, the continuous-time approximation procedure of (6) is given by finding the map \( \{C(U,P) : [0,T] \to M_b \times V_h \times W_h \} \) such that

\[ \begin{align*}
(C(0) &= c_0 \text{ small: } L^2(\Omega)-\text{or } H^1(\Omega)-\text{projection of } c_0 \text{ into } M_b \text{ or some interpolation of } c_0 \text{ into } M_b, \\
\Phi(C(\cdot),\cdot) + (U'V_C,\cdot) + (D(U)\nabla C, \nabla \cdot) = (g(C), \cdot), \quad \forall \nu \in M_b, \quad \tau \in [0,T], \\
A_h(C,U) + B_h(U,P) = (g(C),v)\nu, \quad \forall v \in V_h, \quad \tau \in [0,T], \quad \text{(b.1)}
\end{align*} \]

\[ B_h(U,\varphi) = -(\varphi,\cdot)_h, \quad \varphi \in W_h, \quad \tau \in [0,T]. \]  
(10)

where

\[ \begin{align*}
(a,\beta)_h &= \sum_{K_i \in T_h} \sum_{f_{ij} \in K_i} \alpha_i \beta_j \langle b_{ij}, f_{ij} \rangle, \\
A_h(\alpha,\beta) &= \left( \frac{1}{q(\theta)} \right) \sum_{i,j} \alpha_i \beta_j - \sum_{j} \left( \frac{1}{a_i(\theta)} \right) \alpha_i \beta_j, \\
&= \sum_{K_i \in T_h} \sum_{f_{ij} \in K_i} \alpha_i \beta_j \langle b_{ij}, f_{ij} \rangle,
\end{align*} \]

\[ B_h(\alpha, \varphi) = -(\nabla \cdot \alpha \varphi)_h = -\sum_{K_i \in T_h} \sum_{f_{ij} \in K_i} \alpha_i \beta_j \langle b_{ij}, f_{ij} \rangle. \]

The analysis of the convergence of finite element methods will make use of two useful projections. Let the map \( \{\hat{u}, \hat{b}_i\} : [0,T] \rightarrow V_h \times W_h \) be the projection of the pressure solution \( \{u, p\} \) given by

\[ \begin{align*}
A(c;\hat{u}, \hat{v}) + B(\hat{u}, \hat{v}) &= (\gamma(c), \hat{v}), \quad \forall \hat{v} \in V_h, \quad \text{(a)} \\
B(\hat{u}, \varphi) &= -(\varphi, \hat{u}), \quad \varphi \in W_h. \quad \text{(b)}
\end{align*} \]

Then, by \( \{6\} \), the map exists and (5) implies that

\[ \|u - \hat{u}\| + \|p - \hat{p}\| \leq M_{\|\cdot\|_{L^\infty(H^{s+1}(\Omega))}}\|p\|^{H_{s+1}} \]  
(12)

where \( M \) depends only on uniform bounds for \( a_i(c) \), but not on \( c \) itself.

Next, let \( \hat{c} : [0,T] \rightarrow M_b \) be the projection of \( c \) given by

\[ (D(u)\nabla (\hat{c} - c), \nabla z) + (u \cdot \nabla (\hat{c} - c), z) + (\lambda (\hat{c} - c), z) = 0, \quad z \in M_b, \]  
(13)

where \( \lambda = 1 + q^\tau. \)

Then, at any point \( r \in \Omega \), decomposing \( \nabla \xi \) into orthogonal components \( \alpha \) and \( \beta \), respectively parallel to \( u \) and orthogonal to \( u \), and using the assumption that \( d_i \geq d_i \), by \( \{6\} \),

\[ (D(u)\nabla \xi, \nabla \xi) + (u \cdot \nabla \xi, \xi) + (\lambda \xi, \xi) \geq (\Phi(d_m + d_i |u|)\nabla \xi, \nabla \xi) + (\xi, \xi), \]  
(14)

