The dynamics of nutrient, toxic phytoplankton, nontoxic phytoplankton and zooplankton model

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Abstract

The objective of this paper is to study the dynamical behavior of an aquatic food web system. A mathematical model that includes nutrients, phytoplankton and zooplankton is proposed and analyzed. It is assumed that, the phytoplankton divided into two compartments namely toxic phytoplankton which produces a toxic substance as a defensive strategy against predation by zooplankton, and a nontoxic phytoplankton. All the feeding processes in this food web are formulating according to the Lotka-Volterra functional response. This model is represented mathematically by the set of nonlinear differential equations. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. The occurrence of local bifurcation and Hopf bifurcation are investigated. Finally, numerical simulation is used to study the global dynamics of this model.

Keywords: Aquatic Food Web; HOPF Bifurcation; Local Bifurcation; Stability Analysis.

1. Introduction

Nutrient–phytoplankton–zooplankton NPZ models have been used in oceanography for at least three decades, and are still a common research tool. The NPZ model incorporates one of the simplest sets of dynamics that usefully describe oceanic plankton dynamics [1]. Phytoplankton-zooplankton models have been studied by many authors [2-10]. Some type of phytoplankton produce toxin as a defensive strategy against the predation by zooplankton, these types are known as toxic phytoplankton. In [2], models of nutrient-phytoplankton interaction with a toxic substance that inhibit either the growth rate of phytoplankton, zooplankton, or both trophic levels are proposed and studied. In [3], authors have dealt with a nutrient-plankton model in an aquatic environment in the context of phytoplankton bloom. Roy [4] has constructed a mathematical model for describing the interaction between a nontoxic and a toxic phytoplankton under a single nutrient. Saha and Bandyopadhyay [5] considered a toxin producing phytoplankton–zooplankton model. Since the phytoplankton is a base of all the aquatic food chain and food web systems and most of zooplankton organism depends directly on the phytoplankton in its feeding process. Therefore toxic substances released by toxic phytoplankton play an important role in this context see for example [11]. Phytoplankton organisms are the dominant primary producers in the pelagic environment. They convert inorganic materials into new organic compounds by the process of photosynthesis, starting there by most aquatic food webs [12]. Fan et al. [7] constructed a model to study a NPZ food chain ecosystem involving nontoxic phytoplankton. Rashid and Naji [13] proposed and analyzed NPZ food chain ecosystem model with a toxic phytoplankton. The objective of our model is to determine the interaction between (toxic, nontoxic) phytoplankton and zooplankton under single nutrient in food web ecosystem.

2. Formulation of a mathematical model

In this section, a food web system that contains nutrient, toxic phytoplankton, nontoxic phytoplankton and zooplankton is proposed and analyzed. It is assumed that the density of the nutrient at time T is denoted by N(T), the density of toxic phytoplankton at time T represents by P(T), while P(T) represents the density of the nontoxic phytoplankton at time T. Finally the density of the zooplankton at time T denote by Z(T). Now, in order to formulate the interaction in the above system among these species mathematically following assumptions are obtained:

1) There is a constant concentration of nutrient inter to the system N0 > 0 with constant rate of dilution D > 0. The nutrient uptakes by toxic phytoplankton P1 and nontoxic phytoplankton P2 according to Lotka-Volterra type of functional response with consumption rates α1 > 0 and α2 > 0 respectively, and conversion rates k1 > 0 and k2 > 0 respectively. On the other hand a portion of the dead toxic phytoplankton P1, nontoxic phytoplankton P2 and zooplankton P3 return to the nutrient due to the decomposition operation with rates 0 < m1 < 1, 0 < m2 < 1 and 0 < m3 < 1 respectively.

2) In the absence of nutrient the toxic phytoplankton P1 and the non-toxic phytoplankton P2 decay exponentially due to dilution and natural death rates ε1 > 0 and ε2 > 0 respectively. Further decay facing the toxic phytoplankton P1 and the non-phytoplankton P2 due to the feeding process by zooplankton P3.

3) The zooplankton feeds on the toxic phytoplankton P1 and the non-toxic phytoplankton P2 according to Lotka-Volterra type of functional response with consumption rates β1 > 0 and β2 > 0 respectively, and conversion rates k3 > 0 and k4 > 0 respectively. Further it is assumed that the zooplankton affected by the toxin produced by the toxic phytoplank-
ton $P_i$ during the predation process, with $\theta$ which stand for the liberation rate of toxin substance, $\delta$ the maximum zooplankton in gestation rate for the toxic substance.

Consequently, the dynamics of the above system can be formulated mathematically by the following set of equations: An easy way to comply with the paper formatting requirements is to use this document as a template and simply type your text into it.

$$
dN/dt = D(N_0 - N) - \alpha_1NP_1 - \alpha_2NP_2 + m_1e_1P_1 + m_2e_2P_2 + m_3e_3P_3$$

$$
dP_1/dt = k_1\alpha_1NP_1 - (D + e_1)P_1 - \beta_1P_1P_3$$

$$
dP_2/dt = k_2\alpha_2NP_2 - (D + e_2)P_2 - \beta_2P_2P_3$$

$$
dP_3/dt = k_3\beta_1P_1P_3 + k_4\beta_2P_2P_3 - (D + e_3)P_3 - 3\delta P_3P_3$$

(1)

Note that the above proposed model has eighteen parameters in all, which make the analysis difficult. So, in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

$$
t = DT, x = \frac{N}{N_0}, y = \frac{u_1}{D}, z = \frac{u_2}{D}, w = \frac{m_3\epsilon}{m_2\epsilon}$$

Then the non-dimensional form of system (1) can be written as:

$$
\frac{dx}{dt} = 1 - x - xy - xz + u_1y + u_2z + w = F_1(x, y, z, w)
$$

$$
\frac{dy}{dt} = u_3xy - u_4y - u_3yw = F_2(x, y, z, w)
$$

$$
\frac{dz}{dt} = u_5xz - u_7z - u_6zw = F_3(x, y, z, w)
$$

$$
\frac{dw}{dt} = s_1yw + u_{10}zw - u_{11}w = F_4(x, y, z, w)
$$

Where $s_1 = u_9 - u_{12}$, with initial condition $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$ and $w(0) \geq 0$. It is observed that the number of parameters have been reduced from eighteen in the system (1) to twelve in the system (2). Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the four dimensional space

$$
R_+^4 = \{(x, y, z, w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}
$$

Therefore these functions are Lipschitzian on $R_+^4$, and hence the solution of the system (2) exists and is unique. Moreover, the boundedness of the solution of the system (2) in $R_+^4$ is established in the following theorem.

**Theorem 1:** All the solutions of system (2) are uniformly bounded as $t \to \infty$.

**Proof:** Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2) with non-negative initial condition $(x_0, y_0, z_0, w_0) \in R_+^4$.

Define the function $M(t) = x(t) + y(t) + z(t) + w(t)$, and then take the time derivative of $M(t)$ along the solution of the system (2) we get:

$$
\frac{dM}{dt} \leq 1 - x - u_4 - u_1y - (u_7 - u_2)z - (u_{11} - 1)w
$$

Then $\frac{dM}{dt} + sM \leq 1$, where $s = \min \{1, u_4 - u_1, u_7 - u_2, u_{11} - 1\}$.

Now, by using the comparison theorem on the differential inequality for the initial value $M(0) = M_0$, we get:

$$
M(t) \leq \frac{1}{s} \left( M_0 - \frac{1}{s} \right) e^{-st}
$$

Thus, $\lim_{t \to \infty} M(t) \leq \frac{1}{s}$, and hence $0 \leq M(t) \leq \frac{1}{s}$, $\forall t > 0$. So, all the solutions of system (2) are uniformly bounded as $t \to \infty$.

### 3. Existence of equilibrium points

It is observed that, system (2) has at most seven biologically feasible equilibrium points, namely $E_{i, i = 0, 1, 2, 3, 4, 5, 6}$. The existence conditions for each of these equilibrium points are discussed in the following:

- The single species equilibrium point $E_0 = (1, 0, 0, 0)$ always exists.
- The first planar equilibrium point $E_1 = (x_1, y_1, z_1, 0)$, where:

$$
x_1 = \frac{u_4}{u_1}, \quad y_1 = \frac{u_4 - u_2}{u_4 - u_2}, \quad z_1 = \frac{u_4}{u_4 - u_2}
$$

Exists uniquely in Int.$R_+^2$ (interior of $R_+^2$ ) of $xy$–plane under the following necessary and sufficient condition:

$$
u_4u_5 < u_4 < u_3 \text{ or } u_4u_5 > u_4 > u_3
$$

- The second planar equilibrium point $E_2 = (x_2, y_2, z_2, 0)$, where:

$$
x_2 = \frac{u_4}{u_1} \text{ and } z_2 = \frac{u_4 - u_2}{u_4 - u_2},
$$

Exists uniquely in Int.$R_+^2$ of $xz$–plane under the following necessary and sufficient condition:

$$
u_4u_6 < u_4 < u_5 \text{ or } u_4u_6 > u_4 > u_6
$$

- The first 3D boundary equilibrium point $E_3 = (x_3, y_3, 0, w_3)$, where:

$$
x_3 = \frac{u_4 - u_4}{u_4 - u_4}, \quad y_3 = \frac{u_4}{u_4 - u_4}, \quad w_3 = \frac{u_4(u_4 + u_4)}{u_4(u_4 + u_4)}
$$

