

On common fixed points in generalized Menger spaces

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Abstract

R. Vasuki [1] proved fixed point theorems for expansive mappings in Menger spaces. R. Gujettiya and et al [2] presented an extension of the main result of Vasuki, for four expansive mappings in Menger space. In this article, an important lemma is given to prove that the iteration sequence is Cauchy under suitable condition in Menger probabilistic G-metric space (shortly, MPGM-space). And then, used to obtain three common fixed point theorems for expansive type mappings.

Keywords: Fixed Points; Weakly Compatible; Semi-Compatible; Menger Spaces.

Introduction

The metric fixed point theory plays an important role to detection many applications in other mathematical branches such as differential equation, operation research, mathematical economics fractals, chaos. Different generalizations of the usual notion of a metric space were defined by several mathematicians such as in Dolthinov [3], Czerwik [4], Branceciri [5], Naidu [6] and Huang and Zhang in [7]. Also G-metric space is of these generalizations which introduced by Mustafa and Sims [8]. For more details and results in G- metric space about fixed points, common fixed points and coincidence points or coupled (fixed point, common fixed points and coincidence points) for mappings satisfying various contractive conditions in co G-metric spaces , ordered G-metric spaces and G-cone metric spaces, such as see [9], [10], [11], [12], [13] and [14].

On the other hand, as a generalization of metric space, K. Menger [15], introduced the concept of a probabilistic metric space (briefly, PM-space) where the notion of distance is considered to be statistical or probabilistic. The definition of PM-space corresponds to conjuncture when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. The fundamental importance of PM- theory in probabilistic functional analysis due to its extensive applications in random differential as well as random integral equations, for example, the work due to chang and et al [16]. In the field of fixed point, Sehgal [17] presented an active study about the contraction mapping in PM-spaces. Segal and Bharucha-Reid [18] studied Banach's contraction theorem in complete Menger space. See also [18], [19], [9] and [20]. In [21], Hicks observed that interesting fixed point theorems for contraction mappings on a Menger spaces endowed with a triangular t – norm. Recently, the M. Janfada, A. Janfada and Z. Mollace [22] introduced the structure of probabilistic G-metric spaces and Menger probabilistic G-metric spaces and showed some basic properties about these spaces and then proved some fixed point theorems in it. Abed and Luaibi [23] define a GPM-space and use it to show that proved some fixed point theorem and common fixed point results for convers commuting mappings and weakly compatible mappings in G- Menger space by using implicit conditions.

This paper is included some results about unique common fixed points in probabilistic G-Menger metric space in two various situations.

1. Preliminaries

We be gain with same basic definitions and facts.

Definition 1.1: [8] Let X be a nonempty set and $G: X \times X \times X \rightarrow [0, +\infty)$ be a function for all x, y, z, a in X satisfying the following conditions:

- 1) $G(x, y, z) = 0 \Leftrightarrow x = y = z$
- 2) $0 < G(x, x, y)$ with $x \neq y$
- 3) $G(x, x, y) \leq G(x, y, z)$ with $y \neq z$
- 4) $G(x, y, z) = G(p(x, z, y))$, $p(x, y, z)$ is a permutation of x, y, z
- 5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$.

Then the ordered pair (X, G) is called a generalized metric space or G- metric space.

Definition 1.2: [8] Let (X, G) be a G- metric space. The sequence $\{x_n\}$ is called.

- 1) A G-Cauchy if, $\forall \epsilon > 0$, there is $k \in \mathbb{N}$ such that for all positive integers $n, m, l \geq k$, $G(x_n, x_m, x_l) < \epsilon$.
- 2) A G-convergent to $x \in X$ if, $\forall \epsilon > 0$, there is $k \in \mathbb{N}$ such that for all $n, m \geq k$, $G(x, x_n, x_m) < \epsilon$.

Also, (X, G) is said to be complete G-metric space if every G-Cauchy sequence in X is G- convergent in G .

Definition 1.3: [24] the mapping $\Delta: [0, 1] \times [0, 1] \rightarrow [0, 1]$, is called a continuous t - norm if Δ satisfies the following conditions.