\[ \langle d_i E(u) \nabla \xi, \xi \rangle + \langle d_i E^\perp \nabla \xi, \xi \rangle > k_e d_i = \alpha^2 + d_i \beta^2 \geq d_i \| \nabla \xi \|^2, \]  
(15)

\[ \|c - \hat{c}\| + h_c\|c - \hat{c}\| \leq M\|c\|_{H^{s+1}} + h_c^{H_{s+1}}, \]  
(16)

\[ \|g\|_{L^2(L^\infty(\Omega))} + \|\nabla g\|_{L^2(L^\infty(\Omega))} + \|\hat{c}\|_{L^2(H^{s+1})} \leq M \]  
(18)

3. Lemmas

We point out that the general point of view in Ciarlet [11] for elliptic problems has provided a guide line for our development here. In what follows, let \( S_h \) denote \( M_h \) or \( V_h \) or \( W_h \) as appropriate.

**Lemma 3.1.** \[ \{6\} \] Assume that, for some integer \( s \geq 1, \)

(i) \( \Phi = P_{l_f}(K_f) \),

(ii) the union \( \bigcup_{l_f=1}^{L_f} \{b_{ij}\} \) contains a \( P_{l_f}(K_f) \)-unisolvent subset

and/or the quadrature scheme is exact for the space \( P_{l_f}(K_f) \). Then

\[ M_1\|w\|_{b} \leq \|w\| \leq M_2\|w\|_{b}, \quad w \in S_h, \]

\[ \|w_1, w_2\|_h \leq M\|w_1\|_{h} \|w_2\|_h, \quad w_1, w_2 \in S_h, \quad \text{where} \quad \|w\|_h^2 \equiv (w,w)_h \]

**Lemma 3.2.** \[ \{6\} \] Assume \( \tilde{g} \in C^0(K_f) \). Then for all \( w, v \in S_h, \)

\[ |E_{K_f}(\tilde{g}w, v)| \leq M\|\tilde{g}\|_{L^2(K_f)}\|w\|_{L^2(K_f)}\|v\|_{L^2(K_f)}, \]

where \( E_{K_f}(\cdot) \) is the quadrature error functional in (7).

**Lemma 3.3.** \[ \{6\} \] Assume that, for some integer \( s \geq 1, \Phi = P_{l_f}(K_f) \) and that \( \hat{E}(\hat{\phi}) = 0, \quad \forall \hat{\phi} \in P_{l_f}(K_f) \).

Then there exists a constant \( M \) independent of \( K_f \in T_h \) and \( h_f \) such that for any \( \tilde{g} \in W^{s+1,0}(K_f), \quad \bar{q} \in P_{l_f}(K_f), \quad \bar{q} \neq 0 \in P_{l_f}(K_f), \)

\[ \|E_{K_f}(\tilde{g}w, \bar{q})\| \leq M_{l_f}\|w\|_{W^{s+1,0}(K_f)}\|\tilde{g}\|_{H^{s}(K_f)}\|\bar{q}\|_{H^{s}(K_f)}, \]

where \( h_{f,K_f} = \text{diam}(K_f) \).

**Lemma 3.4.** \[ \{6\} \] Under the same hypotheses as in Lemma 3.3. Furthermore assume that there exists a number \( q_0 \) satisfying \( s + 1 \geq 2q_0 \).

Then there exists a constant \( M \) independent of \( K_f \in T_h \) and \( h_f \) such that for any \( \tilde{g} \in W^{s+1,q_0}(K_f) \) and any \( w \in P_{l_f}(K_f), \)

\[ |E_{K_f}(\tilde{g}w)| \leq M_{l_f}\|w\|_{W^{s+1,q_0}(K_f)}\|\tilde{g}\|_{H^{s}(K_f)}\|w\|_{H^{s}(K_f)}. \]
4. Error Estimates

Theorem. Let \{c, u, p\}, \{C, U, P\}, \{\tilde{u}, \tilde{p}, \tilde{c}\} satisfy (6), (10), (11) and (13), respectively.
Let \(f\) denote \(c\) or \(p\) as appropriate and \(s\) denote \(l\) or \(k\) as appropriate.
Assume that

