International Journal of Applied Mathematical Research, 2 (2) (2013) 188-199
©Science Publishing Corporation
www.sciencepubco.com/index.php/IJAMR

On the construction of the series solution of the Adomian decomposition method

Bissanga Gabriel[1], Bassono Francis[2], Pare Youssouf[2]*, Blaise Some[2]

[1]*University Marien Ngouabi of Brazzaville*
[2]*University of Ouagadougou*
**Corresponding author E-mail:pareyoussouf@yahoo.fr*

Abstract

In this paper, the Adomian decomposition method is used to construct the solution of integral equations and the choice of the first term of the series solution in the algorithm of Adomian is different of the usual one.

Keywords: *Adomian decomposition method, integral equations.*

1 Introduction

The Adomian decomposition method (ADM) is useful to get the solutions of various kinds of problems of ODEs and PDEs [1]-[11]. Here, we use ADM, to investigate the nonlinear integral equations. We construct the series solutions, but the choice of his first term is different of usual procedure.

2 About Volterra and Fredholm integral equations

An integral equation is equation of the following form:

$$\lambda(x)u(x)=\int_G k(x,t)g(u(t))dt+f(x) \tag{1}$$

Where $g(u)$ and $f(x)$ are given functions; $K(x,t)$ the kernel of the integral equation; and $u(x)$ the unknown function in a certain domain $G\subset\mathbb{R}^n$.

Remark-1: If $\lambda(x)=0,\forall x\in G$, (1) is an equation of this kind. If $\lambda(x)\neq 0,\forall x\in G$, (1) is an equation of second kind. If $\lambda(x)=0$ in a certain subset D of G, (1) is an equation of third kind.

2.1 Classification of integral equations

A Fredholm integral equation of the first kind has the following form:

$$\begin{cases}\int_a^b k(x,t)g(u(t))dt=f(x)\\ a\le x\le b;\ a\le t\le b\end{cases} \tag{2}$$

or

$$\begin{cases} \int_a^b k(x,t,u(t))dt = f(x) \\ a \le x \le b;\ a \le t \le b \end{cases} \quad (3)$$

A Fredholm integral equation of second kind has the following form:

$$\begin{cases} u(x) = f(x) + \lambda \int_a^b k(x,t) g(u(t))dt \\ a \le x \le b, a \le t \le b \end{cases} \quad (4)$$

Or

$$\begin{cases} u(x) = f(x) + \lambda \int_a^b k(x,t,u(t))dt \\ a \le x \le b, a \le t \le b \end{cases} \quad (5)$$

A Volterra integral equation of the first kind has the following form:

$$\begin{cases} \int_a^b k(x,t) g(u(t))dt = f(x) \\ a \le x \le b, t \le x \end{cases} \quad (6)$$

or

$$\begin{cases} \int_a^b k(x,t,u(t))dt = f(x) \\ a \le x \le b,\ t \le x \end{cases} \quad (7)$$

A Volterra integral equation of the second kind has the following form:

$$\begin{cases} u(x) = f(x) + \lambda \int_a^x k(x,t) g(u(t))dt \\ a \le x \le b,\ t \le x \end{cases} \quad (8)$$

or

$$\begin{cases} u(x) = f(x) + \lambda \int_a^x k(x,t,u(t))dt \\ a \le x \le b,\ t \le x \end{cases} \quad (9)$$

Remark-2: The Volterra integral equation can be considered like a Fredholm integral equation where $k(x,t) = 0, \forall t > x$.

3 The Adomian decomposition method

Suppose that we need to solve the following equation:

$$Au = f \quad (10)$$

In a real Hilbert space H, where $A : H \to H$ is a linear or a nonlinear operator, $f \in H$ and u is the unknown function. The principle of the ADM is based on the decomposition of the nonlinear operator A in the following form:

$$A = L + R + N$$

where $L + R$ is linear, N nonlinear, L invertible with L^{-1} as inverse. Using that decomposition, equation (10) is equivalent to

$$u = \theta + L^{-1}f - L^{-1}Ru - L^{-1}Nu \tag{11}$$

Where θ verifies $L\theta = 0$. (11) is called the Adomian's fundamental equation or Adomian's canonical form. We look for the solution of (10) in a series expansion form $u = \sum_{n=0}^{+\infty} u_n$ and we consider $Nu = \sum_{n=0}^{+\infty} A_n$ where A_n are special polynomials of variables $u_0, u_1, ..., u_n$ called Adomian polynomials and defined by [1], [2], [3], [4]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{+\infty}\lambda^i u_i\right)\right]_{\lambda=0} ; n = 0,1,2,...$$

