An extension of the paper “Cone metric spaces and fixed point theorems of contractive mappings”

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Abstract

Huang and Zhang [1] established the results of fixed point theorems using contractive method in cone metric space. Later on, Rezapour and Hamlbarani [2] generalize the results [1] with non-normal cones and omitted the assumption of normality. In the present paper we improve these results.

Keywords: Complete cone metric space, common fixed point, contractive condition.

1 Introduction and Preliminaries

Very recently, Huang and Zhang [1] introduce the notion of cone metric spaces. He replaced real number system by ordered Banach space. He also gave the condition in the setting of cone metric spaces. These authors also described the convergence of sequences in the cone metric spaces and introduce the corresponding notion of completeness. The results in [1] were generalized by Sh. Rezapour and R Hambarani in [2] omitting the assumption of normality on the cone. Subsequently, many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cone.

The purpose of this paper is to prove common fixed point theorems for contractive mappings in the context of complete cone metric space omitting the condition of normality of the cone. Our results generalize and improve the results of [2].

First, we recall some standard notations and definitions in cone metric spaces with some of their properties (see [1] and [2]).

Definition 1.1. Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, non-empty and \( P \neq \{0\} \),
(ii) \( ax + by \in P \) for all \( x, y \in P \) and non-negative real numbers \( a, b \),
(iii) \( x \in P \) and \( -x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\} \).

Given a cone \( P \subset E \), we define a partial ordering \( \leq \) on \( E \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). We shall write \( x \ll y \) if \( y - x \in \text{int}P \), \text{int}P denotes the interior of \( P \). The cone \( P \) is called normal if there is a number \( K > 0 \) such that for all \( x, y \in E \),
0 \leq x \leq y \implies \| x \| \leq K \| y \|.

The least positive number $K$ satisfying the above is called the normal constant of $P[1]$. The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n \geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq x_n \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n \to \infty} \| x_n - x \| = 0$. Equivalently, the cone $P$ is called regular if and only if every decreasing sequence which is bounded from below is convergent.

It is well known that every regular cone is normal.

In the following, we always suppose $E$ is a Banach space, $P$ is a cone in $E$ with $Int P \neq \emptyset$ and $\leq$ is partial ordering with respect to $P$.

**Definition 1.2.** ([1]) Let $X$ be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

**Example 1.3.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \infty |x - y|)$, where $\infty \geq 0$ is a constant. Then $(X, d)$ is is a cone metric space [1].

**Definition 1.4.** ([1]) Let $(X, d)$ be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in $X$. Then

(i) $\{x_n\}_{n \geq 1}$ converges to $x$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) $(X, d)$ is a complete cone metric space, if every Cauchy sequence is convergent.

**Lemma 1.5** ([2]) There are no normal cones with normal constant $M < 1$, and for each $k > 1$ there are cones with normal constant $M > k$.

Finally, note that the relations $Int P + Int P \subseteq Int P$ and $\lambda Int P \subseteq Int P(\lambda > 0)$ hold.

## 2 Main Results

In this section, we shall prove some fixed point theorems for pair of contractive maps by using non normality of the cone (by omitting the assumption of normality). We improve some results of [2].

**Theorem 2.1.** Let $(X, d)$ be a complete cone metric space and the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition,

$$d(T_1 x, T_2 y) \leq kd(x, y) \quad \text{for all } x, y \in X$$

where $k \in [0, 1)$ is a constant. Then $T_1$ and $T_2$ have a unique common fixed point in $X$. For each $x \in X$, the iterative sequences $\{T_1^{2n+1} x\}_{n \geq 1}$ and $\{T_2^{2n+2} x\}_{n \geq 1}$ converge to the common fixed point.
Proof. For each $x_0 \in X$ and and $n \geq 1$, set $x_1 = T_1 x_0, x_3 = T_1 x_2 = T_3 x_0, \ldots, x_{2n+1} = T_1 x_{2n} = T_{1}^{2n+1} x_0$.

Similarly, we can have $x_2 = T_2 x_1 = T_2^2 x_0, x_4 = T_2 x_3 = T_2^4 x_0, \ldots, x_{2n+2} = T_2 x_{2n+1} = T_2^{2n+2} x_0$.