(i) \(\tilde{P}_L = P_0(\tilde{K}_f)\),
(ii) the quadrature scheme \(\int_{\tilde{K}_f} \tilde{\phi}(x)dx \approx \sum_{i=1}^{L_k} \tilde{\phi}_i(\tilde{b}_i), \tilde{b}_i\) is exact

for the space \(P_2(\tilde{K}_f)\) and also exact for the space \(P_{2(l-1)}(\tilde{K}_f)\).
and the union \(\bigcup_{i=1}^{L_k} \{\tilde{b}_i\}\) contains a \(P_3(\tilde{K}_f)\)-unisolvent subset.

Then, if \(C(0)\) is determined in such a way that

\[ \|C(0) - \tilde{c}(0)\| \leq M \|c_0\| + 1 \|h^l+1, \text{then for } l \geq 1, \quad k \geq 0 \]

and \(h\) sufficiently small.

\[ \|c - C\|_{L_p} + \|u - U\|_{L_p} + \|p - P\|_{L_p} \leq M\left\| \left( 1 + \|C\|_{L^p(H^{l+1}(\Omega))} + \|c_0\|_{L^p(H^{l+1}(\Omega))} \right) h^{l+1} \right. \]

\[ + \|p\|_{L^p(H^{l+1}(\Omega))} h^{k+1} \right\} \]

Proof. With (12) and (16) known, the convergence analysis will have only to bound

\[ U - \tilde{u}, \quad P - \tilde{p}, \quad \text{and } C - \tilde{c}. \]

Let \(E(\tilde{w}_1 \tilde{w}_2) = (\tilde{w}_1, \tilde{w}_2)_h\). We first consider the estimate of \(U - \tilde{u}\) and \(P - \tilde{p}\). Manipulation of (2.3b), (2.7b) and (11) leads to

(a)

\[ A_h(C; U - \tilde{u}, V) + B_h(V, P - \tilde{p}) = A(c; \tilde{u}, V) - A(C; \tilde{u}, V) + + (\gamma(C) - \gamma(c), V) + E(\frac{1}{a(C)} \tilde{\nu}) + E(-\nu \cdot \tilde{p}) - E(\gamma(c), V), \quad \nu \in V_h \]

(b)

\[ B_h(U - \tilde{u}, \phi) = E(q \phi) + E(-\nu \cdot \tilde{u} \phi), \quad \phi \in V_h \]

Existence and uniqueness of \(U\) and \(P\) can be proved based on ideas of [13, 14]. Hence, as in [6], it follows from assumptions (A), the quasi-regularity of the grid combined with the bound (12) and Lemmas 3.1, 3.3 and 3.4 that

\[ \|U - \tilde{u}\| + \|P - \tilde{p}\| \leq M\left\| \left( 1 + \|\tilde{u}\|_{L^p} \right) c - C \right. \]

\[ + \|\tilde{p}\|_{L^p} + \|C\|_{L^p(H^{l+1}(\Omega))} + \|\gamma\|_{L^p(H^{l+1}(\Omega))} + \|\nu \tilde{\nu}\|_{L^p} \right\} \]

\[ \leq M\left\| c - C \right\| + h^{l+1} \]

where the constant \(M\) depends only on constants in (A).