Where λ is a parameter used by "convenience" and we obtain:

$$\sum_{n=0}^{+\infty} u_n = \theta + L^{-1}f - L^{-1}R\left(\sum_{n=0}^{+\infty} u_n\right) - L^{-1}N\left(\sum_{n=0}^{+\infty} A_n\right) \tag{12}$$

We suppose that the series $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} A_n$ are convergent, and obtain by identification the Adomian algorithm:

$$\begin{cases} u_0 = \theta + L^{-1}f \\ u_1 = -L^{-1}R\left(u_0\right) - L^{-1}A_0 \\ ... \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n ; n \geq 0 \end{cases} \tag{13}$$

In practice, it is often difficult to calculate all the terms of an Adomian series, so we approach the series solution by the truncated series: $\varphi_n = \sum_{i=0}^{n} u_i$ where the choice of n depends on error requirements. It this series converges, the solution of (10) is:

$$u = \lim_{n \to +\infty}\left(\sum_{i=0}^{n} u_i\right) \tag{14}$$

3.1 Application of the ADM to the integral equations

3.1.1 The Adomian algorithm for nonlinear Fredholm integral equation of the second kind

Let's consider the following equation:

$$u(x) = f(x) + \lambda\int_a^b k(x,t)\, g\left(u(t)\right)dt \tag{15}$$

where f, g and k are given functions; and u the unknown function. Let's put

$$Au = u(x) - \lambda\int_a^b k(x,t) g\left(u(t)\right)dt \tag{16}$$

and we obtain :

$$Au = f \tag{17}$$

We suppose that g is nonlinear, we denote $Nu = g(u)$, (17) can be rewritten in the following form:

$$\begin{cases} Au = f(x) \\ \quad = Lu - \lambda\int_a^b k(x,t)N(u(t))dt \end{cases} \tag{18}$$

Where L is the operator identifying. From (18), we have:

$$Lu = f(x) + \lambda\int_a^b k(x,t)N(u(t))dt \tag{19}$$

From (19) we have:

$$\begin{cases} u = L^{-1}f(x) + L^{-1}\left(\lambda\int_a^b k(x,t)N(u(t))dt\right) \\ \quad = f(x) + \lambda\int_a^b k(x,t)N(u(t))dt \end{cases} \tag{20}$$

We suppose that the solution of (15) has the following form:

$$u = \sum_{n=0}^{+\infty} u_n \tag{21}$$

and

$$Nu = \sum_{n=0}^{+\infty} A_n \tag{22}$$

From (20), we have:

$$\sum_{n=0}^{+\infty} u_n = f(x) + \lambda\sum_{n=0}^{+\infty}\left(\int_a^b k(x,t)A_n dt\right) \tag{23}$$

and we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x) = f(x) \\ u_1(x) = \lambda\int_b^b k(x,t)A_0(t)dt \\ \quad \cdots \\ u_n(x) = \lambda\int_a^b k(x,t)A_n(t)dt \end{cases} \tag{24}$$

Remark-3: If in (24), one replaces b by x , we obtain the Adomian algorithm for nonlinear Volterra integral equation of second kind.

3.1.2 The Adomian algorithm for nonlinear Fredholm equation of first kind

Let's consider the following equation:

$$\begin{cases} \int_a^b k(x,t)\,g(u(t))\,dt = f(x) \\ a \le x \le b; a \le t \le b \end{cases}$$

Remark-4: The equation (25) can be considered like Volterra or Fredholm integral equation. Let's make the following transformation:

$$\int_a^b k(x,t)\,g(u(t))\,dt = g(u(x))\int_a^b k(x,t)\,dt + \int_a^b k(x,t)\left[g(u(t)) - g(u(x))\right]dt \quad (26)$$

We denote

$$h(x) = \int_a^b k(x,t)\,dt \quad (27)$$

$$h(x)\,g(u(x)) = f(x) + \int_a^b k(x,t)\left[g(u(x)) - g(u(t))\right]dt \quad (28)$$