- a) $\Delta(r, 1) = r$ for all r in $[0, 1]$;
- b) $\Delta(r, s) = \Delta(s, r)$, for every $r, s \in [0, 1]$;
- c) $\Delta(a, c) \geq \Delta(b, d)$, whenever $a \geq b$ and $c \geq d$, for each a, b, c, d in $[0, 1]$;
- d) Δ is continuous;
- e) $(\Delta(r, \Delta(s, c))) = \Delta(\Delta(r, s), c)$, $(r, s, c \in [0, 1])$.

Example 1.4: [24] The following are the four basic t -norms.

- 1) The minimum: $\Delta_M(r, s) = \min\{r, s\}$.

- 2) The product: $\Delta_P(r, s) = r \cdot s$.
- 3) The Lukasiewicz: $\Delta_L(r, s) = \max\{r + s - 1, 0\}$.
- 4) The weakest :

$$\Delta_D(r, s) = \begin{cases} \min\{r, s\} & \text{if } \max\{r, s\} = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

In respect of the above mentioned t-norms, we have the following ordering:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M. \quad (1.2)$$

Here, we give another version of the definition of probabilistic G-metric space. Our idea depends on the definition in usual G-metric which is different from the definition in [22]:

Definition 1.5: A Menger Probabilistic G-Metric space (briefly, Menger PGM-space) is a triple (X, G, Δ) , where X is a non-empty set, Δ is a continuous t-norm, and G is a mapping from $X \times X \times X$ into L such that, if $G_{x,y,z}$ denotes the value of G at the triple (x, y, z) , the following conditions hold: for all x, y, z in X ,

- 1) $G_{x,y,z}(t) < 1$ for all $t > 0$ if and only if $x \neq y$;
- 2) $G_{x,y,z}(t) = 1$ for all $t > 0$ if and only if $x = y = z$;
- 3) $G_{x,x,y}(t) \geq G_{x,y,z}(t)$
- 4) $G_{x,y,z}(t) = G_{y,z,x}(t) = G_{z,x,y}(t) \dots$,
- 5) $G_{x,y,z}(t+s) \geq \Delta(G_{x,a,a}(t), G_{a,y,z}(s))$ for all $x, y, z, a \in X$ and $t, s \geq 0$,
- 6) If $G_{x,a,a}(t) = G_{a,y,z}(s) = 1$ then $G_{x,y,z}(t+s) = 1$.

Note that, when (X, G, Δ) satisfies the conditions 1,2,3,4 and 6 in definition (1.5) then it is called Probabilistic G-Metric space (briefly, PGM-space)

Definition 1.6: [25] Let (X, G, Δ) be a Menger PGM-space.

- 1) A sequence $\{x_n\}$ in X is said to be PG-convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists positive integer $M(\varepsilon, \lambda)$ such that

$$G_{x_n, x_m, x_m}(\varepsilon) > 1 - \lambda \text{ whenever } m, n \geq M(\varepsilon, \lambda).$$

- 2) A sequence $\{x_n\}$ in X is called PG-Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists positive integer $M(\varepsilon, \lambda)$ such that

$$G_{x_n, x_m, x_l}(\varepsilon) > 1 - \lambda \text{ whenever } n, m, l \geq M(\varepsilon, \lambda).$$

- 3) A Menger PM-space (X, G, Δ) is said to be complete if and only if every PG-Cauchy sequence in X is PG-convergent to a point in X .

Definition 1.7: [25] A pair of maps T and S is called weakly compatible pair if they commute at coincidence points i.e., $Tx = Sx$ implies $TSx = STx$.

Definition 1.8: Self-mappings S and T of a Menger PGM-space (X, G, Δ) are said to be semi-compatible if $G_{STx_n, Tu, Tu}(t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \rightarrow u$ for some u in X , as $n \rightarrow \infty$.

It follows that the pair (S, T) is semi-compatible and $Sy = Ty$ imply $STy = TSY$ by taking $\{x_n\} = y$ and $u = Sy = Ty$.

2. Main results

Firstly, the following lemmas are needed:

Lemma 2.1: Let (X, G, Δ_M) be a Menger PGM-space and $\{x_n\}$ be a sequence in X . If there exists a positive number, $0 < q < 1$ and $t > 0$ and $\Delta_M = \min$

Such that

$$G_{x_n, x_{n+1}, x_m}(q, t) \geq G_{x_{n-1}, x_n, x_{m-1}}(t) \quad (2.1)$$

For all $m \geq n+1, n = 1, 2, 3 \dots$

Then $\{x_n\}$ is a PG-Cauchy sequence in X .