Exists uniquely in Int.$R_+^2$ of $xyw$ – space under the following necessary and sufficient conditions:

$$
s_1 > 0
$$

With

$$
u_4(u_1 + s_1) > u_5(u_1 + s_1)$$

$$
u_4s_1 > u_5(u_1 + s_1)
$$

(5c)

Or

$$
u_4(u_1 + s_1) < u_3(u_1 + s_1)$$

$$
u_4s_1 < u_3(u_1 + s_1)
$$

(5d)

- The second 3D boundary equilibrium point $E_4 = (x_4, 0, z_4, w_4)$, where:
\[ x_4 = \frac{w_{11} + w_{12} + w_{13} + w_{14}}{u_{11} + u_{12} + u_{13} + u_{14}}, \quad z_4 = \frac{u_{11}}{u_{10}}. \]

\[ w_4 = \frac{w_{12} + w_{13} + w_{14}}{u_{11} + u_{12} + u_{13} + u_{14}}. \]  

(6a)

- The third 3D boundary equilibrium point \( E_5 \) = 

\[ (x_5, y_5, z_5, 0), \text{ where:} \]

\[ x_5 = \frac{u_3}{u_2}, \quad y_5 = \frac{u_3 + u_2(u_2 - u_3)}{u_{12} - u_2}, \quad z_5 = \frac{u_3 - u_2 + u_4(u_2 - u_3)}{u_{13} - u_2}. \]  

(7a)

- Existence uniquely in \( \text{Int} \mathbb{R}^2 \) of \( xzw \) space under the following necessary and sufficient conditions:

\[ u_7(u_{10} + u_{11}) > u_6(u_2u_{11} + u_{10}), \]

\[ u_6 > u_0 > u_2u_{11} + u_{10}, \]

\[ u_4 > u_0 > u_0 > u_0 > u_10 + u_{11}. \]  

(6b)

- Or:

\[ u_7(u_{10} + u_{11}) < u_6(u_2u_{11} + u_{10}), \]

\[ u_6 > u_0 > u_2u_{11} + u_{10}, \]

\[ u_4 < u_0 > u_0 > u_10 + u_{11}. \]  

(6c)

- The positive (coexistence) equilibrium point \( E_6 \) = 

\[ (x_6, y_6, z_6, w_6), \text{ where:} \]

\[ x_6 = \frac{s_2}{s_4}, \quad y_6 = \frac{s_3 + s_2(u_3 + s_3)}{u_{12} + u_{13} + u_{14} + u_{10} + u_{11}}, \quad z_6 = \frac{s_2 - s_3}{s_4} \]

\[ \text{where } s_2 = u_4u_7 - u_4u_6, \quad s_3 = u_4u_7 - u_4u_6, \quad s_4 = u_5u_6 - u_3u_9. \]  

(8a)

- Existence uniquely in \( \text{Int} \mathbb{R}^4 \) under the following necessary and sufficient conditions:

\[ u_{11} > u_{10}z_6 \]

\[ \text{With} \]

\[ s_2 > 0, s_3 > 0 \text{ and } s_4 > 0 \]

\[ s_2 < 0, s_3 < 0 \text{ and } s_4 < 0 \]

\[ \text{or} \]

\[ s_2(u_{11} + s_3) > u_{11}s_4 + s_3(s_3 + s_4), \]

\[ u_2s_4 + u_{10}s_2 > s_1s_2 + u_1u_{10}s_4 \]

\[ \text{Or} \]

\[ s_2(s_3 + s_4) < u_{11}s_4 + s_3(s_3 + s_4), \]

\[ u_2s_4 + u_{10}s_2 < s_1s_2 + u_1u_{10}s_4 \]  

(8c)

- The characteristic equation of this Jacobian matrix is given by:

\[ \lambda_{1x} = \frac{u_3(u_4 - u_3)}{u_{11} - u_{10} - u_{11}u_{10}}, \quad \lambda_{1y} = \frac{u_3 - u_{11}u_{10}}{u_{11} - u_{10}} - \lambda_{1y} \]

\[ \lambda_{1z} = \frac{u_3 - u_{11}}{u_{11} - u_{10} - u_{11}u_{10}}, \quad \lambda_{1w} = \frac{u_3 - u_{11}}{u_{11} - u_{10}} - \lambda_{1w} \]

\[ \text{Therefore,} \]

\[ \lambda_{1x} = \frac{u_3 - u_{11}u_{10}}{u_{11} - u_{10}} \quad \lambda_{1y} = \frac{u_3 - u_{11}u_{10}}{u_{11} - u_{10}} \]

\[ \lambda_{1z} = \frac{u_3 - u_{11}}{u_{11} - u_{10} - u_{11}u_{10}}, \quad \lambda_{1w} = \frac{u_3 - u_{11}}{u_{11} - u_{10}} - \lambda_{1w} \]  

(8d)

- The Jacobian matrix of system (2) at \( E_0 \) can be written as:

\[ J_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -u_9 & 0 \\ 0 & 0 & 0 & -u_{11} \end{pmatrix} \]  

(10a)

- Clearly, \( J_0 \) has the following eigenvalues:

\[ \lambda_{0x} = -1, \quad \lambda_{0y} = u_3 - u_4, \quad \lambda_{0z} = u_4 - u_7, \quad \lambda_{0w} = -u_{11} \]

- Therefore all the eigenvalues have negative real parts and hence the equilibrium point \( E_0 \) is locally asymptotically stable provided that

\[ u_4 > u_3 \]

\[ u_7 > u_6 \]

- Otherwise it will be saddle point.

- The Jacobian matrix of system (2) at \( E_1 \) can be written as:

\[ J_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & u_3 - u_{11}u_{10} & 0 \\ 0 & 0 & 0 & u_3 - u_{11} - u_{10}u_{11} \end{pmatrix} \]  

(11a)

- 4. Local stability analysis

In this section, the local stability analysis of system (2) around each of the above equilibrium points is discussed through computing the Jacobain matrix \( \{J_x, y, z, w\} \) of system (2) at each of them.

The general Jacobain matrix of system (2) can be written as follows:

\[ J = \begin{pmatrix} -1+y+z & 0 & 0 & 0 \\ 0 & -u_9 & 0 & 0 \\ u_3 - u_4 - u_{11} & 0 & 0 & -u_9 \\ u_3 - u_4 & 0 & 0 & 0 \end{pmatrix} \]  

(9)

- \( E_0 \) can be written as:

\[ J_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -u_9 & 0 \\ 0 & 0 & 0 & -u_{11} \end{pmatrix} \]  

(10a)
The characteristic equation of \( J \) can be written as:

\[
\lambda^3 + \lambda^2 \Delta_1 + \lambda \Delta_2 + \Delta_3 = 0
\]

where

\[
\Delta_1 = A_1 a_2 - A_3 = a_{21}(a_{11}p_{12} + a_{14}p_{42})
\]

Clearly, the eigenvalue \( \lambda_3 \) in z-direction has negative real part if and only if the following condition holds:

\[
u_6 x_3 < u_7 + u_8 w_3
\]

However, according to existence condition (5c), it is observed that \( A_i > 0; \forall i = 1, 3 \), further \( \Delta > 0 \) if and only if:

\[
u_3 < u_5(1 + y_3)
\]

So, according to Routh-Hurwitz criterion the roots of the third degree polynomial in the characteristic equation have negative real parts and hence the equilibrium point \( E_3 \) is locally asymptotically stable.

The Jacobian matrix of system (2) at \( E_4 \) can be written as:

\[
\begin{bmatrix}
-1 + z_4 & 1 & 0 \\
0 & 0 & -u_8 y_4 \\
0 & u_6 & 0 \\
\end{bmatrix}
= (b_4)_{4 \times 4}
\]

The characteristic equation of \( J_4 \) can be written as:

\[
\left( b_{22} - \lambda_4 \right) \left( \lambda^3 + B_4 \lambda^2 + B_2 \lambda + B_3 \right) = 0
\]

Here

\[
B_1 = -b_1 \text{, } B_2 = -\left( b_{42} b_{34} + b_{41} b_{31} \right) \text{, } B_3 = b_{43} \left( b_{11} p_{34} - b_{31} p_{14} \right)
\]

Further, it is easy to verify that:

\[
\Delta = B_1 B_2 - B_3 = b_{31} \left( b_{11} p_{34} + b_{14} b_{41} \right)
\]

Clearly, the eigenvalue \( \lambda_4 \) in y-direction has negative real part if and only if the following condition holds:

\[
u_3 x_4 < u_4 + u_5 w_4
\]

However, according to existence condition (6b), we obtain that \( B_i > 0; \forall i = 1, 3 \), further \( \Delta > 0 \) if and only if:

\[
u_6 < u_9(1 + z_4)
\]

Now, it is easy to verify that:

\[
u_2 < x_4
\]
Clearly, the equilibrium point $E_5$ has a zero eigenvalue that's its non-hyperbolic point. So, the linearization failed and we will study the stability of $E_5$ by Lyapunov method in the next section.