We denote

$$v(t) = g(u(t)) \quad (29)$$

From (28), we have:

$$h(x)\,v(x) = f(x) + \int^b k(x,t)\left[v(x) - v(t)\right]dt \quad (30)$$

We suppose that $h(x) \neq 0$, and we obtain:

$$v(x) = \frac{f(x)}{h(x)} + \frac{1}{h(x)}\int_a^b k(x,t)\left[v(x) - v(t)\right]dt \quad (31)$$

(31) is an integral equation of second kind in relation to $v(x)$. So we can get $v(x)$ and $u(x) = g^{-1}(v(x))$.

3.1.3 The Adomian integral equation for linear integral equation of second kind

Let's consider the following equation:

$$\begin{cases} u(x) = f(x) + \lambda\int_a^b k(x,t)\,u(t)\,dt \\ a \le x \le b; a \le t \le b \end{cases} \quad (32)$$

Remark-5: The equation (32) can be considered like a Volterra or Fredholm equation. We suppose that the solution of (32) has the following form:

$$u=\sum_{n=0}^{+\infty}u_n \tag{33}$$

Taking (33) in (32), we have:

$$\sum_{n=0}^{+\infty}u_n = f(x)+\lambda\sum_{n=0}^{+\infty}\int_a^b k(x,t)u_n(t)dt \tag{34}$$

and we obtain the following Adomian algorithm:

$$\begin{cases} u_0(x)=f(x) \\ u_1(x)=\lambda\int_a^b k(x,t)u_0(t)dt \\ \ldots \\ u_{n+1}(x)=\lambda\int_a^b k(x,t)u_n(t)dt; n\geq 0 \end{cases}$$

3.1.4 The Adomian Algorithm for linear Fredholm equation of the first kind

Let's consider the following equation:

$$\begin{cases} \int_a^b k(x,t)u(t)dt \\ a\leq x\leq b;\ a\leq t\leq b \end{cases} \tag{35}$$

Remark-6: The equation (35) can be considered like a Volterra equation or Fredholm integral equation.
Let's make the following transformation:

$$\int_a^b k(x,t)u(t)dt=u(x)\int_a^b k(x,t)dt+\int_a^b k(x,t)\left[u(t)-u(x)\right]dt \tag{36}$$

We denote:

$$h(x)=\int_a^b k(x,t)dt \tag{37}$$

Taking (37) in to (36), we can rewrite (35) in the following form:

$$h(x)u(x)=f(x)+\int_a^b k(x,t)\left[u(x)-u(t)\right]dt \tag{38}$$

We suppose that $h(x)\neq 0$, and we obtain:

$$u(x)=\frac{f(x)}{h(x)}+\frac{1}{h(x)}\int_a^b k(x,t)\left[u(x)-u(t)\right]dt \tag{39}$$

(39) is an integral equation of second king in relation to $u(x)$.

4 Application's examples

Example1:

Let's consider the following nonlinear Volterra integral equation of second kind

$$u(x)=\sin x+\frac{1}{8}\sin(2x)-\frac{1}{4}x+\frac{1}{2}\int_0^x u^2(t)\,dt \tag{40}$$

Let's use the Adomian method. We suppose that the solution of (40) has the following form:

$$u(x)=\sum_{n=0}^{+\infty}u_n(x) \tag{41}$$

And

$$Nu=u^2(t)=\sum_{n=0}^{+\infty}A_n(t) \tag{42}$$

From (41) and (42) we have:

$$\sum_{n=0}^{+\infty}u_n=\sin x+\frac{1}{8}\sin(2x)-\frac{1}{4}x+\frac{1}{2}\int_a^b\sum_{n=0}^{+\infty}A_n(t)\,dt \tag{43}$$

According to the standard Adomian algorithm (24), we need to choose $u_0(x)=\sin x+\frac{1}{8}\sin(2x)-\frac{1}{4}x.$ Here we choose $u_0(x)=\sin x$, so we have the following Adomian algorithm:

$$\begin{cases} u_0(x)=\sin x \\ u_1(x)=\frac{1}{8}\sin(2x)-\frac{1}{4}x+\frac{1}{2}\int_0^x A_0(t)\,dt \\ u_n(x)=\frac{1}{2}\int_0^x A_{n-1}(t)\,dt;\forall n\geq 2 \end{cases} \tag{44}$$

