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Research paper

A prey - partially dependent predator with a reserved zone: modelling and analysis

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Abstract

In this paper, a mathematical model consisting of a prey-partially dependent predator has been proposed and analyzed. It is assumed that the prey moving between two types of zones, one is assumed to be a free hunting zone that is known as an unreserved zone and the other is a reserved zone where hunting is prohibited. The predator consumes the prey according to the Beddington-DeAngelis type of functional response. The existence, uniqueness and boundedness of the solution of the system are discussed. The dynamical behavior of the system has been investigated locally as well as globally with the help of Lyapunov function. The persistence conditions of the system are established. Local bifurcation near the equilibrium points has been investigated. Finally, numerical simulation has been used to specify the control parameters and confirm the obtained results.

Keywords: *Local Bifurcation; Prey-Predator; Persistent; Reserved Zone; Stability.*

1. Introduction

It is well known that one of the most important subject in ecology and mathematical modeling in biology is the prey-predator interaction for which many problems still open. Since the pioneer work of Lotka-Volterra model many complex models are developed to study prey-predator interactions involving different types of factors [1]. The existence of reserved zone to protect the prey has become an important factor of prey-predator systems and its effects on stability have been focused in several models [2-5]. Most of these studies have been observed that use of refuges by prey has a stabilizing effect on prey-predator dynamics. Later on, Daga et al [6] investigated the local and global stability of Dubey model given in [3], where the carrying capacity in an unreserved zone is proportional with prey density. They assumed that the predator is wholly dependent on the prey species in an unreserved zone.

Although the predator density dependent migration is not considered a lot in many models of prey refusal, it is observed that refuge has a stabilizing effect on the equilibrium for a simple Lotka-Volterra model. The role of predator density dependent migration in a generalized prey-predator system is investigated by Mukherjee [7]. He obtained the condition which influences the persistence of all the populations in general prey-predator system. Actually the prey population in prey-predator models facing extinction due to the effect of many factors, so in order to keep this species saves from extinction, suitable restrictions on these factors are considered. The construction of reserve zone/refuges and free zone in a given habitat is one of these restrictions, in which the predator density dependent migration of prey population plays a key role for the survival of the populations.

Recently, in our previous paper [8], we proposed and studied a mathematical model consisting of prey-wholly dependent predator with a reserved zone. The stability analysis, persistence and bifurcation are investigated. In this paper however, a mathematical model for a prey - partially dependent predator system with a reserved zone has been proposed and analyzed. It is assumed that the migration of prey species from an unreserved zone to a reserved zone is proportional with the predator density.

2. Model formulation

Consider a prey-predator system in which the predator feeds on variety of food resources including the prey species, which living in a habitat consisting of two zones namely reserved zone and an unreserved zone. In order to formulate the mathematical model that describes the above real system the following hypotheses are adopted:

1) The prey in a reserved zone is capable of reproducing in logistic fashion with carrying capacity $K>0$ and intrinsic growth rate $r_1>0$. While the prey in an unreserved zone / free zone is capable of reproducing in logistic fashion with carrying capacity $L>0$ and intrinsic growth rate $r_2>0$.
2) The transition of prey from an unreserved zone to reserved zone is proportional with a natural moving rate $\alpha>0$ as well as predator density, while the transition in opposite direction is proportional with a natural moving rate $\beta>0$ only. However, the transition of predator species from an unreserved zone is not allowed.
3) In the absence of prey species the predator growth logistically with carrying capacity $M>0$ and intrinsic growth rate $r_3>0$. However it consumes the prey species in an unreserved zone according to Beddington-DeAngelis type of functional response with maximum attack rate $a>0$, half-saturation constant $b>0$ and a scale of the impact of the predator interference that given by $c>0$. Finally, in case of lack of resources the predator will decay exponentially with a death rate given by $d>0$.

Now, let $x(t)$ be the density of prey species in an unreserved zone, $y(t)$ be the density of prey species in a reserved zone and $z(t)$ be the density of predator species at time $t \geq 0$, then according to the above hypothesis the dynamics of the above system can be describe by the following set of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\left(1-\frac{x}{K}\right)-(\alpha+z)x+\beta y-\frac{ax}{b+x+cz}z = F_1(x,y,z) \\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{L}\right)+(\alpha+z)x-\beta y = F_2(x,y,z) \\ \frac{dz}{dt} &= r_3 z\left(1-\frac{z}{M}\right)+\frac{eaxz}{b+x+cz}-dz = F_3(x,y,z) \end{aligned} \tag{1}$$

Here $0<e<1$ represents the conversion rate constant of food from prey to predator. The system is considered with following set of initial conditions $x(t)\geq 0$, $y(t)\geq 0$ and $z(t)\geq 0$. Clearly the interaction functions in the right hand side of system (1) given by the vector $F=(F_1,F_2,F_3)^t$ are continuously differential function on R_+^3, Hence they are Lipschitizian. Therefore the solution of system (1) exists and is unique. Further, all the solutions of system (1) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem 1: *All the solutions of system (1) which initiate in R_+^3 are uniformly bounded.*

Proof: Let $(x(t),y(t),z(t))$ be any solution initiate in R_+^3 and consider the function

$$w(t)=x(t)+y(t)+\frac{1}{e}z(t)$$

By differentiate $w(t)$ with respect to time and then simplifying the resulting terms we get that

$$\frac{dw}{dt}=\frac{dx}{dt}+\frac{dy}{dt}+\frac{1}{e}\frac{dz}{dt}$$

$$\frac{dw}{dt}=(r_1+1)x-\frac{r_1}{K}x^2+(r_2+1)y-\frac{r_2}{L}y^2+\left(\frac{r_3}{e}+1\right)z-\frac{r_3}{eM}z^2-x-y-\left(1+\frac{d}{e}\right)z$$

$$\begin{aligned}\frac{dw}{dt} &\leq (r_1+1)x\left[1-\frac{x}{K(r_1+1)/r_1}\right]+(r_2+1)y\left[1-\frac{y}{L(r_2+1)/r_2}\right] \\ &+\left(\frac{r_3+e}{e}\right)z\left[1-\frac{z}{M(r_3+e)/r_3}\right]-\mu_1\left(x+y+\frac{z}{e}\right)\end{aligned}$$

Where $\mu_1=\min.\{1,(e+d)\}$. Now, since the logistic terms are bounded, then straightforward computation shows that

$$\frac{dw}{dt}+\mu_1 w \leq \frac{K(r_1+1)^2}{4r_1}+\frac{L(r_2+1)^2}{4r_2}+\frac{M(r_3+e)^2}{4er_3}=\mu_2$$

Consequently by using the Gronwall lemma, we obtain that $w(t)=\frac{\mu_2}{\mu_1}$ for sufficiently large t. Hence all the species are uniformly bounded for any initial value in R_+^3.

3. Stability analysis and persistence

There are at most four non-negative equilibrium points of system (1), the existence conditions and stability analyses of them are described below:

The vanishing equilibrium point $E_0=(0,0,0)$ always exists.