The Jacobian matrix of system (2) at $E_6$ can be written as:

$$J_6 = \begin{bmatrix}
-1 & y_6 + z_6 & u_1 - x_6 & u_2 - x_6 & 1 \\
0 & 0 & u_3 y_6 & 0 & 0 \\
0 & 0 & u_6 y_6 & 0 & u_6 z_6 \\
0 & 0 & 0 & u_6 y_6 - u_10 z_6 - u_11 & 0 \\
\end{bmatrix} = (d_0)_{4 \times 4}$$  

(16a)

The characteristic equation of $J_6$ can be written as:

$$\lambda^4 + D_1 \lambda^3 + D_2 \lambda^2 + D_3 \lambda + D_4 = 0$$

(16b)

Here

$$D_1 = -(d_{11} + d_{44})$$
$$D_2 = -(d_{14} d_{44} + d_{12} d_{21} + d_{13} d_{31} + d_{24} d_{14} + d_{34} d_{43})$$
$$D_3 = d_{43} (d_{14} d_{34} - d_{31}) + d_{42} (d_{14} d_{24} - d_{21}) + d_{44} (d_{12} d_{21} - d_{13} d_{31})$$
$$D_4 = d_{12} d_{13} d_{34} d_{43} + d_{13} d_{14} d_{24} d_{43} - d_{12} d_{13} d_{24} d_{34} - d_{12} d_{14} d_{23} d_{43}$$

Consequently,

$$A_1 = D_2 d_2 - D_3 = d_11 f_2 + d_44 f_4 - d_14 d_4 f_0 + \Gamma_4$$

And

$$A_2 = D_3 (D_2 d_2 - D_3) - D_4 f_0^2 = F_1 + F_2 + F_3 + F_4$$

Where

$$F_1 = (d_{14} f_3 + d_{44} f_0) (d_{14} f_2 + d_{44} f_3 + \Gamma_4)$$
$$F_2 = (d_{14} f_3 + d_{44} f_2) (d_{14} f_4 f_0 + \Gamma_4)$$
$$F_3 = -\Gamma_4 (d_{14} d_{44} f_1) (\Gamma_2 + d_{44} f_3 + \Gamma_4)$$
$$F_4 = \Gamma_4^2 f_5 (d_{14} d_{44} - d_{12} d_{43})$$

With

$$\Gamma_1 = d_{11} + d_{44}$$
$$\Gamma_2 = d_{12} d_{21} + d_{13} d_{31}$$
$$\Gamma_3 = d_{24} d_{14} + d_{34} d_{43}$$
$$\Gamma_4 = d_{24} d_{14} + d_{34} d_{43}$$
$$\Gamma_5 = d_{23} d_{34} - d_{31} d_{24}$$
$$\Gamma_6 = d_{13} d_{42} - d_{12} d_{43}$$

u_{11} > s_1 u_6 + u_{10} z_6 

(16c)

$$u_2 > x_6 > \max \left\{ \frac{u_{10} - u_5 s_1}{u_{10} - s_1}, u_1 \right\}$$

(16d)

$$y_6 > \max \left\{ \frac{u_{6} - x_6 - 1}{u_5}, \frac{z_6 - 1}{u_3}, \frac{u_6 z_6 - u_10 z_6 - u_11}{u_{10} - s_1} \right\}$$

(16e)

So, according to Routh-Hurwitz criterion the roots of the third degree polynomial in the characteristic equation have negative real parts and hence the equilibrium point $E_6$ is locally asymptotically stable.

5. Global stability analysis

In this section the global stability for the equilibrium points of system (2) is studied analytically by using the Lyapunov method as shown in the following theorems:

Theorem 2: Assume that, the equilibrium point $E_6$ of system (2) is locally asymptotically stable and the following conditions hold:

$$u_3 < \min \left\{ \frac{u_4}{1 + u_1}, \frac{u_{94} + u_11}{s_1} \right\}$$

(17a)

$$u_6 < \min \left\{ \frac{u_7}{1 + u_2}, \frac{u_{96} + u_11}{u_{10}} \right\}$$

(17b)

Then $E_6$ is globally asymptotically stable in the $R^4_+$. Proof: Consider the following function:

$$V_0 = c_1 (x - 1 - \ln(x)) + c_2 y + c_3 z + c_4 w$$

Where $c_i$, $i = 1, 2, 3, 4$ are positive constants to be determine. Clearly $V_0 : R^4_+ \rightarrow R$ is $C^1$ positive definite function. Now by differentiating $V_0$ with respect to time $t$, we get:

$$\frac{dV_0}{dt} = -c_1 \left( \frac{x - 1}{x} \right)^2 + c_1 (1 + u_1) - c_2 u_4 y + c_1 (1 + u_2) - c_3 u_7 z$$

$$+ c_1 (-c_4 u_1) w + (c_2 u_3 - c_1) x y + (c_3 u_6 - c_1) x z$$

$$+ (c_4 u_1 - c_2 u_5) y w + (c_4 u_10 - c_3 u_8) z w - c_1 \left( u_1 y + u_2 z + w \right)$$

So by choosing the positive constants as below:

$$c_1 = 1, \ c_2 = \frac{1}{u_3}, \ c_3 = \frac{1}{u_6}, \ c_4 = \frac{1}{u_11}$$

We obtain that:

$$\frac{dV_0}{dt} \leq - \left( \frac{x - 1}{x} \right)^2 + \left( 1 + u_1 - \frac{u_4}{u_3} \right) y + \left( 1 + u_2 - \frac{u_7}{u_6} \right) z$$

$$+ \left( \frac{u_1}{u_{11}} - \frac{u_3}{u_3} \right) y w + \left( \frac{u_{10}}{u_{11}} - \frac{u_8}{u_6} \right) z w$$

Now, according to existence condition (8c) it is observed that $D_1 > 0$, $i = 1, 3, 4$, further $\Delta_2 > 0$ if and only if the following conditions hold:
According to conditions (17)(a,b) we have \( \frac{dV_0}{dt} < 0 \). Therefore, \( E_0 \) is globally asymptotically stable in the \( R^4_+ \), and hence the proof is complete.

**Theorem 3:** Assume that, the equilibrium point \( E_1 \) of system (2) is locally asymptotically stable. Then the basin of attraction of \( E_1 \), say \( B(E_1) \subset R^4_+ \), satisfy the following conditions:

\[
(\text{18a}) \quad (u_1 - x_1 + u_3 y_1)^2 \leq (1 + y) \quad (u_4 - u_3 x_1)
\]

\[
(\text{18b}) \quad x_1 + u_2 + \frac{u_6}{u_9} < x < x_1 + \frac{u_1}{u_0}
\]

\[
(\text{18c}) \quad y > y_1 + \frac{s_1}{u_9 u_10}
\]

Proof: Consider the following function:

\[
V_1 = c_1 \left( \frac{(x-x_1)^2}{2} + c_2 (y-y_1)^2 + c_3 x + c_4 w \right)
\]

Where \( c_1, c_2, c_3, c_4 \) are positive constant to be determine. Clearly \( V_1 : R^4_+ \rightarrow R \) is \( C^1 \) positive definite function. Now by differentiating \( V_1 \) with respect to time \( t \), we get:

\[
\frac{dV_1}{dt} = -c_1 (1+y) (-x-x_1) + (c_1 (u_1-x_1) x+u_3 y) (-x-x_1) (y-y_1)
\]

\[
-c_2 (u_4 - u_3 x_1) (y-y_1)^2 - (c_1 u_2 x_1 + c_3 u_7) x
\]

\[
+ (c_1 (x-x_1) + c_2 u_1) w + (c_1 (x_1 + u_2 - x) + c_3 u_6) x z
\]

\[
+ (c_4 x - c_2 u_3 (y-y_1) w + (c_4 u_10 - c_3 u_8) x w
\]

So by choosing the positive constants as below:

\[
c_1 = 1, \quad c_2 = 1, \quad c_3 = \frac{1}{u_8}, \quad c_4 = \frac{1}{u_10}
\]

And according to condition (18a) we obtain that:

\[
\frac{dV_1}{dt} \leq \left[ \sqrt{(1+y)(x-x_1)} - \sqrt{(u_4 - u_3 x_1)} (y-y_1) \right]^2
\]

\[
+ \left( x - x_1 - \frac{u_1}{u_0} \right) w
\]

\[
+ \left( x_1 + u_2 - x + \frac{u_6}{u_8} \right) x z
\]

\[
+ \left( \frac{s_1}{u_10} - u_5 (y-y_1) \right) w
\]

Obviously \( \frac{dV_1}{dt} < 0 \) for every initial point satisfying conditions (18)(b,c) and then \( V_1 \) is a Lyapunov function provided that conditions (18)(a-c) hold. Thus \( E_3 \) is globally asymptotically stable in the interior of \( B(E_3) \), which means that \( B(E_2) \) is the basin of attraction and this completes the proof.