Proof:

It follows from (2.1),

$$\begin{aligned} G_{x_n, x_{n-1}, x_m}((1-q)\varepsilon/2q) &\geq G_{x_{n-1}, x_{n-2}, x_{m-1}}((1-q)\varepsilon/2q^2) \\ &\geq \dots \geq G_{x_2, x_1, x_{m-n+2}}((1-q)\varepsilon/2q^{n-1}). \end{aligned}$$

Since $0 < q < 1$, so for $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $K = M(\varepsilon, \lambda)$ such that

$$G_{x_n, x_{n-1}, x_m}((1-q)\varepsilon/2q) > 1 - \lambda, \text{ for all } n \geq K. \quad (2.2)$$

It is sufficient to prove that for any positive integer p ,

$$G_{x_n, x_{n+p}, x_m}(\varepsilon) \geq 1 - \lambda, n \geq K. \quad (2.3)$$

For $p = 1$, (2.3) holds. Suppose that (2.1) holds for $1 < p \leq k$, then

$$\begin{aligned} G_{x_n, x_{n+k+1}, x_m}(\varepsilon) &\geq G_{x_{n-1}, x_{n+k}, x_{m-1}}(\varepsilon/2) \\ &\geq \min\{G_{x_{n-1}, x_n, x_n}((1-q)\varepsilon/2q), G_{x_n, x_{n+k}, x_{m-1}}(\varepsilon)\} \\ &= \min\{G_{x_n, x_n, x_{n-1}}((1-q)\varepsilon/2q), G_{x_n, x_{n+k}, x_{m-1}}(\varepsilon)\} \\ &\geq \min\{G_{x_n, x_{n-1}, x_m}((1-q)\varepsilon/2q), G_{x_n, x_{n+k}, x_{m-1}}(\varepsilon)\} \end{aligned}$$

[By (3) from definition (0.6.1)]

$$> \min\{1 - \lambda, 1 - \lambda\} = 1 - \lambda, n \geq K.$$

Hence (2.3) holds for $p = k + 1$. Therefore, $\{x_n\}$ is a PG-Cauchy sequence. ■

Lemma 2.3: Let (X, G, Δ) be a Menger PGM-space. If there exists $k \in (0, 1)$ such that for $x, y, z \in X$, $G_{x,y,z}(k, t) = 1$ then $x = y = z$.

From the definition of Menger PGM-space, the proof will be obviously.

The first result:

Theorem 2.4: Let $T, S: X \rightarrow X$ be a weakly compatible mappings on a Menger PGM-space (X, G, Δ_M) with

$$G_{Sx, Sy, Sz}(k, t) \leq G_{Tx, Ty, Tz}(t) \quad (2.4)$$

For all x, y, z in $X, k > 1$ and $t > 0$. Then T and S have a unique common fixed point, whenever $T(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is complete.

Proof:

Let $x_0 \in X$ be an arbitrary point in X such that $y_0 = Sx_0$. Since $T(X) \subseteq S(X)$, so we can choose a point x_1 in X such that $y_1 = Tx_0 = Sx_1$.

In general, choose x_{n+1} such that $y_{n+1} = Tx_n = Sx_{n+1}$, $n = 0, 1, 2, \dots$

Then from (2.4)

$$\begin{aligned} G_{y_{n-1}, y_n, y_{m-1}}(k, t) &= G_{Sx_{n-1}, Sx_n, Sx_{m-1}}(k, t) \\ &\leq G_{Tx_{n-1}, Tx_n, Tx_{m-1}}(t) \\ &= G_{y_n, y_{n+1}, y_m}(t). \end{aligned}$$

Then, $G_{y_n, y_{n+1}, y_m}(q, t) \geq G_{y_{n-1}, y_n, y_{m-1}}(t)$, where $q = 1/k$

By Lemma (2.1), we get $\{y_n\}$ is a PG-Cauchy and hence convergent. If we denote the limit by u , then

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} y_n = u.$$

Since $S(X)$ is complete, there exists a point $p \in X$ such that $Sp = u$.

Now from (2.4)

$$G_{Sp, Sx_n, Sx_n}(k, t) \leq G_{Tp, Tx_n, Tx_n}(t)$$

Taking limit as $n \rightarrow \infty$ we have

$$G_{Sp, u, u}(k, t) \leq G_{Tp, u, u}(t), \text{ which implies that } Tp = u. \text{ Therefore, } Sp = Tp = u.$$

Since S and T are weakly compatible, therefore, $STp = TSp$ i.e., $Su = Tu$.