We now turn to the examination of the concentration equation.
Let \(\eta = c - \tilde{c}\), \(\xi = C - \tilde{c}\) and \(E(\tilde{w}_1 \tilde{w}_2) = (\tilde{w}_1, \tilde{w}_2)\).
Subtract (2.3a) from (2.7a), apply (13), set \(z = \xi\) and use the following relation

\[ (\phi \xi, \xi)_h = \frac{1}{2} \frac{d}{dt} (\phi \xi, \xi)_h \]

To obtain

\[ \frac{1}{2} \frac{d}{dt} (\phi \xi, \xi)_h + (U \cdot \nabla \xi, \xi)_h + (D(U) \nabla \xi, \xi)_h = (\phi \eta, \xi) - (\lambda \eta, \xi) + + - (\nu \cdot u) \Delta \xi - ((D(U) - D(u)) \nabla \xi) + E(\phi \xi \xi) + E(U \cdot \nabla \xi) + + + E(D(U) \nabla \xi) + E(g(C) \xi) \]

\[ = \sum_{i=1}^{R_i} R_i. \]

First, we shall bound the left-hand side of (19). As in [6], it follows from Lemma 3.1 and (15) that

\[ \frac{1}{2} \frac{d}{dt} (\phi \xi, \xi)_h + (D(U) \nabla \xi, \xi)_h \geq \frac{1}{2} \frac{d}{dt} (\phi \xi, \xi)_h + (\phi (d_m + d_t) \nu \xi, \xi)_h \]

\[ \geq M \left\{ 1 + \|\xi\|^2 \right\} \|\xi\|^2 + \|\xi\|^2 \]

(22)

Using the argument of [6], it follows from Lemma 3.1 that

\[ (U \cdot \nabla \xi, \xi)_h = - \frac{1}{2} \frac{d}{dt} (\xi^2, \xi)_h \]

\[ \leq M \left\{ 1 + \|\xi\|^2 \right\} \|\xi\|^2 + \|\xi\|^2 \]

(23)

Hence

\[ \frac{1}{2} \frac{d}{dt} (\phi \xi, \xi)_h + (D(U) \nabla \xi, \xi)_h + (D(U) \nabla \xi, \xi)_h \geq \frac{1}{2} \frac{d}{dt} (\phi \xi, \xi)_h + + + (\phi (d_m + d_t) \nu \xi, \xi)_h \]

\[ \leq M \left\{ 1 + \|\xi\|^2 \right\} \|\xi\|^2 + \|\xi\|^2 \]

(24)

Now, we need to bound the right-hand side of (21). By (18), we have

\[ |R_1| + |R_2| + |R_3| \leq M \left\{ \|\eta\|^2 + \|\eta\|^2 \right. \]

\[ + \|\nu \xi\|_{L^p} + \|\nabla \xi\|_{L^p} + \|\nu \xi\|_{L^p} \]

\[ \leq M \left\{ \|\eta\|^2 + \|\eta\|^2 + \|\nu \xi\|_{L^p} \right\} \]

(25)

Use [6] and (18) to see that

\[ |R_4| = \left| - ((D(U) - D(u)) \nabla \xi) \right| \leq M \|\nabla \xi\|_{L^p} + \|\nabla \xi\|_{L^p} \]

\[ \leq M \left\{ \|\xi\|^2 + \|\xi\|^2 \right\} \]

(26)

By Lemma 3.4 and (18)

\[ |R_6| = |E(\phi \eta \xi)_h| \leq M \left\{ \|\xi\|_{L^p} + \|\xi\|_{L^p} \right\} \]

\[ \leq M \left\{ \|\xi\|^2 + \|\xi\|^2 \right\} \]

(27)

Observe that \(R_7 = E(U \cdot \nabla \xi \xi) = E((U - u) \cdot \nabla \xi) + E(u \cdot \nabla \xi \xi) \)

Thus, using Lemma 3.2 and (18), we see that

\[ |R_7| \leq M \left\{ \|\nabla \xi\|_{L^p} + \|\nabla \xi\|_{L^p} \right\} \]

\[ \leq M \left\{ \|\xi\|^2 + \|\xi\|^2 \right\} \]

(28)
Similar as in estimation of $R_1$, we have

$$
R_8 = E(D(U)\nabla \nabla \xi) = E((D(U) - D(u))\nabla \nabla \xi) + E(D(u)\nabla \nabla \xi)
\leq M\left[\|\nabla \nabla \|_{L^2(\Omega)}\|U - u\|_{L^2(\Omega)}^2 + \|\nabla \xi\|_2 + h^2 \|\xi\|_2 \right]
\leq M\left[\|U - u\|_{L^2(\Omega)}^2 + h^2 \|\xi\|_2 \right] + \varepsilon\|\nabla \xi\|^2.
$$