**Theorem 4:** Assume that, the equilibrium point \( E_2 \) of system (2) is locally asymptotically stable. Then the basin of attraction of \( E_2 \), say \( B(E_2) \subset R^4_+ \), satisfy the following conditions:

\[
(\text{19a}) \quad (u_2 - x_2 + u_6 x_2)^2 \leq (1 + z) \quad (u_7 - u_6 x_2)
\]

\[
(\text{19b}) \quad x_2 + u_1 + \frac{u_3}{u_5} < x < x_2 + \frac{u_1}{u_1}
\]

\[
(\text{19c}) \quad z > \frac{u_10}{u_9 u_8}
\]

Where \( c_1, \ i = 1,2,3,4 \) are positive constants to be determine. Clearly \( V_2 : R^4_+ \rightarrow R \) is \( C^1 \) positive definite function. Now by differentiating \( V_2 \) with respect to time \( t \), we get:

\[
\frac{dV_2}{dt} = -c_1 (1+z) (x-x_2)^2 + (c_1 (u_2 - x_2) + c_6 u_6) (x-x_2) (z-z_2)
\]

\[
-c_3 (u_7 - u_6 x_2) (z-z_2)^2 - (c_1 u_1 x_2 + c_2 u_4) y
\]

\[
+ (c_1 (x-x_2) - c_4 u_1) w + (c_1 (x_2 + u_1 - x) + c_3 u_3) x y
\]

\[
+ (c_4 x_1 - c_2 u_3 (y-y_1) w + (c_4 u_10 - c_3 u_8) x z w
\]

So by choosing the positive constants as below:

\[
c_1 = 1, \quad c_2 = \frac{1}{u_5}, \quad c_3 = 1, \quad c_4 = \frac{1}{u_1}
\]

And according to condition (19a) we obtain that:

\[
\frac{dV_2}{dt} \leq \left[ \sqrt{(1+z)(x-x_2)} - \sqrt{(u_7 - u_6 x_2)} (z-z_2) \right]^2
\]

\[
+ \left( x-x_2 - \frac{u_1}{u_1} \right) w + \left( x_2 + u_1 - x + \frac{u_3}{u_5} \right) x y
\]

\[
+ \left( \frac{u_10}{u_1} - u_8 (z-z_2) \right) x w
\]

Obviously \( \frac{dV_2}{dt} < 0 \) for every initial point satisfying conditions (19)(b,c) and then \( V_2 \) is a Lyapunov function provided that conditions (19)(a-c) hold. Thus \( E_3 \) is globally asymptotically stable in the basin of \( B(E_2) \), which means that \( B(E_2) \) is the basin of attraction and this completes the proof.

**Theorem 5:** Assume that, the equilibrium point \( E_3 \) of system (2) is locally asymptotically stable. Then the basin of attraction of \( E_3 \), say \( B(E_3) \subset R^4_+ \), satisfy the following conditions:

\[
(\text{20a}) \quad (u_1 - x_3 + u_3 y_3)^2 \leq (1 + y) \quad (u_4 + u_5 w - u_3 x_3)
\]

\[
(\text{20b}) \quad (u_1 w - u_5 y_3)^2 \leq (u_4 + u_5 w - u_3 x_3) \quad (u_1 - s_1 y_3)
\]

\[
(\text{20c}) \quad (1+y) \quad (u_1 - s_1 y_3) \geq 1
\]

\[
(\text{20d}) \quad x > x_3 + u_2 + u_6
\]
Proof: Consider the following function:

\[ V_3 = \frac{(x-x_3)^2}{2} + \frac{(y-y_3)^2}{2} + \frac{(w-w_3)^2}{2} \]  

Clearly \( V_3 : \mathbb{R}^4 \to \mathbb{R} \) is \( C^1 \) positive definite function. Now by differentiating \( V_3 \) with respect to time \( t \), and according to conditions (20)(a-c) we obtain that:

\[
\frac{dV_3}{dt} \leq - \left[ \frac{(1+y)}{2} (x-x_3) \sqrt{u_4+u_6w-u_3x_3} \right]^2 \\
- \left[ \frac{(1+y)}{2} (x-x_3) - \sqrt{u_1+u_6y_3} \right]^2 \\
- \left[ \frac{u_4+u_6w-u_3x_3}{2} (y-y_3) - \sqrt{u_1+u_6y_3} \right]^2 \\
+ (x_3 + u_2 - x + u_6) xz + u_10 (w-w_3) zw
\]

Obviously \( \frac{dV_3}{dt} < 0 \), and then \( V_3 \) is a Lyapunov function provided that the given conditions hold. Therefore \( E_3 \) is globally asymptotically stable in the interior of \( B(E_3) \), which means that \( B(E_3) \) is the basin of attraction of \( E_3 \) and the proof is complete.

**Theorem 6:** Assume that, the equilibrium point \( E_4 \) of system (2) is locally asymptotically stable. Then the basin of attraction of \( E_4 \), say \( B(E_4) \subset \mathbb{R}^4_+ \), satisfy the following conditions:

\[ (u_2 - x_4 + u_6z)^2 \leq (1+z) (u_7 + u_9w - u_6x_4) \]  

\[ (u_10w - u_9z)^2 \leq (u_7 + u_9w - u_6x_4) (u_11 - u_10z^4) \]  

\[ (1+z) (u_11 - u_10z^4) \geq 1 \]

\[ x > x_4 + u_1 + u_3 \]

\[ w < w_4 \]

Proof: Consider the following function:

\[ V_4 = \frac{(x-x_4)^2}{2} + \frac{(z-z_4)^2}{2} + \frac{(w-w_4)^2}{2} \]

Clearly \( V_4 : \mathbb{R}^4 \to \mathbb{R} \) is \( C^1 \) positive definite function. Now by differentiating \( V_4 \) with respect to time \( t \), and according to conditions (21)(a-c) we obtain that:

\[
\frac{dV_4}{dt} \leq - \left[ \frac{1+y}{2} (x-x_4) \sqrt{u_7 + u_9w - u_6x_4} (z-z_4) \right]^2 \\
- \left[ \frac{1+y}{2} (x-x_4) - \sqrt{u_11 - u_10z^4} \right]^2 \\
+ \left[ u_4 + u_6w - u_3x_3 \right] (z-z_4) + \left[ u_11 - u_10z^4 \right] (w-w_4)
\]

Obviously \( \frac{dV_4}{dt} < 0 \), and then \( V_4 \) is a Lyapunov function provided that the given conditions hold. Therefore \( E_4 \) is globally asymptotically stable in the interior of \( B(E_4) \), which means that \( B(E_4) \) is the basin of attraction of \( E_4 \) and the proof is complete.

**Theorem 7:** Assume that, the equilibrium point \( E_5 \) of system (2) is locally asymptotically stable. Then the basin of attraction of \( E_5 \), say \( B(E_5) \subset \mathbb{R}^4_+ \), satisfy the following conditions:

\[ (u_1 - x_5 + u_3y)^2 \leq 2(1+y+z) (u_4 - u_3x_5) \]  

\[ (u_2 - x_5 + u_6z)^2 \leq 2(1+y+z) (u_7 - u_6x_5) \]  

\[ u_{11} > x + (u_3y_5 + s_1) y + (u_8z_5 + u_{10}z) \]

Proof: Consider the following function:

\[ V_5 = \frac{(x-x_5)^2}{2} + \frac{(y-y_5)^2}{2} + \frac{(z-z_5)^2}{2} + \frac{(w-w_5)^2}{2} \]

Clearly \( V_5 : \mathbb{R}^4 \to \mathbb{R} \) is \( C^1 \) positive definite function. Now by differentiating \( V_5 \) with respect to time \( t \), and according to conditions (22)(a,b) we obtain that:

\[
\frac{dV_5}{dt} \leq - \left[ \frac{1+y+z}{2} (x-x_5) - \sqrt{u_4 - u_3x_5} (y-y_5) \right]^2 \\
- \left[ \frac{1+y+z}{2} (x-x_5) - \sqrt{u_7 - u_6x_5} (z-z_5) \right]^2 \\
+ \left[ x - u_{11} + s_1 y + u_{10} z \right] w
\]

Obviously \( \frac{dV_5}{dt} < 0 \), and then \( V_5 \) is a Lyapunov function provided that the given conditions hold. Therefore \( E_5 \) is globally asymptotically stable in the interior of \( B(E_5) \), which means that \( B(E_5) \) is the basin of attraction of \( E_5 \) and the proof is complete.

**Theorem 8:** Assume that, the equilibrium point \( E_6 \) of system (2) is locally asymptotically stable. Then the basin of attraction of \( E_6 \), say \( B(E_6) \subset \mathbb{R}^4_+ \), satisfy the following conditions:

\[ (u_1 - x_6 + u_3y)^2 \leq \frac{2}{3} (1+y+z) (u_4 - u_3x_6 + u_5w) \]  

\[ (u_2 - x_6 + u_6z)^2 \leq \frac{2}{3} (1+y+z) (u_7 - u_6x_6 + u_8w) \]  

\[ \frac{4}{9} (1+y+z) (u_{11} - s_1 y_6 - u_{10} z_4) \geq 1 \]  

\[ (s_1 w - u_5 y_6)^2 \leq \frac{2}{3} (u_4 - u_3x_6 + u_5w) (u_{11} - s_1 y_6 - u_{10} z) \]  

\[ (s_1 w - u_5 y_6)^2 \leq \frac{2}{3} (u_4 - u_3x_6 + u_5w) (u_{11} - s_1 y_6 - u_{10} z) \]
\[(u_{10}w-u_8z_6)^2 \leq \frac{2}{3}(u_7-u_6x_6+u_8w)(u_{11}-u_1y_6-u_{10}z)\]  \hspace{1cm} (23e)

Proof: Consider the following function:

\[V_6 = \frac{(x-x_6)^2}{2} + \frac{(y-y_6)^2}{2} + \frac{(z-z_6)^2}{2} + \frac{(w-w_6)^2}{2}\]

Clearly \(V_6; \mathbb{R}^4 \rightarrow \mathbb{R}\) is \(C^1\) positive definite function. Now by differentiating \(V_6\) with respect to time \(t\), and according to conditions (23)(a-e) we obtain that:

\[
d\frac{V_6}{dt} \leq \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
+ \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

\[
- \left[ \sqrt{\frac{(1+y+z)}{3}(x-x_6)-\frac{(y-y_6)}{2}} \right]^2
\]

Obviously \(\frac{dV_6}{dt} < 0\), and then \(V_6\) is a Lyapunov function provided that the given conditions hold. Therefore \(E_0\) is globally asymptotically stable in the interior of \(B(E_0)\), which means that \(B(E_0)\) is the basin of attraction of \(E_0\) and the proof is complete.