The predator free equilibrium point $E_1=(\bar{x},\bar{y},0)$, where

$$\bar{y}=\frac{\bar{x}}{\beta}\left[\frac{r_1\bar{x}}{K}+\alpha-r_1\right] \tag{2}$$

While $\bar{x}$ is a positive root of the third degree polynomial

$$\frac{A_1^2 r_2}{L}x^3+\frac{2A_1A_2r_2}{L}x^2-\left[(r_2-\beta)A_1-\frac{A_2^2 r_2}{L}\right]x-\left[(r_2-\beta)A_2+\alpha\right]=0 \tag{3}$$

Here $A_1=\frac{r_1}{K\beta}>0$ and $A_2=\frac{\alpha-r_1}{\beta}$, exists uniquely in the positive quadrant of $xy-$ plane if and only if one of the following sets of conditions is satisfied

$$\alpha > r_1 \tag{4}$$

Or

$$\begin{aligned} &0<r_2<\frac{\beta r_1}{r_1-\alpha} \\ &(\alpha-r_1)^2 r_2 K < (r_2-\beta)r_1\beta L \\ &\frac{r_1\bar{x}}{K}+\alpha > r_1 \end{aligned} \tag{5}$$

The prey free equilibrium point $E_2=(0,0,\hat{z})$ where

$$\hat{z}=\frac{M}{r_3}(r_3-d) \tag{6a}$$

Exists uniquely on the positive direction of $z-$ axis provided that:

$$r_3 > d \tag{6b}$$

The positive equilibrium point, $E_3=(x^*,y^*,z^*)$ exists uniquely in the interior of R_+^3 ($Int.R_+^3$) provided that there is a positive solution to the following set of algebraic equations.

$$\begin{aligned} &r_1 x\left(1-\frac{x}{K}\right)-(\alpha+z)x+\beta y-\frac{axz}{b+x+cz}=0 \\ &r_2 y\left(1-\frac{y}{L}\right)+(\alpha+z)x-\beta y=0 \\ &r_3\left(1-\frac{z}{M}\right)+\frac{eax}{b+x+cz}-d=0 \end{aligned} \tag{7}$$

Solving the third and second equations with respect to x and y respectively shows that:

$$x^*=\frac{\gamma_1\gamma_2}{eaM-\gamma_1},\quad y^*=\frac{L(r_2-\beta)}{2r_2}+\frac{L}{2r_2}\sqrt{(r_2-\beta)+4\frac{r_3\gamma_1\gamma_2\gamma_3}{L(eaM-\gamma_1)}} \tag{8}$$

Here $\gamma_1=(d-r_3)M+r_3z^*$, $\gamma_2=b+cz^*>0$ and $\gamma_3=\alpha+z^*>0$, however z^* is a positive root of the first equation. Clearly x^* and y^* will be positive provided that

$$\frac{(r_3-d)}{r_3}M<z^*<\frac{ea+(r_3-d)}{r_3}M \tag{9}$$

Now, in order to investigate the local stabilities of the above equilibrium points, we need to consider the Jacobian matrix $DF=J(x,y,z)$ of system (1) that can be written as

$$J(x,y,z)=(C_{ij})_{3\times 3} \tag{10}$$

Where

$C_{11}=r_1-\frac{2r_1}{K}x-(\alpha+z)-\frac{az(b+cz)}{(b+x+cz)^2}$, $C_{12}=\beta$, $C_{13}=-x-\frac{a(b+x)x}{(b+x+cz)^2}$,

$C_{21}=\alpha+z$,

$C_{22}=r_2-\frac{2r_2}{L}y-\beta$, $C_{23}=x$, $C_{31}=\frac{ae(b+cz)z}{(b+x+cz)^2}$, $C_{32}=0$,

$C_{33}=z\left[-\frac{r_3}{M}-\frac{acex}{(b+x+cz)^2}\right]+r_3\left(1-\frac{z}{M}\right)+\frac{eax}{b+x+cz}-d$.

Clearly, straightforward computation shows that the Jacobian matrix near the vanishing equilibrium point $E_0=(0,0,0)$ is

$$J(E_0)=\begin{pmatrix} r_1-\alpha & \beta & 0 \\ \alpha & r_2-\beta & 0 \\ 0 & 0 & r_3-d \end{pmatrix} \tag{11}$$

Thus the characteristic equation can be written as:

$$\left[\lambda^2-\left((r_1-\alpha)+(r_2-\beta)\right)\lambda+(r_1-\alpha)(r_2-\beta)-\alpha\beta\right][r_3-d-\lambda]=0 \tag{12}$$

Hence the eigenvalues of $J(E_0)$ are

$$\lambda_{0x},\lambda_{0y}=\frac{(r_1-\alpha)+(r_2-\beta)}{2} \pm\frac{1}{2}\sqrt{[(r_1-\alpha)+(r_2-\beta)]^2-4[(r_1-\alpha)(r_2-\beta)-\alpha\beta]} \tag{13a}$$

$$\lambda_{0z}=r_3-d \tag{13b}$$

Here λ_{0x}, λ_{0y} and λ_{0z} represent the eigenvalues of $J(E_0)$ in the x-direction, y-direction and z-direction respectively. Clearly all the above eigenvalues will be negative provided that the following conditions hold

$$r_1<\alpha \tag{14a}$$

$$r_2<\beta \tag{14b}$$

$$r_1r_2>r_1\beta+r_2\alpha \tag{14c}$$

$$r_3<d \tag{14d}$$

Since condition (14c) can't satisfy simultaneously with conditions (14a) and (14b), hence $J(E_0)$ has one positive eigenvalues and then E_0 is a saddle point.

The Jacobian matrix of the system (1) near the predator free equilibrium point $E_1=(\bar{x},\bar{y},0)$ can be written as

$$J(E_1)=\begin{pmatrix} r_1-\frac{2r_1}{K}\bar{x}-\alpha & \beta & -\bar{x}\left(1+\frac{a}{b+\bar{x}}\right) \\ \alpha & r_2-\frac{2r_2}{L}\bar{y}-\beta & \bar{x} \\ 0 & 0 & r_3+\frac{ea\bar{x}}{b+\bar{x}}-d \end{pmatrix}=(b_{ij}) \tag{15}$$

Therefore the characteristic equation and the eigenvalues of $J(E_1)$ can be written respectively as

$$\left[\lambda^2-\left[\left(r_1-\frac{2r_1}{K}\bar{x}-\alpha\right)+\left(r_2-\frac{2r_2}{L}\bar{y}-\beta\right)\right]\lambda +\left(r_1-\frac{2r_1}{K}\bar{x}-\alpha\right)\left(r_2-\frac{2r_2}{L}\bar{y}-\beta\right)-\alpha\beta\right]\left[r_3+\left(\frac{ea\bar{x}}{b+\bar{x}}-d\right)-\lambda\right]=0 \tag{16}$$

$$\lambda_{1x},\lambda_{1y}=\frac{b_{11}+b_{22}}{2}\pm\frac{1}{2}\sqrt{[b_{11}+b_{22}]^2-4[b_{11}b_{22}-b_{12}b_{21}]} \tag{17a}$$

$$\lambda_{1z}=b_{33}=r_3+\frac{ea\bar{x}}{b+\bar{x}}-d \tag{17b}$$

Now, since $r_1-\frac{2r_1}{K}\bar{x}-\alpha<0$ due to existence conditions (4-5), thus all these eigenvalues are negative or have negative real parts and hence E_1 is locally asymptotically stable in R^3_+ provided that

$$r_2<\frac{2r_2}{L}\bar{y}+\beta \tag{18a}$$

$$r_3+\frac{ea\bar{x}}{b+\bar{x}}<d \tag{18b}$$

$$\left(r_1-\frac{2r_1}{K}\bar{x}-\alpha\right)\left(r_2-\frac{2r_2}{L}\bar{y}\right)>\left(r_1-\frac{2r_1}{K}\bar{x}\right)\beta \tag{18c}$$

The Jacobian matrix of the system (1) near the prey free equilibrium point $E_2=(0,0,\hat{z})$ can be written as

$$J(E_2)=\begin{pmatrix} r_1-(\alpha+\hat{z})-\frac{a\hat{z}}{(b+c\hat{z})} & \beta & 0 \\ \alpha+\hat{z} & r_2-\beta & 0 \\ \frac{ae\hat{z}}{(b+c\hat{z})} & 0 & -\frac{r_3\hat{z}}{M} \end{pmatrix}=(c_{ij}) \tag{19}$$