Now we will prove that u is a fixed point of S and T . From (2.4)

$$G_{Su, Sx_n, Sx_n}(k, t) \leq G_{Tu, Tx_n, Tx_n}(t)$$

Taking limit as $n \rightarrow \infty$, we have

$$G_{Su, u, u}(k, t) \leq G_{Tu, u, u}(t)$$

Which implies that $Tu = u$. Hence $Tu = Su = u$.

Now, suppose that $u \neq w$ is also another common fixed point of S and T .

Then from (2.4)

$$G_{Su, Sw, Sw}(k, t) \leq G_{Tu, Tw, Tw}(t), \text{ this implies } u = w. \blacksquare$$

Proposition 2.5: If (S, T) is a semi-compatible pair of self-mappings in a Menger PGM-space (X, G, Δ_M) and T is continuous then (S, T) is compatible.

Proof:

Consider a sequence $\{x_n\}$ in X such that $\{Sx_n\} \rightarrow u$ and $\{Tx_n\} \rightarrow u$. As T is continuous we get $TSx_n \rightarrow Tu$. By semi-compatibility of (S, T) , we have $G_{STx_n, Tu, Tu}(\varepsilon) \rightarrow 1$ for all $\varepsilon > 0$, i.e. for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that $G_{STx_n, Tu, Tu}(\varepsilon/2) \rightarrow 1-\lambda$ and $G_{TSx_n, Tu, Tu}(\varepsilon/2) \rightarrow 1-\lambda$ for all $n \geq M(\varepsilon, \lambda)$.

Now $G_{STx_n, TSu, TSu}(\varepsilon) > \Delta \{G_{TSx_n, Tu, Tu}(\varepsilon/2), G_{Tu, TSx_n, TSx_n}(\varepsilon/2)\}$

$$> \Delta \{G_{TSx_n, Tu, Tu}(\varepsilon/2), G_{TSx_n, Tu, Tu}(\varepsilon/2)\}$$

$$> (1-\lambda, 1-\lambda) > 1-\lambda.$$

we get $G_{STx_n, TSu, TSu}(\varepsilon) \rightarrow 1$ for all $\varepsilon > 0$.

Hence, the pair (S, T) is compatible.

Theorem 2.6: Let (X, G, Δ_M) be a complete Menger PGM-space with a probabilistic G -metric is symmetric and the mappings A, B, S and $T: X \rightarrow X$ mappings with

$$G_{Ax, By, By}(k) \leq \min \{G_{Sx, Sx, Ax}(k/d), G_{Ty, Ty, By}(k/e), G_{Sx, Ty, Ty}(k/f)\}$$

For some $k \in (0, 1)$ and for all $x, y \in X$ with $x \neq y$ where $\beta = \min\{d, e, f\} > 1$. Then A, B, S and T have a unique common fixed point in X , if: (i) (A, S) is semi-compatible and (B, T) is weak compatible; (ii) $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$; and (iii) (i) either A or S is continuous.

Proof:

Let $x_0 \in X$, by condition (i) there exist $x_1, x_2 \in X$ such that

$$Tx_0 = Ax_1 = y_0; Sx_1 = Bx_2 = y_1.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Tx_{2n} = Ax_{2n+1} = y_{2n}; Sx_{2n+1} = Bx_{2n+2} = y_{2n+1}$$

Now,

$$\begin{aligned} G_{y_0, y_1, y_1}(k) &= G_{Ax_1, Bx_2, Bx_2}(k) \\ &\leq \min \{G_{Sx_1, Sx_1, Ax_1}(k/d), G_{Tx_2, Tx_2, Bx_2}(k/e), G_{Sx_1, Tx_2, Tx_2}(k/f)\} \\ &\leq \min \{G_{y_1, y_1, y_0}(k/d), G_{y_2, y_2, y_1}(k/e), G_{y_1, y_2, y_2}(k/f)\} \\ &\leq \min \{G_{y_0, y_1, y_1}(k/\beta), G_{y_1, y_2, y_2}(k/\beta), G_{y_1, y_2, y_2}(k/\beta)\}, \end{aligned}$$