6. The local bifurcation analysis

In this section, the local bifurcation near the equilibrium points of the system (2) is investigated by using the Sotomayor’s theorem [14] for local bifurcation. It is well known that the existence of nonhyperbolic equilibrium point is a necessary but not sufficient condition for bifurcation to occur. Now, according to Jacobian matrix of system (2) given in equation (9), it is clear to verify that for any non-zero vector \(V = (v_1, v_2, v_3, v_4)^T\) we have:

\[
D^2F(V,V) = \begin{pmatrix}
-2v_1(v_2+v_1) & 2v_2(v_3v_1-v_4v_2) & 2v_3(v_4v_1-v_3v_4) & 2v_4(v_1v_2+v_3v_4)
\end{pmatrix}^T
\]

\[
= \begin{pmatrix}
2v_3(v_1v_2-v_4v_3) & 2v_4(v_1v_2+v_3v_4)
\end{pmatrix}^T
\]

\[
D^3F(V,V,V) = \begin{pmatrix}
-2v_1(v_2+v_1) & 2v_2(v_3v_1-v_4v_2) & 2v_3(v_4v_1-v_3v_4) & 2v_4(v_1v_2+v_3v_4)
\end{pmatrix}
\]

Here \(D^2\) represent the derivative of Jacobian matrix of system (2), and \(F = (F_1, F_2, F_3, F_4)^T\) with \(F_i\) \(1, 2, 3, 4\) given in system (2). Therefore, \(D^2F(V,V,V) = (0, 0, 0, 0)^T\).

So, according to Sotomayor’s theorem the pitchfork bifurcation does not occur at each point \(E_0\), \(1 = 0, 1, 2, 3, 4, 5, 6\).

**Theorem 9:** Assume that the local stability condition (10b) holds, and let the parameter value \(u_0\) passing through the value \(u_0^* = u_0\) then the system (2) at the equilibrium point \(E_0\) has:

1) No saddle-node bifurcation.
2) Transcritical bifurcation.

Proof: According to the Jacobian matrix \(I_0\) given by Eq. (10a) the system (2) at the equilibrium point \(E_0\) has zero eigenvalue (say \(\lambda_{0z} = 0\)) at \(u = u_0^*\), and the Jacobian matrix \(I_0\) with \(u = u_0^*\) becomes:

\[
I_0 = \begin{pmatrix}
-1 & u_1 - 1 & u_2 - 1 & 1
0 & u_3 - u_4 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & -u_{11}
\end{pmatrix}
\]

Now, let \(V[0] = (v_1^{[0]}, v_2^{[0]}, v_3^{[0]}, v_4^{[0]})^T\) be the eigenvector corresponding to the eigenvalue \(\lambda_{0z} = 0\). Thus \(I_0^TV[0] = 0\), gives \(V[0] = (u_2 - 1)v_3^{[0]}, 0, v_2^{[0]}, v_1^{[0]}\) where \(v_3\) any nonzero real number. Let \(\psi[0] = (\psi_1^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \psi_4^{[0]})^T\) be the eigenvector corresponding to the eigenvalue \(\lambda_{0z} = 0\) of the matrix \(I_0^T\).

Then \(I_0^T\psi[0] = 0\), by solving this equation for \(\psi[0]\) we get \(\psi[0] = (0, 0, v_3^{[0]}, 0)^T\), where \(v_3\) any nonzero real number. Now, consider:

\[
\frac{\partial f}{\partial u} = f_u(X, u) = \begin{pmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
2 & 0 & x & 0
0 & 0 & 0 & 0
\end{pmatrix}
\]

Where \(f_u(X, u)\) represents the derivative of \(f_u(X, u)\) with respect to \(X = (x, y, z, w)^T\). Further, it is observed

\[
Df_u(X, u) = \begin{pmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
2 & 0 & x & 0
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
D^2f_u(X, u)\psi[0] = \begin{pmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
2 & 0 & x & 0
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\psi[0] = (0, 0, v_3^{[0]}, 0)^T
\]

Hence, it is obtain that:

\[
\psi[0] \begin{pmatrix}
D^2f_u(X, u)\psi[0], V[0]
\end{pmatrix}
\]

\[
= (0, 0, 2v_3(u_2 - 1)v_3^{[0]}, 0)^T
\]

Now, by substituting \(V[0] = 0, 0, 0, 0\) in (24) we get

\[
D^2f_u(X, u)\psi[0] = (0, 0, 2v_3(u_2 - 1)v_3^{[0]}, 0)^T
\]

Hence, it is obtain that:

\[
\psi[0] = (0, 0, 2v_3(u_2 - 1)v_3^{[0]}, 0)^T
\]

\[
= 2v_3(u_2 - 1)v_3^{[0]} \neq 0
\]
Since $u_2$ represent a consumption rate then $u_2 - 1 \neq 0$. Thus, according to Sotomayor’s theorem system (2) has transcritical bifurcation at $E_0$ with the parameter $u_6 = u_6^*$. ■

**Theorem 10:** Assume that the local stability condition (11c) hold and suppose that the condition

$$u_1 u_3 + u_4 \neq u_1 u_3 + u_5 \quad (25)$$

Is satisfied. Then when the parameter value $u_{11}$ passing through

$$u_{11} = \frac{s_1 (u_4 - u_3)}{u_3 u_4 - u_4} \quad \text{system (2) at the equilibrium point } E_1 \quad \text{has:}$$

1) No saddle-node bifurcation.

2) Transcritical bifurcation.

Proof: According to the Jacobian matrix $J_1$ given by Eq. (11a) the system (2) at the equilibrium point $E_1$ has zero eigenvalue (say $\lambda_{1w} = 0$) at $u_{11} = u_{11}^*$, and the Jacobian matrix $J_2$ with $u_{11} = u_{11}^*$ becomes:

$$J_2^* = \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}$$

Now, let $V^{[1]} = \begin{pmatrix} v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]} \end{pmatrix}^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1w} = 0$. Thus $J_2^* V^{[1]} = 0$. gives:

$$V^{[1]} = \begin{pmatrix} v_1^{[1]}, \frac{1}{L_2} v_1^{[1]}, 0, v_5^{[1]} \end{pmatrix}^T$$

Where $L_1 = u_2^2 (u_1 (u_3 - u_5) - u_4 + u_5)$, $L_2 = u_5 (u_3 u_4 - u_4)$ and $v_1^{[1]}$ any nonzero real number. Clearly, $u_1 u_3 \neq u_4$ due to the existence condition (3b). Let $\Psi^{[1]} = \begin{pmatrix} \psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}, \psi_4^{[1]} \end{pmatrix}^T$ be the eigenvector associated with the eigenvalue $\lambda_{1w} = 0$ of the matrix $J_1^T$. Then $J_1^T \Psi^{[1]} = 0$, by solving this equation for $\Psi^{[1]}$ we get $\Psi^{[1]} = \begin{pmatrix} 0, 0, \psi_4^{[1]} \end{pmatrix}^T$, where $\psi_4^{[1]}$ any nonzero real number. Now, consider:

$$\frac{df}{du_{11}}(X, u_{11}) = (0, 0, 0, -w)^T$$

Thus, $f_{u_{11}}(X, u_{11}) = (0, 0, 0, 0)^T$ and hence

$$\Psi^{[1]} = f_{u_{11}}(X, u_{11}) = 0$$

So, according to Sotomayor’s theorem the saddle-node bifurcation cannot occur, while the first condition of transcritical bifurcation is satisfied. Now, since:

$$Df_{u_{11}}(X, u_{11}) = \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & -1
\end{pmatrix}$$

Further, it is observed

$$Df_{u_{11}}(E_1, u_{11}^*) V^{[1]} = \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\Psi^{[1]} = f_{u_{11}}(X, u_{11}^*) = \begin{pmatrix} v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]} \end{pmatrix}^T$$

Now, let $\Psi^{[2]} = \begin{pmatrix} v_1^{[2]}, v_2^{[2]}, v_3^{[2]}, v_4^{[2]} \end{pmatrix}^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2w} = 0$. Thus $J_2^* V^{[2]} = 0$. gives:

$$V^{[2]} = \begin{pmatrix} v_1^{[2]}, 0, L_2 v_1^{[2]}, u_6 v_1^{[2]} \end{pmatrix}^T$$