Then
\[
d(x_{2n+1}, x_{2n}) = d(T_1 x_{2n}, T_2 x_{2n-1}) \leq kd(x_{2n}, x_{2n-1}) \\
\leq k^2 d(x_{2n-1}, x_{2n-2}) \leq \cdots \leq k^{2n} d(x_1, x_0).
\]

So for $n > m$,
\[
d(x_{2n}, x_{2m}) = d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + \cdots + d(x_{2m+1}, x_{2m}) \\
\leq (k^{2n-1} + k^{2n-2} + \cdots + k^{2m}) d(x_1, x_0) \leq \frac{k^{2m}}{1-k} d(x_1, x_0)
\]

Let $0 < c$ be given. Choose $\delta > 0$ such that $c + N_\delta (0) \subseteq P$, where $N_\delta (a) = \{ y \in E : \parallel y \parallel < \delta \}$. Also choose a natural number $N_1$ such that $\frac{k^{2m}}{1-k} d(x_1, x_0) \in N_\delta (a)$, for all $m \geq N_1$. Then $\frac{k^{2m}}{1-k} d(x_1, x_0) \ll c$, for all $m \geq N_1$.

Thus
\[
d(x_{2n}, x_{2m}) \ll \frac{k^{2m}}{1-k} d(x_1, x_0) \ll c.
\]

for all $n > m$. Therefore $\{x_{2n}\}_{n \geq 1}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone metric space, there exists $x^* \in X$ such that $x_{2n} \to x^*$. Choose a natural number $N_2$ such that
\[
d(x_{2n}, x^*) \ll \frac{\varepsilon}{2} \text{ for all } n \geq N_2. \text{ Hence}
\]
\[
d(T_1 x^*, x^*) \ll d(T_1 x_{2n}, T_1 x^*) + d(T_1 x_{2n}, x^*) \\
\leq kd(x_{2n}, x^*) + d(x_{2n+1}, x^*) \\
\leq d(x_{2n}, x^*) + d(x_{2n+1}, x^*) \ll \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = c.
\]

for all $n \geq N_2$. Thus, $d(T_1 x^*, x^*) \ll \frac{\varepsilon}{2k^m}$, for all $m \geq 1$. So, $\frac{\varepsilon}{2k^m} - d(T_1 x^*, x^*) \in P$, for all $m \geq 1$. Since $\frac{\varepsilon}{2k^m} \to 0$, (as $m \to \infty$) and $P$ is closed, $-d(T_1 x^*, x^*) \in P$. But $d(T_1 x^*, x^*) \in P$. Therefore $d(T_1 x^*, x^*) = 0$, and so $T_1 x^* = x^*$.

Similarly it can be established that $T_2 x^* = x^*$. Hence $T_1 x^* = x^* = T_2 x^*$. Thus $x^*$ is the common fixed point of pair of maps $T_1$ and $T_2$.

Corollary 2.2 Let $(X, d)$ be a complete cone metric space. Suppose the mappings $T_1, T_2 : X \to X$ satisfy for some positive integer $n$,
\[
d(T_1^{2n+1} x, T_2^{2n+2} y), \ll kd(x, y),
\]

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then $T_1$ and $T_2$ have a unique common fixed point in $X$.

Theorem 2.3 Let $(X, d)$ be a complete cone metric space and the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition
\[
d(T_1 x, T_2 y), \leq k(d(T_1 x, x) + d(T_2 y, y)),
\]

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant.
Then, $T_1$ and $T_2$ have a unique common fixed point in $X$. For each $x \in X$, the iterative sequences $\{T_1^{2n+1}x\}_{n \geq 1}$ and $\{T_2^{2n+2}x\}_{n \geq 1}$ converge to the common fixed point.

**Proof.** For each $x_0 \in X$, and $n \geq 1$, set $x_1 = T_1x_0, x_3 = T_1^3x_0, -\cdots - -\cdots - -\cdots - x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_0, -\cdots -\cdots -\cdots -\cdots -\cdots -$. 

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_0, x_4 = T_2x_3 = T_2^4x_0, -\cdots - -\cdots - -\cdots - x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_0, -\cdots -\cdots -\cdots -\cdots -\cdots -$.