Therefore the characteristic equation and the eigenvalues of $J(E_2)$ can be written respectively as

$$[\lambda^2-(c_{11}+c_{22})\lambda+(c_{11}c_{22}-c_{12}c_{21})]\left[\frac{-r_3\hat{z}}{M}-\lambda\right]=0 \tag{20}$$

$$\lambda_{2x},\lambda_{2y}=\frac{c_{11}+c_{22}}{2}\pm\frac{1}{2}\sqrt{[c_{11}+c_{22}]^2-4[c_{11}c_{22}-c_{12}c_{21}]} \tag{21a}$$

$$\lambda_{2z}=c_{33}=-\frac{r_3\hat{z}}{M} \tag{21b}$$

Clearly all these eigenvalues are negative or have negative real parts and hence E_2 is locally asymptotically stable in R^3_+ provided that

$$r_1<\frac{a\hat{z}}{b+c\hat{z}} \tag{22a}$$

$$r_2<\frac{r_1-\frac{a\hat{z}}{(b+c\hat{z})}}{r_1-(\alpha+\hat{z})-\frac{a\hat{z}}{(b+c\hat{z})}}\beta \tag{22b}$$

Finally, the Jacobian matrix of the system (1) near the positive equilibrium point E_3 can be written as

$$J(E_3)=(a_{ij})_{3\times 3} \tag{23}$$

Where

$a_{11}=r_1-\frac{2r_1}{K}x^*-(\alpha+z^*)-\frac{az^*(b+cz^*)}{(b+x^*+cz^*)^2}$, $a_{12}=\beta>0$,

$a_{13} = -x^*\left[1+\frac{a(b+x^*)}{(b+x^*+cz^*)^2}\right] < 0$,

$a_{21} = \alpha + z^* > 0$, $a_{22} = r_2 - \frac{2r_2}{L}y^* - \beta$, $a_{23} = x^* > 0$,

$a_{31} = \frac{ae(b+cz^*)z^*}{(b+x^*+cz^*)^2} > 0$, $a_{32} = 0$, $a_{33} = -z^*\left[\frac{r_3}{M} + \frac{acex^*}{(b+x^*+cz^*)^2}\right] < 0$

Therefore the characteristic equation of E_2 can be written as follow

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{24}$$

Where

$$A_1 = -(a_{11} + a_{22} + a_{33})$$

$$A_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33}$$

$$A_3 = -a_{33}\left[a_{11}a_{22} - a_{12}a_{21}\right] - a_{31}\left[a_{12}a_{23} - a_{13}a_{22}\right]$$

While

$$\begin{aligned}\Delta = A_1A_2 - A_3 &= -(a_{11}+a_{22})\left[a_{11}a_{22} - a_{12}a_{21}\right] - (a_{11}+a_{33})\left[a_{11}a_{33} - a_{13}a_{31}\right] \\ &\quad - a_{22}a_{33}(a_{22}+a_{33}) - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31}\end{aligned}$$

From the Routh-Hurwitz criterion [1], all the roots (eigenvalues of $J(E_3)$) of Eq. (24) have negative real parts and hence $E_3 = (x^*, y^*, z^*)$ is locally asymptotically stable if and only if A_1 , A_3 and Δ are positive. Therefore in the following theorem we present the sufficient conditions of local stability of E_3 .

Theorem 2: *Suppose that the positive equilibrium point E_3 of system (1) exists in $Int.R^3_+$. Then E_3 is locally asymptotically stable if*

$$K < 2x^* \tag{25a}$$

$$L < 2y^* \tag{25b}$$

Proof: Straightforward computation gives that conditions (25a)-(25b) guarantee that a_{11} and a_{22} are negative, hence by substituting the elements of $J(E_3)$ and then doing simple calculation, we get that A_1 , A_3 and Δ are positive. Hence according to Routh-Hurwitz criterion E_3 is locally asymptotically stable in $Int.R^3_+$.

In the following the global stability of the equilibrium points of system (1) is investigated with the help of Lyapunov method. The results of this study can be summarized in the following theorems.

Theorem 3: *Suppose that the predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$ is locally asymptotically stable in the R^3_+, then it is a globally asymptotically stable provided that*

$$\frac{\beta}{\alpha}\bar{y} < \bar{x} < \frac{b(d-r_3)}{e(a+b)} \tag{26}$$

Proof Consider the following function $V_1 = c_1\left[x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right] + c_2\left[y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right] + c_3 z$, where $c_i; i = 1,2,3$ are positive constants to be determined. Clearly $V_1 : R^3_+ \to R$, is a continuously differentiable positive definite real valued function with $V_1(\bar{x}, \bar{y}, 0) = 0$ and $V_1(x, y, z) > 0$ otherwise. Further, since

$$\frac{dV_1}{dt} = c_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + c_2\left(\frac{y-\bar{y}}{y}\right)\frac{dy}{dt} + c_3\frac{dz}{dt}$$

Then by substituting the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ from system (1) and then simplifying the resulting terms we obtains that

$$\begin{aligned}\frac{dV_1}{dt} &\le -c_1\frac{r_1}{K}(x-\bar{x})^2 - c_2\frac{r_2}{L}(y-\bar{y})^2 \\ &+ \frac{\bar{x}y - x\bar{y}}{x\bar{x}y\bar{y}}\left[(c_1\beta\bar{y} - c_2\alpha\bar{x})xy - (c_1\beta y - c_2\alpha x)\bar{x}\bar{y}\right] \\ &-(c_1 - c_2)xz - \left[c_3(d-r_3) - c_1\bar{x}(1+\tfrac{a}{b})\right]z \\ &-(c_1 - c_3 e)\frac{axz}{R} - c_2\frac{\bar{x}\bar{y}z}{y} - c_3\frac{r_3}{M}z^2\end{aligned}$$

Here $R = b + x + cz$. So, by choosing the positive constants as $c_1 = 1$, $c_2 = \frac{\beta\bar{y}}{\alpha\bar{x}}$ and $c_3 = \frac{a+b}{b(d-r_3)}\bar{x}$, which is positive under the local stability condition (18b). We get that

$$\begin{aligned}\frac{dV_1}{dt} &\le -\frac{r_1}{K}(x-\bar{x})^2 - \frac{\beta\bar{y}}{\alpha\bar{x}}\frac{r_2}{L}(y-\bar{y})^2 - \frac{\beta}{x\bar{x}y}(\bar{x}y - x\bar{y})^2 \\ &-\left(1 - \frac{\beta\bar{y}}{\alpha\bar{x}}\right)xz - \left(1 - \frac{e(a+b)}{b(d-r_3)}\bar{x}\right)\frac{axz}{R}\end{aligned}$$

Clearly, under the given condition $\frac{dV_1}{dt} < 0$ and $\frac{dV_1}{dt} = 0$ at $E_1 = (\bar{x}, \bar{y}, 0)$. Hence V_1 is a Lyapunov function and hence $E_1 = (\bar{x}, \bar{y}, 0)$ is a globally asymptotically stable.

Theorem 4: *Suppose that the prey free equilibrium point $E_2 = (0, 0, \hat{z})$ is a locally asymptotically stable in the R^3_+, then it is a globally asymptotically stable provided that*

$$\frac{r_1K}{4} + \frac{r_2L}{4} < \frac{r_3}{eM}(z-\hat{z})^2 \tag{27}$$

Proof. Consider the following function $V_2 = x + y + \frac{1}{e}\left[z - \hat{z} - \hat{z}ln\left(\frac{z}{\hat{z}}\right)\right]$. Clearly $V_2 : R^3_+ \to R$, is a continuously differentiable positive definite real valued function with $V_2(0,0,\hat{z}) = 0$ and $V_2(x, y, z) > 0$ otherwise. Further, since

$$\frac{dV_2}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{1}{e}\frac{z-\hat{z}}{z}\frac{dz}{dt}$$

Then by substituting the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ from system (1) and then simplifying the resulting terms we obtains that

$$\frac{dV_2}{dt} \le r_1x\left(1 - \frac{x}{K}\right) + r_2y\left(1 - \frac{y}{L}\right) - \frac{r_3}{eM}(z-\hat{z})^2$$

Now by using the boundedness of the logistic terms, it is easy to verify that $\frac{dV_2}{dt} \le 0$, and then $\frac{dV_2}{dt}$ is negative definite function. Therefore according to Lyapunov second theorem E_2 is a globally asymptotically stable in R^3_+.

