International Journal of Applied Mathematical Research, 4 (4) (2015) 519-524
www.sciencepubco.com/index.php/IJAMR
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doi: 10.14419/ijamr.v4i4.4788
Research Paper

Characterization of the generalized Chebyshev-type polynomials of first kind

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Abstract

Orthogonal polynomials have very useful properties in the mathematical problems, so recent years have seen a great deal in the field of approximation theory using orthogonal polynomials. In this paper, we characterize a sequence of the generalized Chebyshev-type polynomials of the first kind $\left\{\mathscr{T}_n^{(M,N)}(x)\right\}_{n\in\mathbb{N}\cup\{0\}}$, which are orthogonal with respect to the measure $\frac{\sqrt{1-x^2}}{\pi}dx + M\delta_{-1} + N\delta_1$, where δ_x is a singular Dirac measure and $M, N \geq 0$. Then we provide a closed form of the constructed polynomials in term of the Bernstein polynomials $B_k^n(x)$. We conclude the paper with some results on the integration of the weighted generalized Chebyshev-type with the Bernstein polynomials.

Keywords: *Bernstein basis, Chebyshev polynomials, Generalized Chebyshev-type polynomials, Orthogonal polynomials.*

1. Introduction

Approximation is essential to many numerical techniques, since it is possible to approximate arbitrary continuous function by a polynomial, at the same time polynomials can be represented in many different bases such as monomial and Bernstein basis.

1.1. Univariate Chebyshev polynomials

Chebyshev polynomial of first kind (Chebyshev-I) of degree $n \geq 0$ in x is defined as $T_n(x) = \cos(n \arccos x)$, $x \in [-1, 1]$. The polynomials $T_n(x)$ of degree n are orthogonal polynomials, except for a constant factor, with respect to the measure $\mathrm{W}(x) = \frac{1}{\sqrt{1-x^2}}$. They are a special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the interrelation is given by

$$T_n(x) = \binom{n-\frac{1}{2}}{n}^{-1} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x). \tag{1}$$

It is worth pointing out that univariate classical orthogonal polynomials are traditional defined on $[-1, 1]$, however, it is more convenient to use $[0, 1]$. For the convenience we recall few representations of univariate Chebyshev-I polynomials. The univariate Chebyshev-I polynomials of degree n in x can be written as [6, 11],

$T_n(x) = \frac{(2n)!!}{(2n-1)!!}\sum_{k=0}^{n}\binom{n-\frac{1}{2}}{n-k}\binom{n-\frac{1}{2}}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}$, which can be transformed in terms of Bernstein basis as $T_n(2x-1) := \frac{2^{2n}(n!)^2}{(2n)!}\sum_{k=0}^{n}(-1)^{n+1}\frac{\binom{n-\frac{1}{2}}{k}\binom{n-\frac{1}{2}}{n-k}}{\binom{n}{k}}B_k^n(x),\ x\in[0,1]$, where $B_k^n(x)$ are the Bernstein polynomials of degree n, $x\in[0,1], k\in\mathbb{N}\cup\{0\}$, defined by $B_k^n(x) = \frac{n!}{k!(n-k)!}x^k(1-x)^{n-k}$.

Note that the double factorial of an integer m, $m!!$, is a generalization of the usual factorial $m!$, and defined as

$$\begin{aligned} m!! &= (m)(m-2)(m-4)\dots(4)(2) && \text{if } m \text{ is even} \\ (2m-1)!! &= (2m-1)(2m-3)(2m-5)\dots(3)(1) && \text{if } m \text{ is odd,} \end{aligned} \tag{2}$$

where $0!! = (-1)!! = 1$. From the definition (2), we can derive the factorial of an integer minus half as

$$\left(r-\frac{1}{2}\right)! = \frac{r!(2r-1)!!\sqrt{\pi}}{(2r)!!}. \tag{3}$$

In addition, the Chebyshev-I polynomials satisfy the orthogonality relation [8]

$$\int_0^1 (x-x^2)^{-\frac{1}{2}}T_n(x)T_m(x)dx = \begin{cases} 0 & \text{if } m\neq n \\ \frac{\pi}{2} & \text{if } m=n=0 \\ \pi & \text{if } m=n\in\mathbb{N} \end{cases}. \tag{4}$$