Then,
\[
d(x_{2n+1}, x_{2n}) = d(T_1x_{2n}, T_2x_{2n-1})
\leq k(d(T_1x_{2n}, x_{2n}) + d(T_2x_{2n-1}, x_{2n-1}))
= k(d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1}))
\]
So,
\[
d(x_{2n+1}, x_{2n}) \leq \frac{k}{1-k}d(x_{2n}, x_{2n-1}) = hd(x_{2n}, x_{2n-1})
\]
Where $h = \frac{k}{1-k}$. For $n > m$,
\[
d(x_{2n}, x_{2n+m}) \leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + \cdots + d(x_{2n+m}, x_{2n}) + \cdots + d(x_{2n+m}, x_{2n}) \leq (h^{2n-1} + h^{2n-2} + \cdots + h^m)d(x_{2n}, x_0) \leq \frac{h^m}{1-h}d(x_{2n}, x_0)
\]
Let $0 < c$ be given. Choose a natural number $N_1$ such that $\frac{h^m}{1-h}d(x_{2n}, x_0) \leq c$, for all $m \geq N_1$. Thus $d(x_{2n}, x_{2n}) \leq c$,

for $n > m$. Therefore $\{x_{2n}\}_{n \geq 1}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone metric space, there exists $x^* \in X$ such that $x_{2n} \to x^*$. Choose a natural number $N_2$ such that
\[
d(x_{2n+1}, x_{2n}) \leq \frac{c(1-k)}{2k} \quad \text{and}
\]
\[
d(x_{2n+1}, x^*) \leq \frac{c(1-k)}{2}, \quad \text{for all } n \geq N_2. \quad \text{Hence}
\]
for $n \geq N_2$ we have
\[
d(T_1x^*, x^*) \leq d(T_1x_{2n}, T_1x^*) + d(T_1x_{2n}, x^*)
\leq k(d(T_1x_{2n}, x_{2n}) + d(T_1x^*, x^*)
\quad + d(x_{2n+1}, x^*)
\]
Thus,
\[
d(T_1x^*, x^*) \leq \frac{1}{1-k}(kd(x_{2n+1}, x_{2n}) + d(x_{2n+1}, x^*))
\leq \frac{c}{2} + \frac{c}{2} = c
\]
Thus, $d(T_1x^*, x^*) \leq \frac{c}{2m}$ for all $m \geq 1$. So $\frac{c}{2m} - d(T_1x^*, x^*) \in P$, for all $m \geq 1$. Since $\frac{c}{2m} \to 0$ (as $m \to \infty$) and $P$ is closed, $-d(T_1x^*, x^*) \in P$. But $d(T_1x^*, x^*) \in P$. Therefore, $d(T_1x^*, x^*) = 0$, and so $T_1x^* = x^*$. Now, if $y^*$ is another fixed point of $T_1$ then $d(x^*, y^*) = d(T_1x^*, T_1y^*) \leq k(d(T_1x^*, x^*) + d(T_1y^*, y^*)) = 0$. 
Hence \( x^* = y^* \). Therefore, the fixed point of \( T_1 \) is unique. Similarly, it can be established that \( T_2 x^* = x^* \). Hence \( T_1 x^* = x^* = T_2 x^* \). Thus \( x^* \) is the common fixed point of pair of maps \( T_1 \) and \( T_2 \).

**Theorem 2.4** Let \( (X, d) \) be a complete cone metric space and the mappings \( T_1, T_2 : X \to X \) satisfy the contractive condition

\[
d(T_1 x, T_2 y), \leq k(d(T_1 x, y) + d(x, T_2 y)),
\]

for all \( x, y \in X \), where \( k \in [0, \frac{1}{2}) \) is a constant.

Then, \( T_1 \) and \( T_2 \) have a unique common fixed point in \( X \). For each \( x \in X \), the iterative sequences \( \{T_1^{2n+1} x\}_{n \geq 1} \) and \( \{T_2^{2n+2} x\}_{n \geq 1} \) converge to the common fixed point.

