Existence and uniqueness of solutions for nonlinear hyperbolic fractional differential equation with integral boundary conditions

Brahim Tellab 1*, Kamel Haouam 2

1 Department of Mathematics, Ouargla University 30000 Ouargla, Algeria
2 Mathematics and Informatics Department, LAMIS Laboratory, Tebessa University, 12000 Tebessa, Algeria
*Corresponding author E-mail: brahintel@yahoo.fr

Abstract

In this paper, we investigate the existence and uniqueness of solutions for second order nonlinear fractional differential equation with integral boundary conditions. Our result is an application of the Banach contraction principle and the Krasnoselskii fixed point theorem.

Keywords: Fractional Derivatives; Contraction Principle; Fixed Point Theorem; Integral Equation.

1. Introduction

Fractional differential equations have been of great interest and attracted many researchers in recent years; this is due to the development of the above cited concept. It has found applications in several different disciplines as physics, engineering, economics, electrochemistry, electromagnetism etc. (See [3, 4, 9, 13, 20]). Such equation have recently proved to be valuable tools in modelling of many phenomena. (See papers [2, 7, 12, 14, 16, 20]). In [10], Benchohra and Ouaar discussed the existence and uniqueness of solutions to the boundary value problem:

\[ C\mathcal{D}_t^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,1], \quad \alpha \in (0, 1], \quad y(0) = \mu_0, \quad y(T) = \nu, \]

(1)

\[ y(0) = \mu_0 \int_0^T y(s)ds = y(T). \]

(2)

\[ C\mathcal{D}_t^\alpha \]

is the Caputo fractional derivative \( f : J \times R \to R \) is continuous function and \( \mu \in R^2 \).

In [8], Sotoris K. Ntouyas investigated the existence and uniqueness of solution of the following problem:

\[ C\mathcal{D}_t^\alpha x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 0 < q < 1, \]

(3)

\[ x(0) = \alpha \Gamma(p+1)/\Gamma(q), \quad 0 < \eta < 1, \]

(4)

\[ C\mathcal{D}_t^\alpha \]

denotes always the Caputo fractional derivative of order \( q \).

\[ f : [0,1] \times R \to R \] is continuous function, \( \alpha \in R \) such that

\[ \alpha \neq \Gamma(p+1)/\Gamma(q), \quad \Gamma \text{ is the Euler function and } \Gamma \%

In this paper, we consider the following nonlinear fractional differential equation with integral boundary conditions:

\[ C\mathcal{D}_t^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,1], \]

(5)

\[ y(0) = \int_0^1 y(s)ds \]

(6)

\[ y(1) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)ds \]

(7)

Where \( C\mathcal{D}_t^\alpha \)

is the Caputo fractional derivative of order \( \alpha \),

\[ 1 < \alpha \leq 2, \quad 0 < \beta \leq 1 \quad \text{and} \quad f : [0,1] \times R \to R \]

is continuous function.

2. Preliminaries

Now, we present some basic definitions and lemmas of fractional calculus which will be used in our theorems [11, 4, and 18].

Definition 2.1: For a differentiable function \( h : [0, +\infty) \to R \), the Caputo derivative of fractional order \( \alpha \) is defined by

\[ C\mathcal{D}_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds, \]

\[ n-1 < \alpha < n, \quad n = [\alpha] + 1, \]

Where \([\alpha] \)

denotes the integer part of \( \alpha \) and \( \Gamma \)

is the gamma function.

Definition 2.2: The Riemann-Liouville fractional integral of order \( \alpha \) is given by

\[ I^\alpha \]

\[ h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \]

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Where $h : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, provided the integral exists?

**Lemma 2.1:** [19] Let $\alpha > 0$, then the differential equation $C^{\alpha} h(t) = 0$ has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^{n-1}, \quad c_1 \in \mathbb{R}, i = 0, 1, 2, \ldots, n - 1, n = \lceil \alpha \rceil + 1.$$

**Lemma 2.2:** [19] Let $\alpha > 0$, then

$$t^\alpha C^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

Where $c_1 \in \mathbb{R}, i = 0, 1, 2, \ldots, n - 1, n = \lceil \alpha \rceil + 1.$

**Lemma 2.3:** Let $1 < \alpha \leq 2$ and let $h : J \times \mathbb{R} \to \mathbb{R}$ be a given continuous function. Then, the boundary-value problem

$$C^{\alpha} y(t) = h(t), \quad t \in J$$

with initial conditions $y(0) = 0, y'(0) = 0$ has a unique solution defined by:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_0 + c_1 t$$

has a unique solution defined by:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_0$$

Where the simplification we obtain:

$$c_1 = -2 \int_0^1 (1-s)^{\alpha-1} h(t) dt.$$  \hfill (13)

Now, we operate (10) and (11), we get

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (s-r)^{\alpha-1} h(r) dr ds$$

that is to say:

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (s-r)^{\alpha-1} h(r) dr ds$$

after the simplification we obtain:

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds + c_0 + c_1$$

which may be written,

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (s-r)^{\alpha-1} h(r) dr ds$$