We obtain:

$$\begin{cases} A_0 = u_0^2 \\ A_1 = 2u_0u_1 \\ A_2 = 2u_0u_2 + u_1^2 \\ A_3 = 2u_0u_3 + 2u_1u_2 \\ A_4 = 2u_0u_4 + 2u_1u_3 + u_2^2 \\ A_5 = 2u_0u_5 + 2u_1u_4 + 2u_2u_3 \\ A_6 = 2u_0u_6 + 2u_1u_5 + 2u_2u_4 + u_3^2 \\ A_7 = 2u_0u_7 + 2u_1u_6 + 2u_2u_5 + 2u_3u_4 \\ \ldots \end{cases} \tag{45}$$

$$\begin{cases} u_0(x) = \sin x \\ u_1(x) = \frac{1}{8}\sin(2x) - \frac{1}{4}x + \frac{1}{2}\int_0^x \sin^2(t)\,dt = 0 \\ A_n = 0, \forall n \geq 1 \Rightarrow u_n = 0, \forall n \geq 1 \end{cases} \tag{46}$$

So the solution of (40) is:

$$u(x) = \sum_{n=0}^{+\infty} u_n(x) = \sin x \tag{47}$$

Remark-7: The ADM gives us the exact solution, but we need to make a good choice of the first term of series solution.

Example2:

Let's consider the following nonlinear Fredholm integral of second kind:

$$u(x) = x^2 - \frac{1}{12} + \frac{1}{2}\int_0^1 tu^2(t)\,dt \tag{48}$$

Let's use the ADM. We suppose that:

$$u(x) = \sum_{n=0}^{+\infty} u_n(x) \tag{49}$$

and

$$Nu = u^2(t) = \sum_{n=0}^{+\infty} A_n(t) \tag{50}$$

(48) - (50) give us:

$$\sum_{n=0}^{+\infty} u_n(x) = x^2 - \frac{1}{12} + \frac{1}{2}\int_0^x t\sum_{n=0}^{+\infty} A_n(t)\,dt \tag{51}$$

According to the standard Adomian algorithm (24), we have:

$$\begin{cases} u_0(x) = x^2 - \dfrac{1}{12} \\ u_n(x) = \dfrac{1}{2}\displaystyle\int_0^1 tA_{n-1}(t)\,dt\,;\, n \geq 1 \end{cases} \tag{52}$$

Which yields

$$\begin{cases} u_0(x) = x^2 - \dfrac{1}{12} \\ u_1(x) = \dfrac{37}{576} \\ u_2(x) = \dfrac{185}{13824} \\ u_3(x) = \dfrac{5069}{1327104} \\ u_4(x) = \dfrac{39035}{31850496} \\ \ldots \end{cases}$$

So the approached solution of (48) is:

$$\begin{cases} u_{Ap}(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots \\ \qquad\quad = x^2 - \dfrac{21325}{31850496} + \ldots \\ \qquad\quad \simeq x^2 - 0.00067 + \ldots \end{cases} \tag{53}$$

Let's make another choice (a good choice) of the first term u_0 of the series solution. We use the following Adomian algorithm:

$$\begin{cases} u_0(x) = x^2 \\ u_1(x) = -\dfrac{1}{12} + \dfrac{1}{2}\displaystyle\int_0^1 tA_0(t)\,dt \\ u_n(x) = \dfrac{1}{2}\displaystyle\int_0^1 tA_{n-1}(t)\,dt\,;\, \forall n \geq 2 \end{cases} \tag{54}$$

We obtain:

$$\begin{cases} u_0(x) = x^2 \\ u_n(x) = 0; \forall n \geq 1 \end{cases} \tag{55}$$

The solution of (48) is:

$$u(x) = \sum_{n=0}^{+\infty} u_n(x) = x^2 = u_{exact}(x) \tag{56}$$

<u>Conclusion:</u> We see that $\left|u_{Ap}(x) - u_{exact}(x)\right| = 0.00067 \simeq 0.$ We remark that the standard Adomian decomposition method approaches enough well the exact solution. We remark too, the ADM gives us the exact solution, through a good choice of the first term of series solution.