Where $\beta = \min\{d, e, f\}$

$$\begin{aligned} &\leq \min \{G_{y_0, y_1, y_1}(k/\beta), G_{y_1, y_2, y_2}(k/\beta)\}, \\ &\leq \min \{G_{y_1, y_2, y_2}(k/\beta)\} = G_{y_1, y_2, y_2}(k/\beta) \end{aligned}$$

$$\text{i.e. } G_{y_0, y_1, y_1}(k) \leq G_{y_1, y_2, y_2}(k/\beta). \text{ Similarly, } G_{y_1, y_2, y_2}(k) \leq G_{y_2, y_3, y_3}(k/\beta)$$

Now, again

$$\begin{aligned} G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k) &= G_{Ax_{2n+1}, Bx_{2n+2}, Bx_{2n+2}}(k) \\ &\leq \min \{G_{Sx_{2n+1}, Sx_{2n+1}, Ax_{2n+1}}(k/d), G_{Tx_{2n+2}, Tx_{2n+2}, Bx_{2n+2}}(k/e), \end{aligned}$$

$$\begin{aligned} &G_{Sx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}}(k/f)\} \\ &\leq \min \{G_{y_{2n+1}, y_{2n+1}, y_{2n}}(k/d), G_{y_{2n+2}, y_{2n+2}, y_{2n+1}}(k/e), \\ &G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/f)\} \\ &\leq \min \{G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k/\beta), G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta), \\ &G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta)\}, \end{aligned}$$

Where $\beta = \min\{d, e, f\}$

$$\begin{aligned} &\leq \min \{G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k/\beta), G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta)\}, \\ &\leq \min \{G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta)\} = G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta) \end{aligned}$$

$$\text{i.e. } G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k) \leq G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta), \beta > 1$$

Now again

$$\begin{aligned} G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k) &= G_{y_{2n+2}, y_{2n+3}, y_{2n+3}}(k) \\ &= G_{Ax_{2n+3}, Bx_{2n+4}, Bx_{2n+4}}(k) \end{aligned}$$

$$\leq \min \{G_{Sx_{2n+3}, Sx_{2n+3}, Ax_{2n+3}}(k/d), G_{Tx_{2n+2}, Tx_{2n+2}, Bx_{2n+2}}(k/e),$$

$$G_{Sx_{2n+3}, Tx_{2n+2}, Tx_{2n+2}}(k/f)\}$$

$$\leq \min \{G_{y_{2n+3}, y_{2n+3}, y_{2n+2}}(k/d), G_{y_{2n+2}, y_{2n+2}, y_{2n+1}}(k/e),$$

$$G_{y_{2n+3}, y_{2n+2}, y_{2n+2}}(k/f)\}$$

$$\leq \min \{G_{y_{2n+2}, y_{2n+3}, y_{2n+3}}(k/\beta), G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta),$$

$$G_{y_{2n+2}, y_{2n+3}, y_{2n+3}}(k/\beta)\},$$

Where $\beta = \min \{d, e, f\}$

Hence, $G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k) \leq G_{y_{2n+2}, y_{2n+3}, y_{2n+3}}(k/\beta)$, as $\beta > 1$

Hence for all n , we have $G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k) \leq G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(k/\beta)$, as $\beta > 1$

Now,

$$G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k) \leq G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(v k) \text{ as } v = 1/\beta, \beta > 1$$

$$G_{y_{2n+1}, y_{2n+2}, y_{2n+2}}(v k) \geq G_{y_{2n}, y_{2n+1}, y_{2n+1}}(k)$$

By lemma (3.1), we get $\{y_n\}$ is a PG-Cauchy sequence in X . Thus there exist some point $z \in X$ to which $\{y_n\}$ converges. Now its subsequences

$$\{Ax_{2n+1}\} \rightarrow z, \{Bx_{2n+2}\} \rightarrow z, \{Sx_{2n+1}\} \rightarrow z, \{Tx_{2n+2}\} \rightarrow z$$

Case I: S is continuous.

In this case, we have

$$SAx_{2n+1} \rightarrow Sz \text{ and } S^2 x_{2n+1} \rightarrow Sz.$$

The semi-compatibility of (A, S) , gives $SAx_{2n+1} \rightarrow Sz$.