**Proof.** For each \( x_o \in X \), and \( n \geq 1 \), set \( x_1 = T_1 x_o, x_3 = T_1 x_2 = T_1^3 x_o, \ldots, x_{2n+1} = T_1 x_{2n} = T_1^{2n+1} x_o, \ldots \)

Similarly, we can have \( x_2 = T_2 x_1 = T_2^2 x_o, x_4 = T_2 x_3 = T_2^4 x_o, \ldots, x_{2n+2} = T_2 x_{2n+1} = T_2^{2n+2} x_o, \ldots \)

Then,

\[
d(x_{2n+1}, x_{2n}) = d(T_1 x_{2n}, T_2 x_{2n-1}) \leq k(d(T_1 x_{2n}, x_{2n-1}) + d(T_2 x_{2n-1}, x_{2n})) \leq k(d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1}))
\]

So,

\[
d(x_{2n+1}, x_{2n}) \leq \frac{k}{1-k} d(x_{2n}, x_{2n-1}) = hd(x_{2n}, x_{2n-1})
\]

Where \( h = \frac{k}{1-k} \). For \( n > m \),

\[
d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + \ldots + d(x_{2m+1}, x_{2m}) \leq (h^{2n-1} + h^{2n-2} + \ldots + h^{2m}) d(x_1, x_0) \leq \frac{k^{2m}}{1-k} d(x_1, x_0)
\]

Let \( 0 < c \) be given. Choose a natural number \( N_1 \) such that \( \frac{k^{2m}}{1-k} d(x_1, x_0) \ll c \), for all \( m \geq N_1 \). Thus \( d(x_{2n}, x_{2m}) \ll c \), for \( n > m \). Therefore \( \{x_{2n}\}_{n \geq 1} \) is a Cauchy sequence in \( (X, d) \). Since \( (X, d) \) is a complete cone metric space, there exists \( x^* \in X \) such that \( x_{2n} \to x^* \). Choose a natural number \( N_2 \) such that

\[
d(x_{2n}, x^*) \ll \frac{c(1-h)}{3} \text{ for all } n \geq N_2
\]

Hence for \( n \geq N_2 \) we have

\[
d(T_1 x^*, x^*) \leq d(T_1 x_{2n}, T_1 x^*) + d(T_1 x_{2n}, x^*) \leq k(d(T_1 x^*, x_{2n}) + d(T_1 x_{2n}, x^*)) + d(x_{2n+1}, x^*) \leq k(d(T_1 x^* x^*) + d(x_{2n}, x^*) + d(x_{2n+1}, x^*))
\]
Thus,
\[ d(T_1 x^*, x^*) \leq \frac{1}{1-k} (kd(x_{2n}, x^*) + d(x_{2n+1}, x^*)) \]
\[ + d(x_{2n+1}, x^*) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \]

Thus, \( d(T_1 x^*, x^*) \ll \frac{c}{3m}, \) for all \( m \geq 1. \) So \( \frac{c}{3m} - d(T_1 x^*, x^*) \in P, \) for all \( m \geq 1. \) Since \( \frac{c}{3m} \to 0 \) \((as \ m \to \infty)\) and \( P \) is closed, \(-d(T_1 x^*, x^*) \in P. \) But \( d(T_1 x^*, x^*) \in P. \) Therefore \( d(T_1 x^*, x^*) = 0 \) and so \( T_1 x^* = x^*. \) Now, if \( y^* \) is another fixed point of \( T_1, \) then

\[ d(x^*, y^*) = d(T_1 x^*, T_1 y^*) \leq k(d(T_1 x^*, y^*) + d(T_1 y^*, x^*)) = 2kd(x^*, y^*) \]

Hence \( d(x^*, y^*) = 0 \) and so \( x^* = y^*. \) Therefore, the fixed point of \( T_1 \) is unique. Similarly, it can be established that \( T_2 x^* = x^*. \) Hence \( T_1 x^* = x^* = T_2 x^*. \) Thus \( x^* \) is the common fixed point of pair of maps \( T_1 \) and \( T_2. \)

**Theorem 2.5** Let \((X, d)\) be a complete cone metric space and the mappings \( T_1, T_2 : X \to X\) satisfy the contractive condition

\[ d(T_1 x, T_2 y), \leq k d(x, y) + ld(x, T_2 y), \]

for all \( x, y \in X, \) where \( k, l \in [0, 1) \) is a constant.