Example 3;

Let's consider the following nonlinear Fredholm-Volterra integro-differential equation:

$$\begin{cases} \dfrac{du(x)}{dx} = f(x) - 2xu(x) + \int_0^x (x+t)u^3(t)dt + \int_0^1 (x-t)u(t)dt \\ u(0) = 1 \end{cases} \quad (57)$$

Where

$$f(x) = \left(-\frac{2}{3}x + \frac{1}{9}\right)e^{3x} + (2x+1)e^x + \left(\frac{4}{3} - e\right)x + \frac{8}{9} \quad (58)$$

Let's use the ADM.
From (57), we have:

$$u(x) = u(0) + \int_0^x f(s)ds - 2\int_0^x su(s)ds + \int_0^x \left(\int_0^s (s+t)u^3(t)dt\right)ds + \int_0^x \left(\int_0^1 (s-t)u(t)dt\right)ds \quad (59)$$

We suppose that the solution of equation (57) has the following form:

$$u(x) = \sum_{n=0}^{+\infty} u_n(x) \quad (60)$$

And

$$Nu = u^3(t) = \sum_{n=0}^{+\infty} A_n(t) \quad (61)$$

From (59)-(61) we have:

$$\begin{cases} \sum_{n=0}^{+\infty} u_n(x) = \left(-\frac{2}{9}x + \frac{1}{9}\right)e^{3x} + (2x-1)e^x + \left(\frac{2}{3} - \frac{1}{2}e\right)x^2 + \\ \frac{8}{9}x + \frac{17}{9} - 2\sum_{n=0}^{+\infty}\int_0^x su_n(s)ds + \sum_{n=0}^{+\infty}\int_0^x \left(\int_0^s (s+t)A_n(t)\right)ds + \\ \sum_{n=0}^{+\infty}\int_0^x \left(\int_0^1 (s-t)u_n(t)\right)ds \end{cases} \quad (62)$$

According to the standard Adomian algorithm (24), we must take:

$$u_0 = \left(-\frac{2}{9}x + \frac{1}{9}\right)e^{3x} + (2x-1)e^x + \left(\frac{2}{3} - \frac{1}{2}e\right)x^2 + \frac{8}{9}x + \frac{17}{9} \quad (63)$$

Here we make another choice of u_0 and use the following Adomian algorithm:

$$\begin{cases} u_0(x) = e^x \\ u_1(x) = \left(-\frac{2}{9}x + \frac{1}{9}\right)e^{3x} + (2x-2)e^x + \left(\frac{2}{3} - \frac{1}{2}e\right)x^2 + \frac{8}{9}x + \frac{17}{9} \\ u_n(x) = -2\int_0^x s u_{n-1}(s)ds + \int_0^x \left(\int_0^s (s+t)A_{n-1}(t)dt\right)ds + \int_0^x \left(\int_0^1 (s-t)u_{n-1}(t)dt\right)ds; \forall n \geq 2 \end{cases} \quad (64)$$

Where

$$\begin{cases} A_0 = u_0^3 \\ A_1 = 3u_0^2 u_1 \\ A_2 = 3u_0 u_1^2 + 3u_0^2 u_2 \\ A_3 = u_1^3 + 6u_0 u_1 u_2 + 3u_0^2 u_3 \\ A_4 = 3u_1^2 u_2 + 3u_0 u_2^2 + 6u_0 u_1 u_3 + 3u_0^2 u_4 \\ A_5 = 3u_1^2 u_3 + 6u_0 u_1 u_4 + 3u_1 u_2^2 + 6u_0 u_2 u_3 + 3u_0^2 u_5 \\ A_6 = 3u_1^2 u_4 + 6u_1 u_2 u_3 + 6u_0 u_1 u_5 + 6u_0 u_2 u_4 + u_2^3 + 3u_0 u_3^2 + 3u_0^2 u_6 \end{cases}$$