Where $L_5 = u_2^2 (u_3 (u_5 - u_4) + u_7 - u_6)$, $L_4 = u_6 (u_2 u_6 - u_7)^2$ and $v_1^{[2]}$ any nonzero real number. Clearly, $u_2 u_6 \neq u_7$ due to the ex-
istence condition (4b). Let \( \psi^{[2]} = (\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]})^T \) be the
eigenvector associated with the eigenvalue \( \lambda_{2w} = 0 \) of the matrix
\( J^T_2 \). Then \( J^T_2 \psi^{[2]} = 0 \), by solving this equation for \( \psi^{[2]} \) we get
\( \psi^{[2]} = (0, 0, 0, \psi_4^{[2]})^T \), where \( \psi_4^{[2]} \) any nonzero real number. Now, consider:
\[
\frac{\partial v}{\partial u_1} - f_{u_1}(X, u_1) = (0, 0, 0, -w)^T
\]
Thus, \( f_{u_1}(E_2, u_{11}) = (0, 0, 0, 0)^T \) and hence
\[
(\psi^{[2]})^T f_{u_1}(E_2, u_{11}) = 0
\]
So, according to Sotomayor’s theorem the saddle-node bifurcation cannot occur, while the first condition of transcritical bifurcation is satisfied. Now, since:
\[
Df_{u_1}(X, u_{11}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]
Therefore,
\[
Df_{u_1}(E_2, u_{11})V^{[2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} V^{[2]} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -v_4^{[2]} \end{pmatrix}
\]
\[
(\psi^{[2]})^T [Df_{u_1}(E_2, u_{11}) V^{[2]}] = (0, 0, 0, -v_4^{[2]})^T = -v_4^{[2]} \psi_4^{[2]} \neq 0
\]
Now, by substituting \( V^{[2]} \) in (24) we get
\[
D^2 f(E_2, u_{11}) (V^{[2]}, V^{[2]}) = \begin{pmatrix} -2 L_3/L_4 \left( v_1^{[2]} \right)^2, 0, 0, 2 u_4 u_6 L_3/L_4 \left( v_1^{[2]} \right)^2 \end{pmatrix}^T
\]
Hence, it is obtained that:
\[
(\psi^{[2]})^T [D^2 f(E_2, u_{11}) (V^{[2]}, V^{[2]})] = 2 u_4 u_6 L_3/L_4 \left( v_1^{[2]} \right)^2 \psi_4^{[2]} \neq 0
\]
Thus, according to Sotomayor’s theorem system (2) has transcritical bifurcation at \( E_2 \) with the parameter \( u_{11} = u_{11}^{**} \).

**Theorem 12:** Assume that the following conditions
\[
\begin{align*}
\gamma_3 - u_6 x_3 > w_3 \\
u_{33} w_3 & \neq \gamma_3 w_3 \\
\epsilon_{13} e_{42} & \neq \epsilon_{12} e_{43}
\end{align*}
\]
Are satisfied. Then when the parameter value \( u_3 \) passing through
\( u_3^{**} = \frac{\gamma_3 - u_6 x_3}{w_3} \), system (2) at the equilibrium point \( E_2 \) has:

1) No saddle-node bifurcation.
2) Transcritical bifurcation.

**Proof:** According to the Jacobian matrix \( J_3 \) given by Eq. (13a) the system (2) at the equilibrium point \( E_3 \) has zero eigenvalue (say \( \lambda_{3w} = 0 \)) at \( u_6 = u_6^* \), and the Jacobian matrix \( J_3 \) with \( u_6 = u_6^* \) becomes:
\[
J_3 = \begin{pmatrix} u_6 \end{pmatrix}
\]
Where \( e_{ij} = a_i \) for all \( i,j = 1, 2, 3, 4 \) with \( e_{11} = 0 \).
\[
L V^{[3]} = \begin{pmatrix} v_1^{[3]} \\ v_2^{[3]} \\ v_3^{[3]} \\ v_4^{[3]} \end{pmatrix} \]
be the eigenvector corresponding to the eigenvalue \( \lambda_{3w} = 0 \). Thus \( J_3 V^{[3]} = 0 \), which gives:
\[
V^{[3]} = \begin{pmatrix} \frac{-e_{33} L_3 v_3^{[3]} - u_{10} v_3^{[3]} - v_3^{[3]} e_{31} L_3 v_3^{[3]}}{L_6} \end{pmatrix}
\]
Where \( L_5 = e_{13} e_{42} - e_{12} e_{43} \), \( L_6 = e_{42} (e_{11} e_{34} - e_{31}) \) and \( v_3^{[3]} \) any nonzero real number. Let \( \psi^{[3]} = (\psi_1^{[3]}, \psi_2^{[3]}, \psi_3^{[3]}, \psi_4^{[3]})^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_{3w} = 0 \) of the matrix \( J_3 \). Then we have \( J_3^T \psi^{[3]} = 0 \), by solving this equation for \( \psi^{[3]} \) we get \( \psi^{[3]} = (0, 0, 0, 0)^T \), where \( \psi_3^{[3]} \) any nonzero real number. Now, consider:
\[
\frac{\partial v}{\partial u_3} = f_{u_3}(X, u_3) = (0, 0, -z, 0)^T
\]
Thus, \( f_{u_3}(E_3, u_3^{**}) = (0, 0, 0, 0)^T \) and hence \( (\psi^{[3]})^T f_{u_3}(E_3, u_3^{**}) = 0 \)
So, according to Sotomayor’s theorem the saddle-node bifurcation cannot occur, while the first condition of transcritical bifurcation is satisfied. Now, since
\[
Df_{u_3}(X, u_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
Further, it is observed
\[
Df_{u_3}(E_3, u_3^{**}) V^{[3]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -w & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{e_{33} L_3 v_3^{[3]}}{L_6} \\ \frac{u_{10} v_3^{[3]}}{L_6} \\ \frac{-v_3^{[3]} e_{31} L_3 v_3^{[3]}}{L_6} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -w v_3^{[3]} & 0 \end{pmatrix}
\]
\[
(\psi^{[3]})^T [Df_{u_3}(E_3, u_3^{**}) V^{[3]}] = \begin{pmatrix} 0 & 0 & -w v_3^{[3]} & 0 \end{pmatrix} = -w v_3^{[3]} \psi_3^{[3]} \neq 0
\]
Now, by substituting \( V^{[3]} \) in (24) we get
\[
D^2 f(E_3, u_3^{**}) (V^{[3]}, V^{[3]}) = (U_1, 0, 0, 0)^T
\]
Where
\[
U_1 = \frac{e_{33} L_3 v_3^{[3]}}{L_6}
\]
u_2 = -2u_{11}L_3(u_{11}u_{21} - u_{12}u_{31}) (v_3^{[1]})^2_{s_1L_4}

Hence, it is obtained that:

\((\Psi^{[2]})^T [D^2f(E_3, u_{12}^0)(v_3^{[1]}, v_3^{[2]})] = U_2\Psi^{[3]}_2 \neq 0\)

Thus, according to Sotomayor’s theorem system (2) has transcritical bifurcation at \(E_2\) with the parameter \(u_{12} = u_{12}^0\).

**Theorem 13:** Assume that the following conditions

\[ u_{34} > u_5 \Psi_4 \]

\[ u_{36} = u_{46} \]

\[ q_{13}q_{42} \neq q_{12}q_{43} \]

Are satisfied. Then when the parameter value \( u_4 \) passing through \( u_4^* = u_{34} - u_5 \Psi_4 \) system (2) at the equilibrium point \( E_4 \) has:

1) No saddle-node bifurcation.

2) Transcritical bifurcation.

Proof: According to the Jacobian matrix \( J_4 \) given by Eq. (14a) the system (2) at the equilibrium point \( E_4 \) has zero eigenvalue (say \( \lambda_{4y} = 0 \)) at \( u_4 = u_4^* \), and the Jacobian matrix \( J_4 \) with \( u_4 = u_4^* \) becomes:

\[ J_4^* = J(u_4 = u_4^*) = (q_{ij})_{4 \times 4} \]

Where \( q_{ij} = h_0 \) for all \( i, j = 1, 2, 3, 4 \) with \( q_{22} = 0 \).

Let \( \Psi^{[4]} = (v_1^{[4]}, v_2^{[4]}, v_3^{[4]}, v_4^{[4]})^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_{4y} = 0 \). Thus \( J_4^{*} \Psi^{[4]} = 0 \), which gives:

\[ \Psi^{[4]} = \begin{pmatrix} -q_{13}L_3v_2^{[4]} - v_2^{[4]} & 0 & 0 & 0 \\ q_{13}L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Where \( L_3 = q_{13}q_{43} - q_{12}q_{42} \cdot L_4 = q_{13}(q_{13}q_{43} - q_{31}) \) and \( v_2^{[4]} \) any nonzero real number. Let \( \Psi^{[4]} = \begin{pmatrix} \psi^{[4]}_1, \psi^{[4]}_2, \psi^{[4]}_3, \psi^{[4]}_4 \end{pmatrix}^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_{4y} = 0 \) of the matrix \( J_4^{*} \). Then we have \( J_4^{*} \Psi^{[4]} = 0 \), by solving this equation for \( \Psi^{[4]} \) we get \( \Psi^{[4]} = (0, v_2^{[4]}, 0, 0)^T \), where \( v_2^{[4]} \) any nonzero real number. Now, consider:

\[ \frac{\partial f}{\partial u_4} = f_{u_4}(X, u_4) = (0, -y, 0, 0)^T \]

Thus, \( f_{u_4}(E_4, u_4^0) = (0, 0, 0, 0)^T \) and hence \( (\Psi^{[4]})^T f_{u_4}(E_4, u_4^0) = 0 \).