Then, \( T_1 \) and \( T_2 \) have a unique common fixed point in \( X. \) whenever \( k + l < 1. \)

**Proof.** For each \( x_o \in X, \) and \( n \geq 1, \) set

\[ x_1 = T_1 x_o, x_3 = T_1 x_2 = T_1^3 x_o, \ldots \]

\[ x_{2n+1} = T_1 x_{2n} = T_1^{2n+1} x_o. \]

Similarly, we can have

\[ x_2 = T_2 x_1, x_4 = T_2 x_3 = T_2^3 x_o, \ldots \]

\[ x_{2n+2} = T_2 x_{2n+1} = T_2^{2n+2} x_o. \]

Then,

\[ d(x_{2n+1}, x_{2n}) = d(T_1 x_{2n}, T_2 x_{2n-1}) \]

\[ \leq k d(x_{2n}, x_{2n-1}) + ld(T_2 x_{2n-1}, x_{2n}) \]

\[ = k d(x_{2n}, x_{2n-1}) \leq k^2 d(x_{2n}, x_{2n-1}) \]

Thus for \( n > m, \) we have

\[ d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + \ldots \]

\[ + d(x_{2m+1}, x_{2m}) \leq (k^{2m-1} + k^{2m-2} + \ldots + k^2 + k) d(x_1, x_o) \]

\[ \leq \frac{k^{2m}}{1-k} d(x_1, x_o) \]

Let \( 0 \ll c \) be given. Choose a natural number \( N_1 \) such that \( \frac{k^{2m}}{1-k} d(x_1, x_o) \ll c, \) for all \( m \geq N_1. \) Thus

\[ d(x_{2n}, x_{2m}) \ll c, \]

for \( n > m. \) Therefore \( \{x_{2n}\}_{n \geq 1} \) is a Cauchy sequence in \((X, d)\) is a complete cone metric space, there exists \( x^* \in X \) such that \( x_{2n} \to x^*. \) Choose a natural number \( N_2 \) such that

\[ d(x_{2n}, x^*) \ll \frac{c}{3} \text{ for all } n \geq N_2 \]
Hence, for \( n \geq N_2 \) we have

\[
\begin{align*}
\d(T_1 x^*, x^*) & \leq \d(x_{2n}, T_1 x^*) + \d(x_{2n}, x^*) \\
& = \d(T_1 x_{2n-1}, T_1 x^*) + \d(x_{2n}, x^*) \\
& \leq k \d(x_{2n-1}, x^*) + l \d(T_1 x_{2n-1}, x^*) + \d(x_{2n}, x^*) \\
& \leq \d(x_{2n-1}, x^*) + \d(x_{2n}, x^*) + \d(x_{2n}, x^*).
\end{align*}
\]

So,

\[
\d(T_1 x^*, x^*) \leq \frac{\xi}{4} + \frac{\xi}{4} + \frac{\xi}{4} = c.
\]

Thus,

\[
\d(T_1 x^*, x^*) \leq \frac{c}{2m}, \text{ for all } m \geq 1.
\]

Hence \( \frac{c}{2m} - \d(T_1 x^*, x^*) \in P \), for all \( m \geq 1 \). Since \( \frac{c}{2m} \to 0 \) (as \( m \to \infty \)) and \( P \) is closed, \( -\d(T_1 x^*, x^*) \in P \). But, \( \d(T_1 x^*, x^*) \in P \). Therefore \( \d(T_1 x^*, x^*) = 0 \), and so \( T_1 x^* = x^* \). Now, if \( y^* \) is another fixed point of \( T_1 \) and \( k + l < 1 \), then

\[
\d(x^*, y^*) = \d(T_1 x^*, y^*) \leq k \d(x^*, y^*) + l \d(T_1 x^*, y^*)
= (k + l)\d(x^*, y^*).
\]

Hence \( \d(x^*, y^*) = 0 \) and so \( x^* = y^* \). Therefore, the fixed point of \( T_1 \) is unique whenever \( k + l < 1 \). Similarly, it can be established that \( T_2 x^* = x^* \). Hence \( T_1 x^* = x^* = T_2 x^* \). Thus \( x^* \) is the unique common fixed point of pair of maps \( T_1 \) and \( T_2 \), whenever \( k + l < 1 \).

References