Note that

$$
R_9 = E(g(C)\xi) = E(g(C) - g(c))\xi) + E(g(c)\xi).
$$

Thus, using Lemma 3.4 and (2.1b) to see that

$$
R_9 \leq Mh^{2} \left[\left\|\xi_{t} - h\xi_{t} - g(c)\right\|_1 + \|g(C)\xi\|_1\right]
\leq Mh^{2} \left[\left\|\xi_{t} - g(c)\right\|_1 + \|g(C)\xi\|_1\right]
\leq M\left[\|\xi\|_2 \right].
$$

Then, combine the above estimates $|R_i|$, $1 \leq i \leq 9$, and use (12), (16), (17) and (20) to obtain

$$
9 \sum_{i=1}^{9} |R_i| \leq M\left[\left(\|c\|_2 \right)^2 + \|c\|_1 \right]h^{2} \|\xi\|_2 \left[\|\xi\|_2 + \|\nabla \xi\|_2 \right] + \varepsilon\|\nabla \xi\|^2.
$$

Then, (24) and (25) imply that

$$
\frac{d}{dt} (\phi \xi, \xi) + (\phi (du + d(U)) - \xi) \nabla \xi, \nabla \xi)
\leq M\left[\left(\|\xi\|_2 \right)^2 + \left(\|c\|_2 \right)^2 + \|c\|_1 \right]h^{2} \|\xi\|_2 \left[\|\xi\|_2 + \|\nabla \xi\|_2 \right]
+ \|\xi\|^2 + \|\nabla \xi\|^2.
$$

where $M$ depends on certain lower norms of the solution of the differential problem but not on the solution of the approximation problem.

Make the induction hypothesis that

$$
\|\xi\|_{L^2(\Omega)} \leq 1;
$$

certainly, for any reasonable choice of the initial condition (27) holds for $t = 0$. Thus (27) will hold for $t \leq T_0$ for some $T_0 > 0$; we shall show for $h = (h_1, h_2, h_3)$ sufficiently small that $T_0 = \infty$ and convergence will take place asymptotically at an optimal rate.

Integrate (26) in time and assume that

$$
\|\xi(0)\| \leq M\left[c_0\right]_1 + 1h^{2}.
$$

Then, it follows from (26), (27) and Gronwall’s Lemma that

$$
\|\xi\|^2 \leq M\left[\left(\|\xi\|_2 \right)^2 + \left(\|c\|_2 \right)^2 + \|c\|_1 \right]h^{2} \|\xi\|_2 \left[\|\xi\|_2 + \|\nabla \xi\|_2 \right]
+ \|\xi\|^2 + \|\nabla \xi\|^2.
$$

thus, use (28) to obtain

$$
\|\xi\|_{L^2(\Omega)} \leq M\left[\left(\|\xi\|_2 \right)^2 + \|c\|_2 \right]h^{2} \|\nabla \xi\|_2 \left[\|\xi\|_2 + \|\nabla \xi\|_2 \right]
\left[\|\xi\|^2 + \|\nabla \xi\|^2\right].
$$

To complete the argument, note that (30) implies that the induction hypothesis (27) holds for small $h$.

Therefore use (30) with the inequalities (12), (16), (20) and the triangle inequality to obtain

$$
\|c - C\|_{L^2(\Omega)} + \|u - U\|_{L^2(\Omega)} + \|P - P\|_{L^2(\Omega)}
\leq M\left[\left(\|\xi\|_2 \right)^2 + \|c\|_2 \right]h^{2} \|\nabla \xi\|_2 \left[\|\xi\|_2 + \|\nabla \xi\|_2 \right]
\left[\|\xi\|^2 + \|\nabla \xi\|^2\right].
$$

References