Step 1. By putting $x = Sx_{2n+1}$, $y = x_{2n+2}$ in (iv), we have

$$G_{ASx_{2n+1}, Bx_{2n+2}, Bx_{2n+2}}(k)$$

$$\leq \min \{G_{SSx_{2n+1}, SSx_{2n+1}, ASx_{2n+1}}(k/d), G_{Tx_{2n+2}, Tx_{2n+2}, Bx_{2n+2}}(k/e),$$

$$G_{SSx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}}(k/f)\}$$

$$\leq \min \{G_{SSx_{2n+1}, SSx_{2n+1}, ASx_{2n+1}}(k/\beta), G_{Tx_{2n+2}, Tx_{2n+2}, Bx_{2n+2}}(k/\beta),$$

$$G_{SSx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}}(k/\beta)\},$$

Where $\beta = \min \{d, e, f\}$

Letting $n \rightarrow \infty$, we have

$$G_{Sz, z, z}(k) \leq \min \{G_{Sz, Sz, Sz}(k/\beta), G_{z, z, z}(k/\beta), G_{Sz, z, z}(k/\beta)\}$$

Thus,

$$G_{Sz, z, z}(k) \leq \min \{G_{Sz, z, z}(k/\beta)\} \Rightarrow G_{Sz, z, z}(k) \leq G_{Sz, z, z}(k/\beta)$$

Since $\beta > 1$, therefore $1/\beta \in (0, 1)$.

Using lemma (2.3), $Sz = z$.

Step 2. By putting $x = z$, $y = x_{2n+2}$ in (iv), we have

$$G_{Az, Bx_{2n+2}, Bx_{2n+2}}(k) \leq \min \{G_{Sz, Sz, Az}(k/d), G_{Tx_{2n+2}, Tx_{2n+2}, Bx_{2n+2}}(k/e)$$

$$, G_{Sz, Tx_{2n+2}, Tx_{2n+2}}(k/f)\}$$

$$\leq \min \{G_{Sz, Sz, Az}(k/\beta), G_{Tx_{2n+2}, Tx_{2n+2}, Bx_{2n+2}}(k/\beta),$$

$$G_{Sz, Tx_{2n+2}, Tx_{2n+2}}(k/\beta)\}$$

Where $\beta = \min \{d, e, f\}$

Letting $n \rightarrow \infty$, we have

$$G_{Az, z, z}(k) \leq \min \{G_{Sz, Sz, Az}(k/\beta), G_{z, z, z}(k/\beta), G_{Sz, z, z}(k/\beta)\},$$

$$\leq \min \{G_{z, z, z}(k/\beta), G_{z, z, z}(k/\beta), G_{z, z, z}(k/\beta)\},$$

$$\leq G_{z, z, z}(k/\beta)$$

i.e. $G_{Az, z, z}(k) \leq G_{Az, z, z}(k/\beta)$, as $\beta > 1$.

Hence, $Az = z$. Thus, $Sz = z = Az$.

As $T(X) \subseteq A(X)$, then there exists $w \in X$ such that $Tw = Az$ for some $z \in X$.

Therefore, $z = Az = Sz = Tw$.

Step 3. By putting $x = x_{2n+1}$, $y = w$ in (iv), we have

$$G_{Ax_{2n+1}, Bw, Bw}(k) \leq \min \{G_{Sx_{2n+1}, Sx_{2n+1}, Ax_{2n+1}}(k/d), G_{Tw, Tw, Bw}(k/e)$$

$$, G_{Sx_{2n+1}, Tw, Tw}(k/f)\}$$

$$\leq \min \{G_{Sx_{2n+1}, Sx_{2n+1}, Ax_{2n+1}}(k/\beta), G_{Tw, Tw, Bw}(k/\beta),$$

$$G_{Sx_{2n+1}, Tw, Tw}(k/\beta)\}$$

Letting $n \rightarrow \infty$, we have

$$G_{z, Bw, Bw}(k) \leq \min \{G_{z, z, z}(k/d), G_{Tw, Tw, Bw}(k/e), G_{z, Tw, Tw}(k/f)\}$$

$$\leq \min \{G_{z, z, z}(k/\beta), G_{z, z, Bw}(k/\beta), G_{z, z, z}(k/\beta)\}$$

$$\leq G_{z, z, Bw}(k/\beta)$$

$$\leq G_{z, Bw, Bw}(k/\beta)$$

i.e. $G_{z, Bw, Bw}(k) \leq G_{z, Bw, Bw}(k/\beta)$, as $\beta > 1$.