So, according to Sotomayor’s theorem the saddle-node bifurcation cannot occur, while the first condition of transcritical bifurcation is satisfied. Now, since

\[ Df_{u_4}(X, u_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Further, it is observed

\[ Df_{u_4}(E_4, u_4^0)\Psi^{[4]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} v_2^{[4]} \end{pmatrix} \]

\[ (\Psi^{[4]})^T [Df_{u_4}(E_4, u_4^0)\Psi^{[4]}] = (0, 0, 0, 0) \]

Now, by substituting \( \Psi^{[4]} \) in (24) we get

\[ D^2f(E_4, u_4^0)(\Psi^{[4]}, \Psi^{[4]}) = (u_3, u_4, 0, u_5)^T \]

Where

\[ U_4 = 2q_{13}L_3(1-u_{10})v_2^{[4]} \]

\[ U_4 = -2u_{11}L_3(u_{11}u_{21} - u_{12}u_{31})v_2^{[4]} \]

Hence, it is obtained that:

\[ (\Psi^{[4]})^T [D^2f(E_4, u_4^0)(\Psi^{[4]}, \Psi^{[4]})] = U_4\Psi^{[4]} \neq 0 \]

Thus, according to Sotomayor’s theorem system (2) has transcritical bifurcation at \( E_4 \) with the parameter \( u_4 = u_4^* \).

**Remark:** According to Sotomayor’s theorem system (2) has no bifurcation at the nonhyperbolic equilibrium point \( E_4 \) for different parameter values, and that ensure the nonhyperbolic is a necessary but not sufficient condition for bifurcation to occur.

**Theorem 14:** system (2) has no bifurcation at equilibrium point \( E_6 \).

Proof: According to the Jacobian matrix \( J_6 \) given by Eq. (16a) the system (2) at the equilibrium point \( E_6 \) has zero eigenvalue (say \( \lambda_{6y} = 0 \)) at \( u_6 = u_6^0 \) and the Jacobian matrix \( J_6 \) with \( u_6 = u_6^0 \) becomes:

\[ J_6^* = J(u_6 = u_6^0) = (h_{ij})_{4 \times 4} \]

Where \( h_{ij} = d_{ij} \) for all \( i, j = 1, 2, 3, 4 \) with \( h_{12} = u_{12}^0 \).

Let \( \Psi^{[6]} = \begin{pmatrix} \psi^{[6]}_1, \psi^{[6]}_2, \psi^{[6]}_3, \psi^{[6]}_4 \end{pmatrix}^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_{6y} = 0 \). Thus \( J_6 \Psi^{[6]} = 0 \), which gives:

\[ \Psi^{[6]} = \begin{pmatrix} \psi^{[6]}_1, \psi^{[6]}_2, \psi^{[6]}_3, \psi^{[6]}_4 \end{pmatrix} \]

Where \( L_{60} = h_{34}h_{12} - h_{32}h_{14} \), \( L_{61} = h_{34}h_{12} - h_{32}h_{14} \) and \( \psi^{[6]} \) any nonzero real number. Let \( \Psi^{[6]} = \begin{pmatrix} \psi^{[6]}_1, \psi^{[6]}_2, \psi^{[6]}_3, \psi^{[6]}_4 \end{pmatrix}^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_{6y} = 0 \). Thus \( J_6^{*} \Psi^{[6]} = 0 \), which gives:

\[ \Psi^{[6]} = \begin{pmatrix} \psi^{[6]}_1, \psi^{[6]}_2, \psi^{[6]}_3, \psi^{[6]}_4 \end{pmatrix} \]

Where

Further, it is observed

\[ Df_{u_4}(E_4, u_4^0)\Psi^{[4]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} v_2^{[4]} \end{pmatrix} \]

\[ (\Psi^{[4]})^T [Df_{u_4}(E_4, u_4^0)\Psi^{[4]}] = (0, 0, 0, 0)^T \]

\[ (\Psi^{[4]})^T [D^2f(E_4, u_4^0)(\Psi^{[4]}, \Psi^{[4]})] = (0, 0, 0, 0)^T \]

\[ (\Psi^{[4]})^T [D^2f(E_4, u_4^0)(\Psi^{[4]}, \Psi^{[4]})] = (0, 0, 0, 0)^T \]

\[ (\Psi^{[4]})^T [D^2f(E_4, u_4^0)(\Psi^{[4]}, \Psi^{[4]})] = (0, 0, 0, 0)^T \]

\[ (\Psi^{[4]})^T [D^2f(E_4, u_4^0)(\Psi^{[4]}, \Psi^{[4]})] = (0, 0, 0, 0)^T \]
\[ \psi^{[6]} = \left( 0, -\frac{h_{21} \psi^{[6]}_3, \psi^{[6]}_y}{h_{34}}, 0 \right)^T \]

Where \( \psi^{[6]}_3 \) any non-zero real number. Now, consider:
\[ \frac{\partial f}{\partial u_6} = f_u(X, u_6) = (0, 0, 0, 0)^T \]
Thus, \( f_u(E_6, u_6) = (0, 0, 0, 0)^T \) and hence
\[ (\psi^{[6]})^T f_u(E_6, u_6) = xz \psi^{[6]}_3. \]

Now, by substituting \( \psi^{[6]} \) in (24) we get
\[ D^2 f(E_6, u_6)^{(\psi^{[6]}, \psi^{[6]})} = (U_6, 0, 0, U_7)^T \]
Where
\[ U_6 = -\frac{u_2(2L_0 + (h_{21} + h_{34})L_0)}{u_6L_{11}} \left( \psi^{[6]} \right)^2 \]
\[ U_7 = -\frac{s_3(h_{34}L_0 + L_{11}) + u_6(h_{34}L_0 + L_{11})}{L_{11}} \left( \psi^{[6]} \right)^2 \]

Hence, it is obtained that:
\[ (\psi^{[6]})^T D^2 f(E_6, u_6)^{(\psi^{[6]}, \psi^{[6]})} = 0 \]

Thus, according to Sotomayor’s theorem system (2) has no bifurcation at \( E_6 \) with the parameter \( u_6 = u_6^* \).

7. The Hopf bifurcation analysis

In this section, the occurrence of Hopf bifurcation of system (2) near the positive equilibrium point \( E_6 \) is studied below.

**Theorem 15:** Assume that the following conditions are hold:
\[ \Gamma_1 > \frac{d_2 d_4 d_{41}}{d_3 \Gamma_6} \]  
\[ \Gamma_2 > \frac{d_1 (d_1 d_4 d_{41} + d_4 \Gamma_6) + d_4 (d_4 d_4 d_{41} - d_1 \Gamma_6 + \Gamma_6)}{d_4 \gamma_3} \]  

(29a)

(29b)

Then system (2) possesses a Hopf bifurcation around the equilibrium point \( E_6 \) when the parameter \( u_5 \) passes through \( u_5^* \), where

\[ u_5^* = \frac{1}{2 y_6 R_1} \left( -R_2 + \sqrt{R_2^2 - 4 R_1 R_3} \right) \]

With

\[ R_1 = d_1 d_4 d_{41} \]
\[ R_2 = d_1 d_4 d_4 \Gamma_6 + d_4 (d_4 d_4 d_{41} + d_4 \Gamma_6) - 2 d_1 d_4 d_{41} d_4 d_4 d_4 \]
\[ -d_4 (d_4 \Gamma_6 + d_4 \gamma_3) \]
\[ R_3 = -\Gamma_2 \Gamma_6 \left( d_3 d_4 d_{41} (\Gamma_2 + d_4 d_4 d_4 d_4 d_4) - d_4 \Gamma_6 + \Gamma_6 \right) \]
\[ +d_3 d_4 d_4 \right) \Gamma_6 - d_1 d_4 d_4 d_{41} \right) \Gamma_6 \]

Proof: According to the Hopf bifurcation theorem, the Hopf bifurcation can occur provided that:
\[ \Delta_2 > 0, \Delta_2 (u^*_i) = 0. \]

Therefore we obtain that \( \Delta_2 = 0 \) gives
\[ u_5^* y_6 R_1 + u_5 y_6 R_2 + R_3 = 0 \]

(29c)

Then Eq. (29c) has a unique positive root
\[ u_5^* = \frac{1}{2 y_6 R_1} \left( -R_2 + \sqrt{R_2^2 - 4 R_1 R_3} \right) \]

Provided that the conditions (29)(a,b) hold. Now, at \( u_5 = u_5^* \) the characteristic equation can be written as:
\[ \left( \lambda_i^* + \frac{D_i}{D_1} \right) \left( \lambda_i^* + D_i \lambda_6 + \frac{\Delta_i}{D_1} \right) = 0 \]

Which has four roots
\[ \lambda_{i1,2} = \pm i \sqrt{\frac{D_i}{D_1}}, \quad \lambda_{i3,4} = \frac{1}{2} \left( -D_i \pm \sqrt{D_i^2 - 4 \Delta_i} \right) \]

Clearly, at \( u_5 = u_5^* \) there are two pure imaginary eigenvalues \( \lambda_{i1,2} \) and \( \lambda_{i3,4} \) and two eigenvalues which are real and negative.