Finally, in the following theorem the conditions of globally asymptotically stable for a positive equilibrium point are established.

Theorem 5: *Suppose that the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ is locally asymptotically stable in the R^3_+, then it is a globally asymptotically stable provided that*

$$\frac{az^*}{bR^*} < \frac{r_1}{K} \tag{28a}$$

$$\xi_{12}^2 < \xi_{11}\xi_{22} \tag{28b}$$
$$\xi_{13}^2 < \xi_{11}\xi_{33} \tag{28c}$$
$$\xi_{23}^2 < \xi_{22}\xi_{33} \tag{28d}$$

Here $R^* = b + x^* + cz^*$, $\xi_{11} = \frac{1}{2}\left(\frac{r_1}{K} - \frac{az^*}{bR^*}\right)$, $\xi_{12} = \frac{\beta y^* z}{\alpha x^* y}$, $\xi_{22} = \frac{\beta y^*}{2\alpha x^*}\left(\frac{r_2}{L} - \frac{x^* z^*}{yy^*}\right)$, $\xi_{13} = \frac{ae(b+cz^*)}{RR^*} - \left(1 + \frac{a(b+x^*)}{RR^*}\right)$, $\xi_{23} = \frac{\beta y^*}{\alpha y}$ and $\xi_{33} = \frac{1}{2}\left(\frac{r_3}{M} + \frac{aecx^*}{RR^*}\right)$.

Proof Consider the following function

$$V_3 = d_1\left[x - x^* - x^* ln\left(\frac{x}{x^*}\right)\right] + d_2\left[y - y^* - y^* ln\left(\frac{y}{y^*}\right)\right] + d_3\left[z - z^* - z^* ln\left(\frac{z}{z^*}\right)\right],$$

where $d_i; i = 1,2,3$ are positive constants to be determined. Clearly $V_3 : R_+^3 \to R$, is a continuously differentiable positive definite real valued function with $V_3(x^*, y^*, z^*) = 0$ and $V_3(x, y, z) > 0$ otherwise. Further, since

$$\frac{dV_3}{dt} = d_1\left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + d_2\left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + d_3\left(\frac{z - z^*}{z}\right)\frac{dz}{dt}$$

Then by substituting the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ from system (1) and then simplifying the resulting terms we obtains that

$$\begin{aligned}\frac{dV_3}{dt} &= -d_1\left[\frac{r_1}{K} - \frac{az^*}{bR^*}\right](x - x^*)^2 - d_2\left[\frac{r_2}{L} + \frac{x^* z^*}{yy^*}\right](y - y^*)^2 \\ &-d_3\left[\frac{r_3}{M} + \frac{aecx^*}{RR^*}\right](z - z^*)^2 \\ &+\left[d_3\frac{ae(b + cz^*)}{RR^*} - d_1\left(1 + \frac{a(b + x^*)}{RR^*}\right)\right](x - x^*)(z - z^*) \\ &+d_2\frac{z}{y}(x - x^*)(y - y^*) + d_2\frac{x^*}{y}(y - y^*)(z - z^*) \\ &-\frac{x^* y - xy^*}{xx^* yy^*}\left[x^* y^*(d_1\beta y - d_2\alpha x) + xy(d_2\alpha x^* - d_1\beta y^*)\right]\end{aligned}$$

By choosing the positive constants as $d_1 = 1, d_2 = \frac{\beta y^*}{\alpha x^*}, d_3 = 1$, and using the given conditions we get after some algebraic manipulation that:

$$\begin{aligned}\frac{dV_3}{dt} \le &-\left[\sqrt{\xi_{11}}(x - x^*) - \sqrt{\xi_{22}}(y - y^*)\right]^2 - \left[\sqrt{\xi_{11}}(x - x^*) - \sqrt{\xi_{33}}(z - z^*)\right]^2 \\ &-\left[\sqrt{\xi_{22}}(y - y^*) - \sqrt{\xi_{33}}(z - z^*)\right]^2 - \frac{\beta}{xx^* y}\left(x^* y - xy^*\right)^2\end{aligned}$$

Clearly, $\frac{dV_3}{dt} \le 0$ under the given conditions and $\frac{dV_3}{dt} = 0$ at the positive equilibrium point E_3. Then V_3 is a Lyapunov function and hence E_3 is a globally asymptotically stable.

In the following the persistence of system (1) is investigated. It is well known that an ecological system persists if and only if each species persists. Mathematically this means that the solution of the system do not has omega limit set in the boundary planes. Consequently, to establish the persistence conditions of system (1), we need to show whether there is a periodic dynamics in the $xy-$ plane or not. Straightforward computation shows that in the absence of predator system (1) reduces to the following subsystem in the interior of $xy-$ plane:

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1 - \frac{x}{K}\right) - \alpha x + \beta y = g_1(x, y) \\ \frac{dy}{dt} &= r_2 y\left(1 - \frac{y}{L}\right) + \alpha x - \beta y = g_2(x, y)\end{aligned} \tag{29}$$

This 2D system (29) has a unique positive equilibrium point $E_1 = (\bar{x}, \bar{y})$ in the interior of positive quadrant of $xy-$ plane, which is globally asymptotically stable [8]. Consequently, in the following theorem, the necessary and sufficient conditions, which guarantee the uniform persistence of system (1), are derived.

Theorem 6: *Suppose that the boundary points E_1 and E_2 exist, and let the following conditions hold*

$$r_3 + \frac{ea\bar{x}}{b + \bar{x}} < d \tag{30a}$$
$$\left(r_1 - (\alpha + \hat{z}) - \frac{a\hat{z}}{b + c\hat{z}}\right) r_2 - \beta\left(r_1 - \frac{a\hat{z}}{b + c\hat{z}}\right) < 0 \tag{30b}$$
$$\left(r_1 - \frac{2r_1\bar{x}}{K} - \alpha\right)\left(r_2 - \frac{2r_2\bar{y}}{L}\right) - \beta\left(r_1 - \frac{2r_1\bar{x}}{K}\right) < 0 \tag{30c}$$

Then system (1) is uniformly persistent.

Proof Suppose that w is any point in the positive octant and let $o(w)$ is the orbit through w. Let $\Omega(w)$ represents the omega limit set of the orbit through w. Since system (1) is bounded, $\Omega(w)$ is bounded.

We first claim that $E_0 \notin \Omega(w)$. If $E_0 \in \Omega(w)$, then according to Butler-McGehee lemma [9], there exists a point $u \in \Omega(w) \cap W^s(E_0)$ where $W^s(E_0)$ represents the stable manifold of E_0. Now since the $o(u)$ lies in $\Omega(w)$ and $W^s(E_0)$ is the $z-$ axis (the eigenvalue of Jacobian matrix at E_0 in the z −direction is negative due to condition (30a)) then $o(u)$ is unbounded orbit which leads to contradiction.

Now our claim is that $E_1 \notin \Omega(w)$, otherwise $E_1 \in \Omega(w)$. Since E_1 is a saddle point with stable manifold in zx −plane or yz −plane due to conditions (30a) and (30c), hence again by Butler-McGehee lemma, there is a point $u \in \Omega(w) \cap W^s(E_1)$, where $W^s(E_1)$ represents the stable manifold of E_1. Now since the $o(u)$ lies in the $\Omega(w)$ and $W^s(E_1)$ is xz −plane (similarly in case of yz −plane), hence $o(u)$ is unbounded orbit lies in the $\Omega(w)$, which leads to contradiction.

Similarly, if $E_2 \in \Omega(w)$, a contradiction will occurs due to condition (30b).