As (B, T) is weak compatible, we have $TBw = BTw \Rightarrow Bz = Tz$.

Step 4. By putting $x = z$, $y = z$ in (iv), we have

$$G_{Az, Bz, Bz}(k) \leq \min \{G_{Sz, Sz, Az}(k/d), G_{Tz, Tz, Bz}(k/e), G_{Sz, Tz, Tz}(k/f)\}$$

$$\leq \min \{G_{Sz, Sz, Az}(k/\beta), G_{Tz, Tz, Bz}(k/\beta), G_{Sz, Tz, Tz}(k/\beta)\}$$

$$\leq \min \{G_{z, z, z}(k/\beta), G_{z, z, Bz}(k/\beta), G_{z, Bz, Bz}(k/\beta)\}$$

$$\leq \min \{G_{z, z, z}(k/\beta), G_{z, Bz, Bz}(k/\beta), G_{z, Bz, Bz}(k/\beta)\}$$

$$G_{Az, Bz, Bz}(k) = G_{z, Bz, Bz}(k) \leq G_{z, Bz, Bz}(k/\beta), \text{ Thus } z = Bz.$$

Therefore $z = Bz = Tz$. Hence, $z = Az = Sz = Bz = Tz$.

Therefore, z is a common fixed point of A, B, S and T .

Case II. A is continuous.

In this case, we have $ASx_{2n+1} \rightarrow Az$ and $A^2x_{2n+1} \rightarrow Az$. and the semi-compatibility of (A, S) gives $ASx_{2n+1} \rightarrow Sz$. By uniqueness of limit in Menger space, we get $Az = Sz$.

Step 5. By putting $x = z$, $y = x_{2n+1}$ in (IV), we have

$$G_{Az, Bx_{2n+1}, Bx_{2n+1}}(k) \leq \min \{G_{Sz, Sz, Az}(k/d),$$

$$G_{Tx_{2n+1}, Tx_{2n+1}, Bx_{2n+1}}(k/e), G_{Sz, Tx_{2n+1}, Tx_{2n+1}}(k/f)\}$$

$$\leq \min \{G_{Sz,Sz,Az}(k/\beta), G_{Tx_{2n+1},Tx_{2n+1},Bx_{2n+1}}(k/\beta), G_{Sz,Tx_{2n+1},Tx_{2n+1}}(k/\beta)\}$$

$$As \beta = \min \{d, e, f\}.$$

Letting $n \rightarrow \infty$, we have

$$G_{Az,z,z}(k) \leq \min \{G_{Sz,Sz,Az}(k/\beta), G_{z,z,z}(k/\beta), G_{Sz,z,z}(k/\beta)\}$$

$$\leq \min \{G_{Az,Az,Az}(k/\beta), G_{z,z,z}(k/\beta), G_{Az,z,z}(k/\beta)\}$$

$$\text{i.e. } G_{Az,z,z}(k) \leq G_{Az,z,z}(k/\beta), \text{ Thus, } z = Az.$$

Hence, $z = Az = Sz = Bz = Tz$, that is, z is a common fixed point of A, B, S and T .

Now, let u be another common fixed point of A, B, S and T , then $u = Au = Su = Bu = Tu$.

So, by putting $x = u$ and $y = z$ in (iv), we have

$$G_{Au,Bz,Bz}(k) \leq \min \{G_{Su,Su,Au}(k/d), G_{Tz,Tz,Bz}(k/e), G_{Su,Tz,Tz}(k/f)\}$$

$$\leq \min \{G_{Su,Su,Au}(k/\beta), G_{Tz,Tz,Bz}(k/\beta), G_{Su,Tz,Tz}(k/\beta)\}$$

$$\Rightarrow G_{u,z,z}(k) \leq \min \{G_{u,u,u}(k/\beta), G_{z,z,z}(k/\beta), G_{u,z,z}(k/\beta)\} \leq G_{u,z,z}(k/\beta)$$

$$\text{Hence, } G_{u,z,z}(k) \leq G_{u,z,z}(k/\beta), k > 0, \beta > 1. \Rightarrow u = z.$$

Therefore, z is the unique common fixed point of A, B, S and T . ■

Remark 2.7: Since every metric space is a Menger space, all results in Menger space, with some suitable modifications, can be applied to metric spaces, such as theorems 3.1, and 3.2 in [26].

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