Now, for all values of \( u_5 \) in the neighborhood of \( u_5^* \), the roots in general of the following form:
\[ \lambda_{i1,2} = \alpha_i + i \alpha_i, \lambda_{i3,4} = \alpha_i - i \alpha_i, \lambda_{i3,4} = \frac{1}{2} \left( -D_i \pm \sqrt{D_i^2 - 4 \Delta_i} \right) \]

Clearly, \( \text{Re} \left( \lambda_{i3,4} (u_{i5}) \right) \mid_{u_{i5}=u^*_5} = \alpha_i (u_i^*) = 0, k = 1,2 \]

That means the first condition of necessary and sufficient conditions for Hopf bifurcation is satisfied at \( u_5 = u_5^* \). Now, to verify the transversality condition we must prove that
\[ \text{Re} (u_{i5}) \text{Re} (u_{i5}) + \text{Im} (u_{i5}) \text{Re} (u_{i5}) \neq 0. \]

Note that for \( u_5 = u_5^* \) we have:
\[ \alpha_i = 0, \quad \alpha_i = \frac{D_i}{D_1}, \quad \text{Im} (u_{i5}) \text{Re} (u_{i5}) + \text{Im} (u_{i5}) \text{Re} (u_{i5}) \neq 0. \]

Then
\[ \text{Re} (u_{i5}) = -D_i (u_{i5}) \]
\[ \text{Im} (u_{i5}) = 2 \frac{D_i (u_{i5})}{D_1} (D_i - 2 D_i) \]
\[ \text{Re} (u_{i5}) = -D_i (u_{i5}) - D_i (u_{i5}) D_i (u_{i5}) \]
\[ \text{Im} (u_{i5}) = \alpha_i (u_i^*) \left( D_i (u_{i5}) D_i (u_{i5}) \right) \]

Where:
\[ D_i = \frac{\text{d}D_i}{\text{d}u_{i5}} \mid_{u_{i5}=u^*_5} = 0 \]
\[ D_i = \frac{\text{d}D_i}{\text{d}u_{i5}} \mid_{u_{i5}=u^*_5} = y_6 d_{42} \]
\[ D_i = \frac{\text{d}D_i}{\text{d}u_{i5}} \mid_{u_{i5}=u^*_5} = y_6 d_{41} \]
\[ D_i = \frac{\text{d}D_i}{\text{d}u_{i5}} \mid_{u_{i5}=u^*_5} = y_6 d_{41} \Gamma_6 \]
Then we have
\[ \Theta(u_5)\bar{v}(u_5) + \Gamma(u_5)\bar{\nu}(u_5) = \]
\[ 2\gamma_s D_s D_{11} \left[ d_1 \left( D_2 - 2 \frac{D_1}{D_{11}} \right) - D_{11} D_{11} \right] \neq 0 \]

So, we obtain that the Hopf bifurcation occurs around the equilibrium point \( E_5 \) at the parameter \( u_5 = u_5^* \).

8. Numerical analysis

In this section the dynamical behavior of system (2) is studied numerically starting at different sets of initial points with different sets of parameters values. The objectives of this study are: first specify the control parameters on the dynamical behavior of system (2) and second ensure our obtained analytical results. It is observed that, for the following set of hypothetical parameters:

\[
\begin{align*}
&u_1 = 0.1, u_2 = 0.5, u_4 = 0.1, u_5 = 0.4, u_6 = 0.5, \\
&u_7 = 0.1, u_8 = 0.4, u_{10} = 0.2, u_{11} = 0.2, s_1 = 0.2
\end{align*}
\]

(29)

The solution of system (2) approaches asymptotically to the positive equilibrium point \( E_4 = (1.13, 0.46, 0.53, 1.16) \) and this is confirming our obtained analytical results as shown in Fig. (1).

![Fig. 1: Time series of the solution of system (2) that approaches asymptotically to the stable positive equilibrium point (1.13, 0.46, 0.53, 1.16)](image)

Now, in order to specify the control parameters values of system (2), the system is solved numerically for the data given in (29) with varying one parameter each time. It is observed that, for the data given in (29) with varying one of the parameter values \( u_1, u_2, u_{10} \) and \( u_{11} \), there is no change in the dynamical behavior of system (2) and the system still approaches to the positive equilibrium point and hence these parameters are not control parameters. It is observed that for the data as given in (29) with \( u_3 < 0.5 \), the solution of system (2) approaches asymptotically to \( E_4 \) as shown in Fig. (2a), however for \( u_3 > 0.5 \), the solution of system (2) approaches asymptotically to \( E_5 \) as shown in Fig. (2b). The solution of system (2) has similar behavior as that of varying \( u_3 \) when \( u_3 \) passing through 0.4.

![Fig. 2: Time series of the solution of system (2) for the data given by (29) with (a) \( u_3 = 0.49 \), which approaches to (1.26, 0.99, 0.99, 1.33) in the interior of positive octant of xzw-space, (b) \( u_3 = 0.51 \), which approaches to (1.31, 0.99, 0, 1.42) in the interior of positive octant of xzw-space](image)

For the data given in (29) with \( u_3 < 0.4 \), the solution of system (2) approaches asymptotically to \( E_3 \) as shown in Fig. (3a), however for \( u_3 > 0.4 \), the solution of system (2) approaches asymptotically to \( E_4 \) as shown in Fig. (3b), the solution of system (2) has similar behavior as that of varying \( u_3 \) when \( u_3 \) passing through 0.5.

![Fig. 3: Time series of the solution of system (2) for the data given by (29) with (a) \( u_3 = 0.39 \), which approaches to (1.31, 0.99, 0, 1.42) in the interior of positive octant of xzw-space, (b) \( u_3 = 0.41 \), which approaches to (1.26, 0, 0.99, 1.33) in the interior of positive octant of xzw-space](image)

For the data given in (29) with \( u_3 > 0.1 \), the solution of system (2) approaches asymptotically to \( E_4 \) as shown in Fig. (4a). However, for \( u_3 > 0.1 \) with the rest of parameters as given in (29), the solution of system (2) approaches asymptotically to \( E_4 \) as shown in Fig. (4b).

![Fig. 4: Time series of the solution of system (2) for the data given by (29) with (a) \( u_3 = 0.2 \), which approaches to (1.26, 0, 1, 1.33) in the interior of positive octant of xzw-space, (b) \( u_3 = 0.1 \), which approaches to (1.26, 1, 0, 1.33) in the interior of positive octant of xzw-space](image)

For the data given in (29) with \( s_3 \leq -0.22 \), the solution of system (2) approaches asymptotically to \( E_5 \) as shown in Fig. (5).
tem (2) has been discussed. The dynamical behavior of system (2) has been investigated locally as well as globally. To understand the effect of varying each parameter on the dynamics of system (2) and to confirm our obtained analytical results, system (2) has been solved numerically for a biological feasible set of hypothetical parameters values and the following results are obtained:

1) For the set of hypothetical parameters values given by (29) system (2) approaches asymptotically to stable positive equilibrium point, and hence the food web system coexists (persist).

2) It is observed that varying the parameters: u_{11}, u_2 which stand for conversion rate from death (toxic, nontoxic) phytoplankton to nutrient, the consumption rate from nontoxic phytoplankton to zoooplankton u_4 and the zooplankton natural death rate u_{11}, do not have any effect on the dynamical behavior of system (2) and the system still approaches to a positive equilibrium point.

3) As the consumption rates from nutrient to toxic phytoplankton u_3 decreases from a critical value (0.5) keeping other parameters fixed as in (29) then the toxic phytoplankton faces extinction and the solution of system (2) approaches asymptotically to equilibrium point E_2 in the Int. \( R^+_2 \) of xzw-space. While increasing u_3 from that critical value will causes extinction in the nontoxic phytoplankton species and the solution of system (2) approaches asymptotically to equilibrium point E_2 in the Int. \( R^+_2 \) of xzw-space. It is observed that the consumption rate u_3 has the same effect as u_5 with different critical value. Clearly, these critical values are bifurcation points.

4) As the consumption rate from toxic phytoplankton to zoo- plankton u_5 decreases from a critical value (0.4) keeping other parameters fixed as in (29) then the nontoxic phytoplankton faces extinction and the solution of system (2) approaches asymptotically to equilibrium point E_2 in the Int. \( R^+_2 \) of xzw-space. While increasing u_5 from that critical value will causes extinction in the toxic phytoplankton species and the solution of system (2) approaches asymptotically to equilibrium point E_2 in the Int. \( R^+_2 \) of xzw-space. It is observed that the consumption rate u_5 has the same effect as u_7 with different critical value. Clearly, these critical values are bifurcation points.

5) As the toxic phytoplankton natural death rate u_7 increases from acritical value (0.1) keeping other parameters fixed as in (29) then again the toxic phytoplankton faces extinction and the solution of system (2) approaches asymptotically to equilibrium point E_2 that means the system losses the persistence. Otherwise the solution still approaches to the positive equilibrium point. However, increasing nontoxic phytoplankton natural death rate u_7 from the same critical value with the other parameters as given in (29) has extinction effect in the nontoxic phytoplankton and the system approaches asymptotically to E_2 again that means the system losses the persistence. Otherwise the solution still approaches to the positive equilibrium point. Finally, these critical values represent bifurcation points.

6) Gradually decreasing the parameter s_1 from the critical value (−0.23) which stand for the difference between the consumption rate from toxic phytoplankton and the liberation rate of toxin substance, causes extinction in the zooplankton species and the system approaches to E_2 in the Int. \( R^+_2 \) of xzw-space. Hence, the system (2) bifurcate at that critical point.

7) As increasing the parameters u_4, u_7 with s_1 ≤ −0.22 causes extinction effect in phytoplankton (toxic, nontoxic) and zooplankton and the system approaches to E_2 on the x − axis. However decreasing the value of u_4 and increasing u_7 with s_1 ≤ −0.22 causes extinction effect in the nontoxic phytoplankton and zooplankton and the system approaches to E_1 on xz − plane. While increasing the value of u_4 and decreasing u_7 with s_1 ≤ −0.22 causes extinction effect in the
toxic phytoplankton and zooplankton and the system approaches to $E_2$ on $xz$—plane.

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References