2. Generalized Chebyshev-type polynomials

In this section we characterize the generalized Chebyshev-type polynomials of first kind, $\mathscr{T}_r^{(M,N)}(x)$, then we provide explicit closed form as a linear combination of Bernstein polynomials $B_i^r(x)$. We conclude this section with the closed form of the integration of the weighted generalized Chebyshev-type with respect to the Bernstein polynomials.

Using the fact (1) and similar construction of the results in [5, 7], and for $M,N\geq 0$, define the sequence of the generalized Chebyshev-type polynomials $\left\{\mathscr{T}_n^{(M,N)}(x)\right\}_{n\in\mathbb{N}\cup\{0\}}$ as

$$\mathscr{T}_n^{(M,N)}(x) = \frac{(2n)!}{2^{2n}(n!)^2}T_n(x) + MQ_n(x) + NR_n(x) + MNS_n(x),\ n\in\mathbb{N}\cup\{0\} \tag{5}$$

where for $n\in\mathbb{N}$

$$Q_n(x) = \frac{(2n)!}{2^{2n-1}(n!)^3}\left[n^2T_n(x) - \frac{1}{2}(x-1)DT_n(x)\right], \tag{6}$$

$$R_n(x) = \frac{(2n)!}{2^{2n-1}(n!)^3}\left[n^2T_n(x) - \frac{1}{2}(x+1)DT_n(x)\right], \tag{7}$$

and

$$S_n(x) = \frac{(2n)!}{2^{2n-2}(n!)^3(n-1)!}[n^2T_n(x) - xDT_n(x)]. \tag{8}$$

By using $(x^2-1)D^2T_n(x) = n^2T_n(x) - xDT_n(x)$, we find that

$$S_n(x) = \frac{4(2n-1)!!}{n!(n-1)!(2n)!!}(x^2-1)D^2T_n(x),\ n\in\mathbb{N}. \tag{9}$$

It is clear that $Q_0(x) = R_0(x) = S_0(x) = 0$. Furthermore, for $n\in\mathbb{N}\cup\{0\}$ the generalized Chebyshev-type polynomials satisfy the symmetry relation [7], $\mathscr{T}_n^{(M,N)}(x) = (-1)^n\mathscr{T}_n^{(N,M)}(-x)$, which implies that $Q_n(x) = (-1)^nR_n(-x)$ and $S_n(x) = (-1)^nS_n(-x)$. From (6) and (7) it follows that for $n\in\mathbb{N}$

$$Q_n(1) = \frac{2(2n-1)!!}{(n-1)!(2n)!!}T_n(1) \quad\text{and}\quad R_n(-1) = \frac{2(2n-1)!!}{(n-1)!(2n)!!}T_n(-1).$$

Note that (6),(7) and (8) imply that for $n \in \mathbb{N}$, we have

$$Q_n(x) = \sum_{k=0}^{n} \frac{(2k)!q_k}{2^{2k}(k!)^2} T_k(x) \quad \text{with} \quad q_k = \frac{4}{(2k-3)(k-1)!}, \tag{10}$$

$$R_n(x) = \sum_{k=0}^{n} \frac{(2k)!r_k}{2^{2k}(k!)^2} T_k(x) \quad \text{with} \quad r_k = \frac{4}{(2k-3)(k-1)!}, \tag{11}$$

and

$$S_n(x) = \sum_{k=0}^{n} \frac{(2k)!s_k}{2^{2k}(k!)^2} T_k(x) \quad \text{with} \quad s_k = \frac{4}{(k-1)!(k-2)!}. \tag{12}$$