Therefor $\Omega(w)$ doesn't intersect any of the boundary planes of axis and then system (1) is persistent. In addition since system (1) is bounded then according to theorem of Butler et al [10], system (1) becomes uniformly persistent.

4. The local bifurcation analysis

In this section, an application of the Sotomayor's theorem [11] is used to investigate the occurrence of the local bifurcation near the equilibrium points of system (1). Since the existence of a non-hyperbolic equilibrium point is a necessary but not sufficient condition for bifurcation to occurs, a parameter that makes the Jacobian matrix has a zero real part eigenvalue will be adopted as a candidate bifurcation parameter as shown in the following theorems.

Consider now the Jacobian matrix of system (1) at (x, y, z) that given by Eq. (10). Then, with a straightforward computation, it is easy to verify that

$$D^2F(x, y, z)(V, V) = (\bar{\bar{d}}_{ij})_{3\times1} \tag{31}$$

Here

$$\begin{aligned}\bar{\bar{d}}_{11} &= -2\left(\frac{r_1}{K} - \frac{a(b + cz)z}{R^3}\right)v_1^2 - 2v_1v_3 \\ &-\frac{2av_1v_3}{R^3}\left[b^2 + bx + bcz + 2cxz\right] + \frac{2acv_3^2}{R^3}(b + x)x \\ \bar{\bar{d}}_{21} &= 2v_1v_3 - \frac{2r_2}{L}v_2^2 \\ \bar{\bar{d}}_{31} &= -\frac{2aev_1^2}{R^3}(b + cz)z + \frac{2aev_1v_3}{R^3}\left[b^2 + bx + bcz + 2cxz\right] \\ &-2\left[\frac{r_3}{M} + \frac{acex(b + x)}{R^3}\right]v_3^2\end{aligned}$$

Here $V=(v_1,v_2,v_3)^t$ is any vector in R_+^3. Moreover

$$D^3F(x,y,z)(V,V,V)=(\overline{\overline{\overline{d}}}_{ij})_{3\times1} \tag{32}$$

Here

$$\overline{\overline{\overline{d}}}_{11}=-\frac{6av_1^3}{R^4}(b+cz)z+\frac{6av_1^2v_3}{R^4}\left[b^2+bx+2cxz-(cz)^2\right]$$
$$-\frac{2acv_1v_3^2}{R^4}\left[b^2-x^2+bcz+2cxz\right]-\frac{6ac^2v_3^3}{R^4}(b+x)x$$

$$\overline{\overline{\overline{d}}}_{21}=0$$

$$\overline{\overline{\overline{d}}}_{31}=\frac{6aev_1^3}{R^4}(b+cz)z-\frac{6aev_1^2v_3}{R^4}\left[b^2+bx+2cxz+(cz)^2\right]$$
$$-\frac{6acev_1v_3^2}{R^4}\left[b^2-x^2+bcz+2cxz\right]+\frac{6ac^2ev_3^3}{R^4}(b+x)x$$

Theorem 7: *Assume that conditions (18a) and (18c) hold and let the parameter d passes through the value $\tilde{d}=r_3+\frac{ae\bar{x}}{b+\bar{x}}$, then system (1) near the predator free equilibrium point $E_1=(\bar{x},\bar{y},0)$ has*

1) No saddle-node bifurcation.
2) Transcritical bifurcation provided that

$$abeM\left(\frac{\tilde{b}_{13}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{23}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}\right)\neq r_3(b+\bar{x})^2+aceM\bar{x} \tag{33a}$$

3) Pitchfork bifurcation provided that

$$abeM\left(\frac{\tilde{b}_{13}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{23}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}\right)=r_3(b+\bar{x})^2+aceM\bar{x} \tag{33b}$$

Proof According to the Jacobian matrix at the predator free equilibrium point $J(E_1)$ that given by Eq. (15) and their characteristic equation that given in Eq. (16), it's easy to verify that $J(E_1)$ has zero eigenvalue $\tilde{\lambda}=0$ at $\tilde{d}=r_3+\frac{ae\bar{x}}{b+\bar{x}}$ and hence E_1 will be a non-hyperbolic point. Let $\tilde{V}=(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\tilde{\lambda}=0$ of the Jacobian matrix $\tilde{J}=J(E_1,\tilde{d})$, then

$$\left[\tilde{J}-\tilde{\lambda}I\right]\tilde{V}=0\Rightarrow\quad \tilde{V}=\left(\frac{\tilde{b}_{13}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{23}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}\tilde{v}_3,-\frac{\tilde{b}_{11}\tilde{b}_{23}-\tilde{b}_{13}\tilde{b}_{21}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}\tilde{v}_3,\tilde{v}_3\right)^t$$

where $\tilde{v}_3$ represents any nonzero real number and $\tilde{b}_{ij}=b_{ij};\ \forall i,j=1,2,3$ in Eq. (15) with $\tilde{b}_{33}=0$. Clearly we have $\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}\neq0$ due to conditions (18a) and (18c).

Let $\tilde{\Psi}=(\tilde{\psi}_1,\tilde{\psi}_2,\tilde{\psi}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\tilde{\lambda}=0$ of the transpose of Jacobian matrix $\tilde{J}^t=J^t(E_1,\tilde{d})$, then

$$\left[\tilde{J}^t-\tilde{\lambda}I\right]\tilde{\Psi}=0\Rightarrow\quad \tilde{\Psi}=\left(0,0,\tilde{\psi}_3\right)^t$$

where $\tilde{\psi}_3$ represents any nonzero real number.

Now let $X=(x,y,z)$ then since

$$F_d(X,d)=\begin{pmatrix}0\\0\\-z\end{pmatrix}\Rightarrow F_d(E_1,\tilde{d})=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Here $F_d(X,d)$ represents the derivative of $F=(F_1,F_2,F_3)^t$ with respect to d. Then we get that

$$\tilde{\Psi}^tF_d(E_1,\tilde{d})=0$$

Thus according to the Sotomayor's theorem for local bifurcation, the saddle-node bifurcation can't occur while the first condition of transcritical and pitchfork bifurcation is satisfied. Further, since

$$DF_d(X,d)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}\Rightarrow DF_d(E_1,\tilde{d})=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

Here $DF_d(X,d)$ represents the derivative of $F_d(X,d)$ with respect to $X=(x,y,z)$, consequently we get

$$\tilde{\Psi}^t\left[DF_d(E_1,\tilde{d})\tilde{V}\right]=-\tilde{\psi}_3\tilde{v}_3\neq0$$

Moreover, by substituting E_1, $\tilde{d}$ and $\tilde{V}$ in Eq. (31), it is observed that

$$\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]=\frac{2\tilde{v}_3^2\tilde{\psi}_3}{M(b+\bar{x})^2}\left[\begin{array}{l}abeM\left(\frac{\tilde{b}_{13}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{23}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}\right)\\-r_3(b+\bar{x})^2-aceM\bar{x}\end{array}\right]$$

Cleary, if condition (33a) holds then $\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]\neq0$ and hence transcritical bifurcation occurs. However if condition (33b) holds; then $\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]=0$, and hence the transcritical bifurcation can't occur. Further by substituting E_1, $\tilde{d}$ and $\tilde{V}$ in Eq. (32), it is observed that

$$\tilde{\Psi}^t[D^3F(E_1,\tilde{d})(\tilde{V},\tilde{V},\tilde{V})]=$$
$$-\frac{6ae\tilde{v}_3^3\tilde{\psi}_3}{(b+\bar{x})^3}\left[\frac{\tilde{b}_{13}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{23}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}+c\right]\left[b\,\frac{\tilde{b}_{13}\tilde{b}_{22}-\tilde{b}_{12}\tilde{b}_{23}}{\tilde{b}_{12}\tilde{b}_{21}-\tilde{b}_{11}\tilde{b}_{22}}-c\bar{x}\right]\neq0$$

Therefore pitchfork bifurcation occurs and the proof is complete.