(14) becomes:

$$\gamma_1 y_0 + \gamma_2 c_1$$

Using (13), we obtain

$$c_0 = \frac{1}{\gamma_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds$$

- $\frac{1}{\gamma_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds + \frac{2 \gamma_2}{\gamma_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds.$
A combination of (11), (13) and (15) leads to

\[ y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds \]

\[ + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^t (t-s)^{\alpha-1} (r-s)^{\beta-1} h(s) \, ds \, dr \]

\[ - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds \]

\[ + \left[ \frac{2\gamma_2}{\Gamma(\alpha) \Gamma(\beta)} - \frac{2r}{\alpha \Gamma(\alpha)} \right] \int_0^t (t-s)^{\alpha-1} h(s) \, ds, \]

i.e.,

\[ y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds + \]

\[ \int_0^t \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} (r-s)^{\beta-1} (t-s)^{\alpha-1} \, ds \, dr - \]

\[ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \left[ \frac{2\gamma_2}{\Gamma(\alpha) \Gamma(\beta)} - \frac{2r}{\alpha \Gamma(\alpha)} \right] \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \, ds \quad \blacksquare \]

3. Existence and uniqueness result

**Theorem 3.1**: (Fixed point theorem of Banach) [6] Let \( X \) be a Banach space and \( T : X \to X \) a contracting mapping. Then \( T \) has a unique fixed point i.e.,

\[ T \exists x \in X : Tx = x. \]

Our first result of existence is based on theorem of Banach contracting application

**Theorem 3.2**: Suppose that the function \( f : [0,1] \times R \to R \) is continuous and there is a constant \( L > 0 \) such that:

\[ |f(t, x) - f(t, y)| \leq L |x - y|, t \in [0,1], x, y \in R. \]

If \( LA < 1 \), then the boundary value problem (5)-(7) has a unique solution, where

\[ A = \frac{1}{\Gamma(\alpha+1)} \left[ \frac{B(\beta, \alpha)}{\Gamma(\alpha+\beta) \Gamma(\beta)} + \frac{1}{\Gamma(\alpha+1)} \right] \]

\[ + \frac{2\gamma_2}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+2)} + \frac{2}{\alpha \Gamma(\alpha+2)} \]

(16)

**Proof of theorem 3.2**

We define the operator \( F \), by:

\[ (Fx)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds \]

\[ + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^t (t-s)^{\alpha-1} (r-s)^{\beta-1} f(s, y(s)) \, ds \, dr \]

\[ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \]

\[ + \left[ \frac{2\gamma_2}{\Gamma(\alpha) \Gamma(\beta)} - \frac{2r}{\alpha \Gamma(\alpha)} \right] \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) \, ds, \quad t \in [0,1]. \]

If we put \( \sup_{t \in [0,1]} |f(t, 0)| = M \) we show that \( FB_p \subset B_p \), where \( B_p = \{ y \in C([-1,1], \mathbb{R}) : y \leq \rho \} \) and \( \rho \geq \frac{MA}{1-LA} \)

Let \( y \in B_p, \ t \in [0,1] \). this leads to write:

\[ \|Fy(t)\| \leq \sup_{t \in [0,1]} \left[ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds \right] \]

\[ + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^t (t-s)^{\alpha-1} (r-s)^{\beta-1} f(s, y(s)) \, ds \, dr \]

\[ + \left[ \frac{2\gamma_2}{\Gamma(\alpha) \Gamma(\beta)} - \frac{2r}{\alpha \Gamma(\alpha)} \right] \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} f(s, y(s)) \, ds \]

\[ + \frac{2r}{\alpha \Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} (1-s)^{\alpha-1} f(s, y(s)) \, ds \]

\[ \leq \sup_{t \in [0,1]} \left[ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \left( \|f(s, y(s)) - f(s, 0)\| + \|f(s, 0)\| \right) ds \right] \]

\[ + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^t (t-s)^{\alpha-1} (r-s)^{\beta-1} f(s, y(s)) \, ds \, dr \]

\[ + \left[ \frac{2\gamma_2}{\Gamma(\alpha) \Gamma(\beta)} - \frac{2r}{\alpha \Gamma(\alpha)} \right] \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} \]

\[ \|f(s, y(s)) - f(s, 0)\| + \|f(s, 0)\| \, ds \]

\[ + \left[ \frac{2\gamma_2}{\Gamma(\alpha) \Gamma(\beta)} - \frac{2r}{\alpha \Gamma(\alpha)} \right] \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} \]

\[ \|f(s, y(s)) - f(s, 0)\| + \|f(s, 0)\| \, ds \]

\[ \leq \sup_{t \in [0,1]} \left[ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \left( \|f(s, y(s)) - f(s, 0)\| + \|f(s, 0)\| \right) ds \right] \]

(18)

To put \( a = \frac{r - 1}{1 - r} = \), i.e.,

\[ 1 - r = (1-u)(1-s), \ dr = (1-s) \, du \]

we obtain:

\[ \int_0^1 (1-s)^{\beta-1} (r-s)^{\alpha-1} dr \]

\[ = \int_0^1 (1-s)^{\alpha+\beta-1} (1-u)^{\beta-1} du \]