Therefore, for $M, N \geq 0$ the generalized Chebyshev-type polynomials $\left\{\mathscr{T}_n^{(M,N)}(x)\right\}_{n\in\mathbb{N}\cup\{0\}}$ are orthogonal on the interval $[-1, 1]$ with respect to the measure

$$\frac{\sqrt{1-x^2}}{\pi} dx + M\delta_{-1} + N\delta_1, \tag{13}$$

where δ_x is a singular Dirac measure, which can be written as

$$\mathscr{T}_n^{(M,N)}(x) = \frac{(2n-1)!!}{(2n)!!} T_n(x) + \sum_{k=0}^{n} \frac{(2k)!\lambda_k}{2^{2k}(k!)^2} T_k(x) \tag{14}$$

where

$$\lambda_k = Mq_k + Nr_k + MNs_k. \tag{15}$$

2.1. Characterization using Bernstein basis

The Bernstein polynomials have been studied thoroughly and there exist many great enduring works on theses polynomials [2]. They are known for their analytic and geometric properties [1, 4], where the basis are known to be optimally stable. They are all non-negative, $B_i^n(x) \geq 0$, $x \in [0, 1]$, form a partition of unity (normalization) $\sum_{k=0}^{n} B_k^n(x) = 1$, satisfy symmetry relation $B_i^n(x) = B_{n-i}^n(1-x)$, have a single unique maximum of $\binom{n}{i} i^i n^{-n} (n-i)^{n-i}$ at $x = \frac{i}{n}$, $i = 0, \ldots, n$, and their roots are $x = 0, 1$ with multiplicities. The Bernstein polynomials of degree n can be defined by combining two Bernstein polynomials of degree $n-1$. That is, the kth nth-degree Bernstein polynomial defined by the following recurrence relation $B_k^n(x) = (1-x)B_k^{n-1}(x) + xB_{k-1}^{n-1}(x), \quad k = 0, \ldots, n; n \geq 1$ where $B_0^0(x) = 0$ and $B_k^n(x) = 0$ for $k < 0$ or $k > n$. For more details, see Farouki [2].

In addition, it is possible to write Bernstein polynomial of degree r where $r \leq n$ in terms of Bernstein polynomials of degree n using the following degree elevation [3]:

$$B_k^r(x) = \sum_{i=k}^{n-r+k} \frac{\binom{r}{k}\binom{n-r}{i-k}}{\binom{n}{i}} B_i^n(x), \quad k = 0, 1, \ldots, r. \tag{16}$$

Now, to write a generalized Chebyshev-type polynomial $\mathscr{T}_r^{(M,N)}(x)$ of degree r as a linear combination of the Bernstein polynomial basis $B_i^r(x), i = 0, 1, \ldots, r$ of degree r in explicit closed form, we begin with substituting the value of $B_k^n(x)$ into (14) to get

$$\mathscr{T}_r^{(M,N)}(x) = \frac{(2r-1)!!}{(2r)!!} \sum_{i=0}^{r} (-1)^{r-i} \eta_{i,r} B_i^r(x) + \sum_{k=0}^{r} \frac{(2k)!\lambda_k}{2^{2k}(k!)^2} \sum_{j=0}^{k} (-1)^{k-j} \eta_{j,k} B_j^k(x).$$

where $\eta_{i,r} = \frac{\binom{r-\frac{1}{2}}{i}\binom{r-\frac{1}{2}}{r-i}}{\binom{r}{i}}, \quad i = 0, 1, \ldots, r.$

This shows that the generalized Chebyshev-type polynomial $\mathscr{T}_r^{(M,N)}(x)$ of degree r can be written in the Bernstein basis form. Now, by expanding the right-hand side and using (3) with some simplifications, we have

$$\eta_{i,r} = \frac{\left(r-\frac{1}{2}\right)!\left(r-\frac{1}{2}\right)!}{(i-\frac{1}{2})!(r-i-\frac{1}{2})!r!} = \frac{(2r-1)!!(2r-1)!!}{2^r r!(2i-1)!!(2(r-i)-1)!!}.$$

Using the fact $(2n)! = (2n-1)!!2^n n!$ we get $\eta_{i,r} = \frac{\binom{2r}{r}\binom{2r}{2i}}{2^{2r}\binom{r}{i}}$, where $\eta_{0,r} = \frac{1}{2^{2r}}\binom{2r}{r}$. With simple combinatorial simplifications we have $\eta_{i-1,r} = \frac{(i-\frac{1}{2})}{(r-i+\frac{1}{2})}\eta_{i,r}$. Thus we have the following theorem.