Theorem 8: *Assume that conditions (22a) and (22b) hold and let the parameter passes through the value $\hat{\beta}=\frac{\hat{c}_{11}r_2}{(r_1-\frac{a\hat{z}}{b+c\hat{z}})}$, then system* (1) near the prey free equilibrium point $E_2=(0,0,\hat{z})$ has

1) No saddle-node bifurcation.
2) Transcritical bifurcation provided that

$$\frac{r_1}{K}\varphi_1^2\gamma_1+\frac{r_2}{L}\neq\frac{a\varphi_1\gamma_1}{(b+c\hat{z})^2}(\varphi_1\hat{z}-b\varphi_2)+\varphi_1\varphi_2(1-\gamma_1) \tag{34a}$$

3) Pitchfork bifurcation provided that

$$\frac{r_1}{K}\varphi_1^2\gamma_1+\frac{r_2}{L}=\frac{a\varphi_1\gamma_1}{(b+c\hat{z})^2}(\varphi_1\hat{z}-b\varphi_2)+\varphi_1\varphi_2(1-\gamma_1) \tag{34b}$$

$$3\varphi_1^2\hat{z}+3c\varphi_1\varphi_2\hat{z}+cb\varphi_2^2\neq3b\varphi_1\varphi_2 \tag{34c}$$

Here $\varphi_1=-\frac{\hat{c}_{12}}{\hat{c}_{11}},\ \varphi_2=\frac{\hat{c}_{12}\hat{c}_{31}}{\hat{c}_{11}\hat{c}_{33}},\gamma_1=-\frac{\hat{c}_{21}}{\hat{c}_{11}}$ and $\hat{c}_{ij}=c_{ij}$ for all $i,j=1,2,3$ in Eq. (19), where $\beta=\hat{\beta}$.

Proof According to the Jacobian matrix at the prey free equilibrium point $J(E_2)$ that given by Eq. (19) and their characteristic equation that given in Eq. (20), it's easy to verify that $J(E_2)$ has zero eigenvalue $\hat{\lambda}=0$ at $\beta=\hat{\beta}$ and hence E_2 will be a non-hyperbolic point. Let $\hat{V}=(\hat{v}_1,\hat{v}_2,\hat{v}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\hat{\lambda}=0$ of the Jacobian matrix $\hat{J}=J(E_2,\hat{\beta})$, then

$$\left[\hat{J}-\hat{\lambda}I\right]\hat{V}=0 \Rightarrow \hat{V}=\left(\varphi_1\hat{v}_2,\hat{v}_2,\varphi_2\hat{v}_2\right)^t$$

Where $\hat{v}_2$ represents any nonzero real number.

Let $\hat{\Psi}=(\hat{\psi}_1,\hat{\psi}_2,\hat{\psi}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\hat{\lambda}=0$ of the transpose of Jacobian matrix $\hat{J}^t=J^t(E_2,\hat{\beta})$, then

$$\left[\hat{J}^t-\hat{\lambda}I\right]\hat{\Psi}=0 \Rightarrow \hat{\Psi}=\left(\gamma_1\hat{\psi}_2,\hat{\psi}_2,0\right)^t$$

where $\hat{\psi}_2$ represents any nonzero real number.

Now let $X=(x,y,z)$ then since

$$F_\beta(X,\beta)=\begin{pmatrix} y\\ y\\ 0\end{pmatrix} \Rightarrow F_\beta(E_2,\hat{\beta})=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Here $F_\beta(X,\beta)$ represents the derivative of $F=(F_1,F_2,F_3)^t$ with respect to β. Then we get that

$$\hat{\Psi}^t F_\beta(E_2,\hat{\beta})=0$$

Thus according to the Sotomayor's theorem for local bifurcation, the saddle-node bifurcation can't occur while the first condition of transcritical and pitchfork bifurcation is satisfied. Further, since

$$DF_\beta(X,\beta)=\begin{pmatrix}0&1&0\\0&1&0\\0&0&0\end{pmatrix} \Rightarrow DF_\beta(E_2,\hat{\beta})=\begin{pmatrix}0&1&0\\0&1&0\\0&0&0\end{pmatrix}$$

Here $DF_\beta(X,\beta)$ represents the derivative of $F_\beta(X,\beta)$ with respect to $X=(x,y,z)$, consequently we get

$$\tilde{\Psi}^t\left[DF_d(E_1,\tilde{d})\tilde{V}\right]=-\tilde{\psi}_3\tilde{v}_3\neq 0$$

Moreover, by substituting E_2, $\hat{\beta}$ and $\hat{V}$ in Eq. (31), it is observed that

$$\hat{\Psi}^t[D^2F(E_2,\hat{\beta})(\hat{V},\hat{V})]=-2\hat{v}_2\hat{\psi}_2\begin{bmatrix}\frac{r_1}{K}\varphi_1^2\gamma_1+\frac{r_2}{L}-\frac{a\varphi_1\gamma_1}{(b+c\hat{z})^2}(\varphi_1\hat{z}-b\varphi_2)\\ -\varphi_1\varphi_2(1-\gamma_1)\end{bmatrix}$$

Cleary, if condition (34a) holds then $\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]\neq 0$ and hence transcritical bifurcation occurs. However if condition (34b) holds, then $\tilde{\Psi}^t[D^2F(E_1,\tilde{d})(\tilde{V},\tilde{V})]=0$, and hence the transcritical bifurcation can't occur. Further by substituting E_2, $\hat{\beta}$ and $\hat{V}$ in Eq. (32), it is observed that

$$\tilde{\Psi}^t[D^3F(E_1,\tilde{d})(\tilde{V},\tilde{V},\tilde{V})]=-\frac{2a\varphi_1\gamma_1\hat{v}_2^3\hat{\psi}_2}{(b+c\hat{z})^3}\left[3\varphi_1^2\hat{z}+3c\varphi_1\varphi_2\hat{z}+cb\varphi_2^2-3b\varphi_1\varphi_2\right]$$

Therefore $\tilde{\Psi}^t[D^3F(E_1,\tilde{d})(\tilde{V},\tilde{V},\tilde{V})]\neq 0$ and hence pitchfork bifurcation occurs if condition (34c) holds and then the proof is complete.

Theorem 9: *Assume that condition (25a) holds together with the following condition and let the parameter L passes through the value* $\breve{L}=\dfrac{2r_2y^*\Lambda_1}{(r_2-\beta)\Lambda_1+\breve{a}_{12}\Lambda_2}$.

$$(r_2-\beta)\Lambda_1+\breve{a}_{12}\Lambda_2<0 \tag{35}$$

where $\Lambda_1=\breve{a}_{13}\breve{a}_{31}-\breve{a}_{11}\breve{a}_{33}<0$; $\Lambda_2=\breve{a}_{21}\breve{a}_{33}-\breve{a}_{23}\breve{a}_{31}<0$ and $\breve{a}_{ij}=a_{ij};\ \forall i,j=1,2,3$ in Eq. (23) while $\breve{a}_{22}=r_2-\frac{2r_2}{\breve{L}}y^*-\beta$. Then system (1) near the positive equilibrium point $E_3=(x^*,y^*,z^*)$ has a saddle-node bifurcation but neither transcritical bifurcation nor pitchfork bifurcation can occur.

Proof According to the Jacobian matrix at the positive equilibrium point $J(E_3)$ that given by Eq. (23) and their characteristic equation given in Eq. (24), it's observed that

$$A_3=(r_2-\beta)\Lambda_1-\frac{2r_2y^*}{L}\Lambda_1+a_{12}\Lambda_2$$

Thus it is easy to verify that $A_3=0$ and hence $J(E_3)$ has zero eigenvalue $\breve{\lambda}=0$ at the parameter value $\breve{L}$, which is positive under condition (35). Hence E_3 is a non-hyperbolic equilibrium point.