(19)
By substitution in (18), and after simplification, we get

\[
\| F(y) - (Fy) \| \\
\leq \sup_{t \in [0,1]} \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \| f(s, x(s)) - f(s, y(s)) \| ds \\
+ \frac{2[2]}{\Gamma(\alpha + 1) \Gamma(\beta)} \int_0^1 (1-r)^{\beta-1} (1-x)^{\alpha-1} \right. \\
\left. \int_0^1 (1-s)^{\alpha-1} | f(s, x(s)) - f(s, y(s)) | ds \\
+ \frac{2[2]}{\Gamma(\alpha + 1) \Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} | f(s, x(s)) - f(s, y(s)) | ds \\
+ \frac{2}{\Gamma(\alpha + 1) \Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} | f(s, x(s)) - f(s, y(s)) | ds \\
\leq L\| x - y \| \right. \\
\left. + \frac{1}{\Gamma(\alpha + 1)} + \frac{2[2]}{\Gamma(\alpha + 1) \Gamma(\beta)} \\
+ \frac{2}{\Gamma(\alpha + 1) \Gamma(\beta)} \\
= LA \| x - y \| \\
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\]
\[ \rho^* \geq \left\| \frac{1}{\Gamma(\alpha+1)} \right\| \frac{B(\beta, \alpha)}{\Gamma(\alpha+\beta)} + \frac{2|b_2|}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha+1)} \right\| \]

and we consider the set

\[ B_{\rho^*} = \left\{ y \in C\left([-0,1], R \right) : \|y\| \leq \rho^* \right\}. \]

We define two operators \( P \) and \( Q \) on \( B_{\rho^*} \) by:

\[ (Py)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds, \quad t \in [0,1] \]

\[ (Qy)(t) = \frac{1}{\gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-r)^{\beta-1}(t-r)^{\alpha-1}drds + \frac{2|b_2|}{\Gamma(\alpha+1)} \int (t-s)^{\alpha-1}f(s, y(s))ds \]

Let \( x, y \in B_{\rho^*} \), we have

\[ \|Px + Qy\| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1}ds \]

\[ + \left\| \frac{1}{\Gamma(\alpha+2)} \right\| \frac{B(\beta, \alpha)}{\Gamma(\alpha+\beta)} + \frac{2|b_2|}{\Gamma(\alpha+1)} \int_0^1 (t-s)^{\alpha-1}ds \]

\[ \leq \left\| \frac{1}{\Gamma(\alpha+1)} \right\| \frac{B(\beta, \alpha)}{\Gamma(\alpha+\beta)} + \frac{2|b_2|}{\Gamma(\alpha+2)} \int_0^1 (t-s)^{\alpha-1}ds \]

\[ \leq \rho^* \]

Then, \( Px + Qy \in B_{\rho^*} \), we have

\[ \|Qx - Qy\| \leq L \|x - y\| \times \left\{ \frac{B(\beta, \alpha)}{\Gamma(\alpha+\beta)} + \frac{1}{\gamma(\alpha)\Gamma(\beta)} \right\} \]

By exploiting (25), we deduce that \( Q \) is a contraction. According to the definition of the operator \( P \), we deduce that the continuity of \( f \) implies that of \( P \). In addition, we have:

\[ \|P^*\| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1}ds \]

which implies that \( P \) is uniformly bounded.

Now we show that \( P \) is compact. We have

\[ (Py)(t_1) - (Py)(t_2) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t_1-s)^{\alpha-1} f(s, y(s))ds \]

\[ - \frac{1}{\Gamma(\alpha)} \int_0^1 (t_2-s)^{\alpha-1} f(s, y(s))ds \]

Taking into account the condition \((H_1)\), we set

\[ f^* = \sup_{(t, x) \in [0,1] \times B_{\rho^*}} |f(t, x)| \]

Then we can write

\[ \|Py(t_1) - (Py)(t_2)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} f(s, y(s))ds \]

\[ + \int_0^1 (t_2-s)^{\alpha-1} f(s, y(s))ds \]

\[ \leq \int_0^1 (t_1-s)^{\alpha-1} f(s, y(s))ds \]

\[ + \int_0^1 (t_2-s)^{\alpha-1} f(s, y(s))ds \]

a simple calculation leads to:

\[ \|Py(t_1) - (Py)(t_2)\| \leq \frac{f^*}{\Gamma(\alpha+1)} \left| t_2 - t_1 \right|^\alpha \]

(27)

The second member of (27) is independent of \( y \) and tends to zero when \( t_2 - t_1 \rightarrow 0 \), so \( P \) is equicontinuous. Using the Arzela-Ascoli theorem, we deduce that \( P \) is compact in \( B_{\rho^*} \). Thus all the assumptions of the fixed point theorem of Krasnoselskii are satisfied. Which implies that the boundary value problem (5)-(7) has a unique solution on \([0,1]\).

References