We obtain:

$$\begin{cases} u_1(x) = \left(-\frac{2}{9}x + \frac{1}{9}\right)e^{3x} + (2x-2)e^x + \left(\frac{2}{3} - \frac{1}{2}e\right)x^2 + \frac{8}{9}x + \frac{17}{9} - \\ 2\int_0^x s u_0(s)ds + \int_0^x \left(\int_0^s (s+t)A_0(t)dt\right)ds + \int_0^x \left(\int_0^s (s-t)u_0(t)dt\right)ds \end{cases}$$

So, we get:

$$\begin{cases} u_1(x) = \left(-\frac{2}{9}x + \frac{1}{9}\right)e^{3x} + (2x-2)e^x - 2\int_0^x s e^s ds + \left(\frac{2}{3} - \frac{1}{2}e\right)x^2 + \\ \frac{8}{9}x + \frac{17}{9} + \int_0^x \left(\int_0^s (s+t)e^{3t}dt\right)ds + \int_0^x \left(\int_0^s (s-t)e^t dt\right)ds \\ = 0 \end{cases}$$

$$A_1 = 3u_0^2 u_1 = 0$$

$$u_2(x) = -2\int_0^x s u_1(s)ds + \int_0^x \left(\int_0^s (s+t)A_1(t)dt\right)ds + \int_0^x \left(\int_0^1 (s-t)u_1(t)dt\right)ds = 0$$

and we can easily get :

$$\begin{cases} A_n(x) = 0; \forall n \geq 1 \\ u_n(x) = 0; \forall n \geq 1 \end{cases} \quad (65)$$

So the solution of (57) is:

$$u(x) = \sum_{n=0}^{+\infty} u_n(x) = e^x \quad (66)$$

5 Conclusion

In this paper we showed that the Adomian decomposition method is useful so solve the integral equations. In some cases, this method gives us the exact solution. However, the choice of the first term in the algorithm of Adomian is not standard; this choice is sometimes determinant for the convergence of the approached solution to the exact solution.

References

[1] K. ABBAOUI, Les fondements de la méthode décompositionnelle d'Adomian et application a la résolution de problèmes issus de la biologie et de la médicine. Thèse de doctorat de l'Université Paris VI. Octobre 1995.

[2] K. ABBAOUI and Y. CHERRUAULT, Convergence of Adomian method applied to differential equations. Math. Comput. Modelling (28, 5), pp 103-109, 1994.

[3] K. ABBAOUI and Y. CHERRUAULT, Convergence of Adomian applied to non linear equations. Math. Comput> Modelling (20,9), pp60-73, 1994.

[4] K. ABBAOUI and Y. CHERRUAULT, The Decomposition method applied to the Cauchy problem. Kybernetes, (28,1), pp68-74, 1999.

[5] N. NGARHASTA, B.SOME, K.ABBAOUI and Y.CHERRUAULT, New numerical study of Adomian method applied to a diffusion model. Kybernetes, Vol.31, no1,pp61-75,2002.

[6] Gabriel BISSANGA, Application of Adomian decomposition method to solving the Duffing equation. Comparison with perturbation method. Procceedings of the Fourth International Workshop on Contemporary Problems in Mathematical Physics, Cotonou Benin, Nov. 2005. World Scientific Publishing Co. Pte. Ltd. 2006.

[7] Gabriel BISSANGA. A-K NSEMI, Application of Adomian decomposition method to solving the Van Der Pol equation and comparison with the regular perturbation method. Proceedings of the Five international workshop on contemporary problems in mathematical physics, Cotonou-Benin, eds. J. Govaerts, M.N. Hounkounou (International Chair in Mathematical Physics and Applications, ICMP-UNESCO Chair, University of Abomey-Calavy. 072 BP 50 Cotonou, Republic of Benin, December 2008).

[8] Pierre BAKI-TANGOU, Gabriel BISSANGA. Application of Adomian method to solving the Duffing-Van Der Pol equation. Communications in Mathematical Analysis. Vol. 4, No.2.pp.30-40(2008).

[9] D. Lesnic, A computational algebraic investigation of the decomposition method for time-dependent problems. Applied Mathematics and Computation 119(2001) 197-206.

[10] D. Lesnic, Blow-up solutions obtained using the decomposition method. Chaos, Solitons 28(2006) 776-787.

[11] Chengri Jin and Mingzhu Liu, A new modification of Adomian decomposition method for solving a kind of evolution equation. Applied Mathematics and Computation 169 (2005) 953-962.