Let $\breve{V}=(\breve{v}_1,\breve{v}_2,\breve{v}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\breve{\lambda}=0$ of the Jacobian matrix $\breve{J}=J(E_3,\breve{L})=(\breve{a}_{ij})_{3\times3}$. Then

$$\left[\breve{J}-\breve{\lambda}I\right]\breve{V}=0 \Rightarrow \breve{V}=\left(-\frac{\breve{a}_{33}}{\breve{a}_{31}}\breve{v}_3,\frac{\breve{a}_{21}\breve{a}_{33}-\breve{a}_{23}\breve{a}_{31}}{\breve{a}_{22}\breve{a}_{31}}\breve{v}_3,\breve{v}_3\right)^t=\left(\theta_1\breve{v}_3,\theta_2\breve{v}_3,\breve{v}_3\right)^t$$

Here $\breve{v}_3$ is any nonzero real number and $\breve{a}_{21}\breve{a}_{33}-\breve{a}_{23}\breve{a}_{31}<0$ always.

Let $\breve{\Psi}=(\breve{\psi}_1,\breve{\psi}_2,\breve{\psi}_3)^t$ be the eigenvector that associated with the zero eigenvalue $\breve{\lambda}=0$ of the transpose of Jacobian matrix $\breve{J}^t=J^t(E_3,\breve{L})$, then

$$\left[\breve{J}^t-\breve{\lambda}I\right]\breve{\Psi}=0 \Rightarrow$$

$$\breve{\Psi}=\left(-\frac{\breve{a}_{22}}{\breve{a}_{12}}\breve{\psi}_2,\breve{\psi}_2,\frac{\breve{a}_{13}\breve{a}_{22}-\breve{a}_{12}\breve{a}_{23}}{\breve{a}_{12}\breve{a}_{33}}\breve{\psi}_2\right)^t=\left(\mu_1\breve{\psi}_2,\breve{\psi}_2,\mu_2\breve{\psi}_2\right)^t$$

Here $\breve{\psi}_2$ represents any nonzero real number.

Now let $X=(x,y,z)$ then since

$$F_L(X,L)=\begin{pmatrix}0\\ \frac{r_2y^2}{L^2}\\ 0\end{pmatrix} \Rightarrow F_L(E_3,\breve{L})=\begin{pmatrix}0\\ \frac{r_2y^{*2}}{\breve{L}^2}\\ 0\end{pmatrix}$$

Here $F_L(X,L)$ represents the derivative of $F=(F_1,F_2,F_3)^t$ with respect to L. Then we get that

$$\breve{\Psi}^tF_L(E_3,\breve{L})=\frac{r_2y^{*2}}{\breve{L}}\breve{\psi}_2\neq 0$$

Thus according to the Sotomayor's theorem for local bifurcation, the transcritical and pitchfork bifurcation can't occur while the first condition of saddle-node bifurcation is satisfied. Further, straightforward computation gives that

$$D^2F(E_3,\breve{L})(\breve{V},\breve{V})=(\breve{\breve{d}}_{ij})_{3\times1}$$

where

$$\breve{\breve{d}}_{11} = -2\left[\frac{r_1}{K} - \frac{a(b+cz^*)z^*}{R^{*3}}\right]\theta_1^2\breve{v}_3^2 - 2\theta_1\breve{v}_3^2 - \frac{2a\theta_1\breve{v}_3^2}{R^{*3}}\left[b^2+bx^*+bcz^*+2cx^*z^*\right] + \frac{2ac\breve{v}_3^2}{R^{*3}}(b+x^*)x^*$$

$$\breve{\breve{d}}_{21} = 2\theta_1\breve{v}_3^2 - \frac{2r_2}{\breve{L}}\theta_2^2\breve{v}_3^2$$

$$\breve{\breve{d}}_{31} = -\frac{2ae\theta_1^2\breve{v}_3^2}{R^{*3}}(b+cz^*)z^* + \frac{2ae\theta_1\breve{v}_3^2}{R^{*3}}\left[b^2+bx^*+bcz^*+2cx^*z^*\right] - 2\left[\frac{r_3}{M} + \frac{ace}{R^{*3}}(b+x^*)x^*\right]\breve{v}_3^2$$

Hence we obtain that

$$\begin{aligned}\breve{\Psi}^t D^2F(E_2,\breve{L})(\breve{V},\breve{V}) &= \mu_1\breve{\psi}_2\breve{\breve{d}}_{11} + \breve{\psi}_2\breve{\breve{d}}_{21} + \mu_2\breve{\psi}_2\breve{\breve{d}}_{31} \\ &= -2\breve{\psi}_2\breve{v}_3^2\left[\frac{r_1}{K}\theta_1^2\mu_1 + \frac{r_2}{\breve{L}}\theta_2^2 + \frac{r_3}{M}\mu 2 + \theta_1(\mu_1-1) \right.\\ &\left. \quad + \frac{a(e\mu_2-\mu_1)}{R^{*3}}\begin{pmatrix}\theta_1^2(b+cz^*)z^* + c(b+x^*)x^* \\ -\theta_1(b^2+bx^*+bcz^*+2cx^*z^*)\end{pmatrix}\right]\end{aligned}$$

Straightforward computation shows that $\breve{\Psi}^t D^2F(E_2,\breve{L})(\breve{V},\breve{V}) \neq 0$. Hence system (1) has saddle-node bifurcation at E_1 with the bifurcation point given by $\breve{L}$.

Now before we go further to study the dynamical behavior of system (1) numerically, we have to explain that the above bifurcation parameters are functions of different other parameters of system (1) and hence the bifurcation may occurs in case of varying more than one of those parameters.

5. Numerical simulation

In this section, the numerical simulation is used to study the global dynamics of system (1) and specify the control parameters of the system, those parameters which affect the dynamics of the system as varying them. Therefore system (1) is solved numerically for different sets of initial points and different sets of parameters. It's observed that for the following set of hypothetical data system (1) approaches asymptotically to the global stable positive equilibrium point as shown in Fig. (1).

$$\begin{aligned}&r_1 = 1.5, K = 200, \alpha = 0.5, \beta = 0.9, a = 0.5, b = 10, c = 0.1 \\ &r_2 = 0.75, L = 100, e = 0.75, d = 0.1, r_3 = 0.25, M = 20.\end{aligned} \quad (36)$$

Clearly the solution of system (1) approaches asymptotically to the positive equilibrium point, represented by $E_3 = (4.71, 105.1, 20.43)$, for the data given in (36) starting from different sets of initial points and this is confirm our obtain analytical results regarding to existence and global stability of this point.

Now, as the natural death rate parameter of the predator increases in the range $d \geq 0.62$, the positive equilibrium point loses its stability and the solution of system (1) approaches asymptotically to the predator free equilibrium point E_1 as shown in the typical figure given by Fig. (2). However decreasing the value of this parameter doesn't affect the solution of system (1) and its still approaches to the positive equilibrium point. Further it is observed that varying the other parameters, one at a time, in (36) doesn't affect the dynamical behavior of system (1).

According to the Fig. (3), the solution of system (1) approaches asymptotically to the prey free equilibrium point as the parameters of system (1) satisfy the obtained stability conditions. Finally, decreasing the value of natural moving rate in the range $\beta < 0.4$ keeping other parameters fixed as given in (36) with $r_1 = 0.25$ and causes survival of the prey species and the solution of system (1) return to approaches asymptotically to the positive equilibrium point as shown in Fig. (4).