Theorem 2.1. *For $M, N \geq 0$, the generalized Chebyshev-type polynomials $\mathscr{T}_n^{(M,N)}(x)$ of degree n have the following Bernstein representation:*

$$\mathscr{T}_n^{(M,N)}(x) = \frac{(2n-1)!!}{(2n)!!}\sum_{i=0}^{n}(-1)^{n-i}\eta_{i,n}B_i^n(x) + \sum_{k=0}^{n}\frac{(2k)!\lambda_k}{2^{2k}(k!)^2}\sum_{j=0}^{k}(-1)^{k-j}\eta_{j,k}B_j^k(x) \tag{17}$$

where $\lambda_k = Mq_k + Nr_k + MNs_k$ and $\eta_{i,n} = \frac{\binom{2n}{n}\binom{2n}{2i}}{2^{2n}\binom{n}{i}}$, $i = 0, 1, \ldots, n$ where $\eta_{0,n} = \frac{1}{2^{2n}}\binom{2n}{n}$. Moreover, the coefficients $\eta_{i,n}$ satisfy the recurrence relation

$$\eta_{i,n} = \frac{(2n-2i+1)}{(2i-1)}\eta_{i-1,n}, \quad i = 1, \ldots, n. \tag{18}$$

It is worth mentioning that Bernstein polynomials can be differentiated and integrated easily as

$$\frac{d}{dx}B_k^n(x) = n[B_{k-1}^{n-1}(x) - B_k^{n-1}(x)],\ n \geq 1, \text{ and } \int_0^1 B_k^n(x)dx = \frac{1}{n+1},\ k = 0, 1, \ldots, n.$$

Rababah [9] provided some results concerning integrals of univariate Chebyshev-I and Bernstein polynomials. In the following we consider integration of the weighted generalized Chebyshev-type with Bernstein polynomials,

$$I = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}B_r^n(x)\mathscr{T}_i^{(M,N)}(x)dx.$$

By using (17), the integral can be simplified to

$$I = \int_0^1 (1-x)^{-\frac{1}{2}}x^{-\frac{1}{2}}\binom{n}{r}x^r(1-x)^{n-r}\frac{(2i)!}{2^{2i}(i!)^2}\sum_{k=0}^{i}(-1)^{i-k}\frac{\binom{i-\frac{1}{2}}{k}\binom{i-\frac{1}{2}}{i-k}}{\binom{i}{k}}B_k^i(x)$$
$$+\sum_{d=0}^{i}\lambda_d\int_0^1(1-x)^{n-r-\frac{1}{2}}x^{r-\frac{1}{2}}\binom{n}{r}\frac{(2d)!}{2^{2d}(d!)^2}\sum_{j=0}^{d}(-1)^{d-j}\frac{\binom{d-\frac{1}{2}}{j}\binom{d-\frac{1}{2}}{d-j}}{\binom{d}{j}}B_j^d(x)dx,$$

where λ_d defined in (15). By reordering the terms we get

$$I = \binom{n}{r}\frac{(2i)!}{2^{2i}(i!)^2}\sum_{k=0}^{i}(-1)^{i-k}\binom{i-\frac{1}{2}}{k}\binom{i-\frac{1}{2}}{i-k}\int_0^1 x^{r+k-\frac{1}{2}}(1-x)^{n+i-r-k-\frac{1}{2}}dx$$
$$+\sum_{d=0}^{i}\binom{n}{r}\frac{(2d)!\lambda_d}{2^{2d}(d!)^2}\sum_{j=0}^{d}(-1)^{d-j}\binom{d-\frac{1}{2}}{j}\binom{d-\frac{1}{2}}{d-j}\int_0^1 x^{r+j-\frac{1}{2}}(1-x)^{n+d-r-j-\frac{1}{2}}dx.$$

The integrals in the last equation are the Beta functions $B(x_i, y_i)$ with $x_1 = r+k+\frac{1}{2}$, $y_1 = n+i-r-k+\frac{1}{2}$, $x_2 = r+j+\frac{1}{2}$, and $y_2 = n+d-r-j+\frac{1}{2}$.