Keeping the above in view, it is clear that the solution of system (1) is affected by varying the parameters: predator natural death rate (d), the prey natural moving rate from reserved zone to unreserved zone (β) and intrinsic growth rates r_1 and r_2 or equivalently the prey carrying capacity in reserved zone (L) that depends on them as in theorem (8)

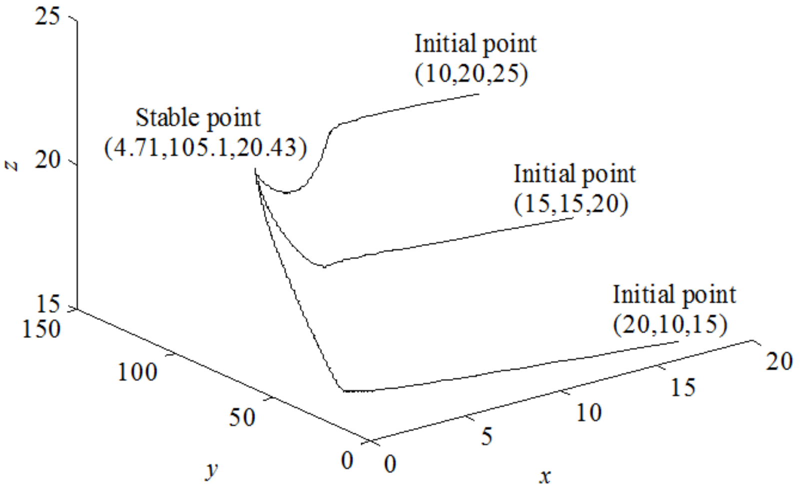

Fig. 1: The Solution of System (1) Approaches Asymptotically to the Positive Equilibrium Point Starting from Different Initial Points.

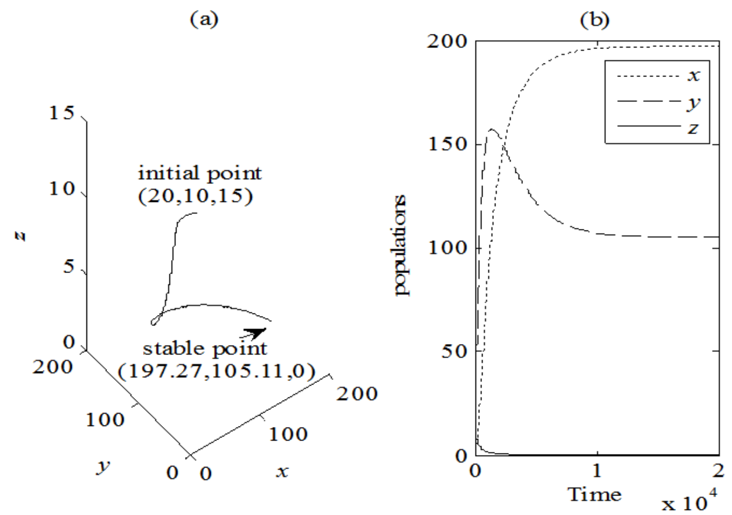

Fig. 2: The Solution of System (1) for the Data Given in (36) With $d = 0.65$. (A) 3D Predator Free Equilibrium Point Attractor $E_1 = (197.27, 105.11, 0)$. (B) Time Series of the Attractor in (A).

Moreover, for the data given in (36) with $r_1 = 0.25$ and system (1) approaches asymptotically to the prey free equilibrium point, represented by $E_2 = (0, 0, 12)$, as shown in Fig. (3) Below.

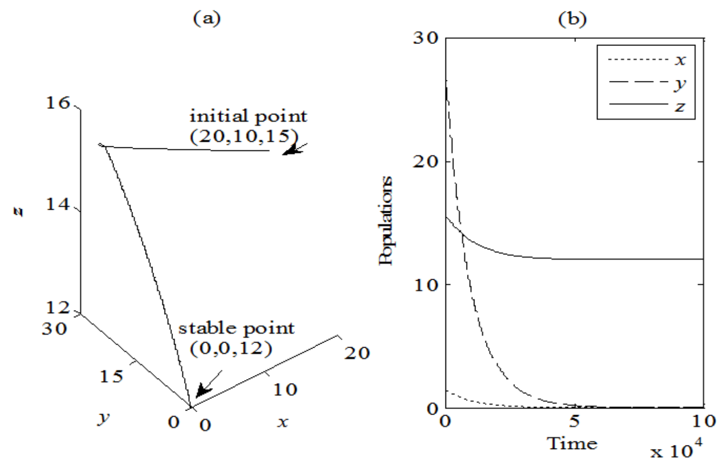

Fig. 3: The Solution of System (1) for the Data Given in (36) with $r_1 = 0.25$ and. (A) 3D Prey Free Equilibrium Point Attractor. (B) Time Series of the Attractor in (A).

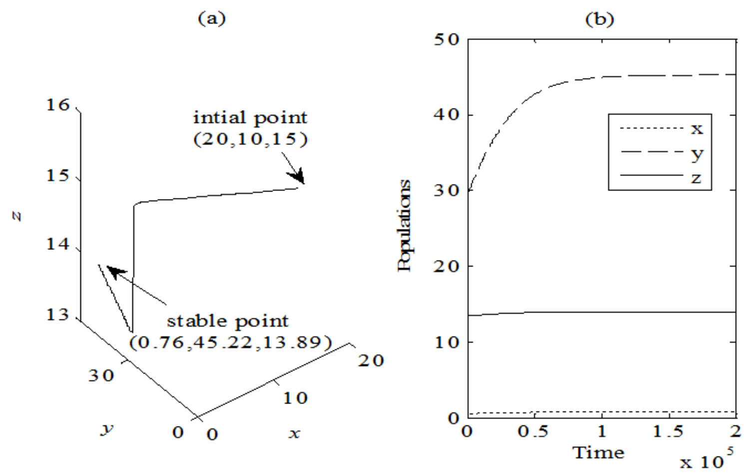

Fig. 4: The Solution of System (1) for the Data Given in (36) with $r_1 = 0.25$, and $\beta = 0.25$. (A) 3D Positive Equilibrium Point Attractor $E_3 = (0.76, 45.22, 13.89)$. (B) Time Series of the Attractor in (A).

6. Discussion

In this paper, a mathematical model has been proposed and analyzed to study the prey-predator system consisting of a predator that depends partially on the prey in an unreserved zone. It is assumed that the habitat consisting of an unreserved zone and a reserved zone. The predator is consumed the prey according to the Beddington-DeAngelis type of functional response. The dynamical behavior of the proposed model represented by system (1) has been investigated locally as well as globally. Local bifurcation near the equilibrium points has been investigated. It is observed that the system has at most four nonnegative equilibrium points, the vanishing equilibrium point that always exists and an unstable saddle point; the predator free equilibrium point; the prey free equilibrium point and the coexistence (positive) equilibrium point. The local and global stability of all these points are investigated analytically. The local bifurcations near them are also studied. Finally the global dynamics of system (1) is investigated numerically for the biologically feasible hypothetical data that given in (36) and the obtained results can be summarized as below:

1) System (1) doesn't approach to periodic dynamics, instead of that it approaches asymptotically to one of its nonnegative equilibrium points.
2) The solution of system (1) approaches asymptotically to the positive equilibrium point starting from different sets of initial points, which indicate to globally asymptotically stable of the positive equilibrium point and coexistence of all species.
3) Increasing the natural death rate of the predator causes extinction in predator species and the solution of system (1) approaches asymptotically to the predator free equilibrium point. Thus this parameter represents a bifurcation parameter of the system.
4) Decreasing the intrinsic growth rates of prey species in both the zones (reserved and unreserved zone) causes extinction in prey species from both the zones while the predator still survive depending on other sources of food and hence the solution of system (1) approaches asymptotically to the prey free equilibrium point. Consequently these parameters play the role of bifurcation parameter.
5) In addition to the hypothesis adopted in point (4), decreasing the natural moving rate of the prey from a reserved zone to an unreserved has a coexistence effects on the system and the solution again approaches asymptotically to the positive equilibrium point. So, this parameter represents a bifurcation parameter too of the system.

Keeping the above in view, by comparing the obtained results with those obtained in our previous paper [8], it is observed that adding other food resources to the predator in a habitat having a reserved zone has a stabilizing effect on the system dynamics due to expanding the ranges of stability of the positive equilibrium point.

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