Hence, the following theorem provides a closed form of the integration of the weighted generalized Chebyshev-type with respect to the Bernstein polynomials.

Theorem 2.2. *Let $B_r^n(x)$ be the Bernstein polynomial of degree n and $\mathscr{T}_i^{(M,N)}(x)$ be the generalized Chebyshev-type polynomial of degree i, then for $i, r = 0, 1, \ldots, n$ we have*

$$\int_0^1 (x-x^2)^{-\frac{1}{2}} B_r^n(x)\mathscr{T}_i^{(M,N)}(x)dx = \binom{n}{r}\frac{(2i)!}{2^{2i}(i!)^2}\sum_{k=0}^{i}(-1)^{i-k}\binom{i-\frac{1}{2}}{k}\binom{i-\frac{1}{2}}{i-k}B(r+k+\frac{1}{2}, n+i-r-k+\frac{1}{2})$$
$$+\sum_{d=0}^{i}\binom{n}{r}\frac{(2d)!\lambda_d}{2^{2d}(d!)^2}\sum_{j=0}^{d}(-1)^{d-j}\binom{d-\frac{1}{2}}{j}\binom{d-\frac{1}{2}}{d-j}B(r+j+\frac{1}{2}, n+d-r-j+\frac{1}{2})$$

where λ_k defined in (15) and $B(x, y)$ is the Beta function.

3. Applications

The analytic and geometric properties of the Bernstein polynomials made them important for the development of Bézier curves and surfaces in Computer Aided Geometric Design. The Bernstein polynomials are the standard basis for the Bézier representations of curves and surfaces in Computer Aided Geometric Design. However, the Bernstein polynomials are not orthogonal and could not be used effectively in the least-squares approximation [10]. So, the method of least squares approximation accompanied by orthogonal polynomials has been introduced.

Definition 3.1. *For a function $f(x)$, continuous on $[0, 1]$ the least square approximation requires finding a polynomial (Least-Squares Polynomial) $p_n(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x)$ that minimize the error*

$$E(a_0, a_1, \ldots, a_n) = \int_0^1 [f(x) - p_n(x)]^2 dx.$$

For minimization, the partial derivatives must satisfy $\frac{\partial E}{\partial a_i} = 0, i = 0, \ldots, n$. These conditions give rise to a system of $(n+1)$ normal equations in $(n+1)$ unknowns: $a_0, a_1, \ldots, a_n$. Solution of these equations will yield the unknowns of the least-squares polynomial $p_n(x)$. By choosing $\varphi_i(x) = x^i$, then the coefficients of the normal equations give the Hilbert matrix that has round-off error difficulties and notoriously ill-condition for even modest values of n. However, choosing $\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\}$ to be orthogonal simplifies the least-squares approximation problem, the matrix of the normal equations will be diagonal, which make the numerical calculations more efficient. See [10] for more details on the least squares approximations.

4. Conclusion

We used the elegant analytic and geometric properties of the Bernstein polynomials to characterize the generalized Chebyshev-type polynomials of first kind, $\mathscr{T}_r^{(M,N)}(x)$, and provided explicit closed form as a linear combination of Bernstein polynomials $B_i^r(x)$. We concluded the paper with the closed form of the integration of the weighted generalized Chebyshev-type with respect to the Bernstein polynomials.

Acknowledgements

The author would like to thank the referees for their valuable comments which helped to improve the manuscript.

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