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Application of $\frac{G'}{G}$ −expansion method to two concert problems

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Abstract

In this work G'/G -expansion method is used to obtain exact solutions of the Jimbo-Miwa (JM) and Tzitzeica-Dodd_Bullough (TDB) equations. It is shown that G'/G -expansion method is straightforward and concise, and its applications are promising.

Keywords: G'/G -*expansion method, Jimbo-Miwa equation, Tzitzeica-Dodd_Bullough equation, Exact solutions.*

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1 Introduction

The nonlinear equations of mathematical physics are major subjects in physics and engineering, and various powerful methods have been presented, such as the tanh method [1-2], sine–cosine method [3], homotopy perturbation method [4-5], variational iteration method [6-7], Adomian decomposition method [8], Exp-function method [9-11], and many others [12-13]. Very recently, Wang et al. [14] introduced a new method called the G'/G expansion method to look for travelling wave solutions of nonlinear evolution equations. The G'/G expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in G'/G and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (ODE).

The present paper is motivated by the desire to extend the G'/G expansion method to the Jimbo-Miwa and TDB equations. Jimbo- Miwa equation is of particular interest in science and has been known in the following form

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \tag{1}$$

where $u = u(x, y, z, t)$. Eq. (1) is investigated by Jimbo and Miwa and its certain soliton solutions are obtained [15]. Then, it is studied by several authors regarding its solutions, symmetries and integrability properties [16–18]. Wazwaz [19, 20] employed the tanh–coth and Hirota's bilinear methods to solve Eq. (1).

The second equation that will be considered is the Tzitzeica–Dodd–Bullough (TDB) equation
[21], in the following form

$$u_{xt} = e^{-u} + e^{-2u}. \tag{2}$$

This equation plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. TDB equation has been solved by using exp-function method in [22].

2 The $\frac{G'}{G}$ – expansion method

Consider a nonlinear partial differential equation, in two independent variables say x and t , in the form

$$p(u, u_t, u_x, u_{xx}, u_{tt}, \dots) = 0. \tag{3}$$

Where $u = u(x,t)$ is an unknown function, p is non-linear equation in $u = u(x,t)$ and its various partial derivatives. To apply the method following steps showed be fallowed.

Step 1. Using the transformation

$$\xi = x - wt, \tag{4}$$

where w is constant ,we can rewrite Eq. (3) as the following nonlinear ODE:

$$Q(u, u', u'', u''', \dots) = 0. \tag{5}$$

Where the superscripts denote the derivatives with respect to ξ .

Step 2. Suppose that the solution of ODE (5) can be expressed by a polynomial in G'/G as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{6}$$

where $G = G(\xi)$ satisfies the second order LODE in the form as fallows

$$G'' + \lambda G' + \mu G = 0. \tag{7}$$

$\alpha_i, \lambda,$ and μ are constants to be determined later with $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance the highest order derivatives and highest order nonlinear appearing in ODE (5).

Step 3. Substituting Eq. (6) into Eq. (5) and using the second order LODE, Eq. (7) yields an algebraic equation involving powers of G'/G . Equating the coefficient of each power of G'/G to zero gives a set of algebraic equations for determining $\alpha_i, w, \lambda,$ *and* μ .

Step 4. Assuming that the constants $\alpha_i, w, \lambda,$ *and* μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (7), depending on the sign of $\Delta = \lambda^2 - 4\mu,$ are well known for us, by substituting $\alpha_i, w,$ and the general solutions of Eq. (7) into Eq. (6), solutions of the nonlinear evolution Eq. (3) can be obtained.

3 Application to the JM equation

To apply G'/G -expansion method on Eq. (1), let's introduce a variable ξ , defined as

$$\xi = x + y + z - wt. \tag{8}$$

So, Eq. (1) turns to the following ordinary differential equation,

$$u^{(4)} + 6u'u'' - (2w + 3)u'' = 0, \tag{9}$$

where w is constant to be determined. By integrating from Eq. (9), we obtain

$$u''' + 3u'^2 - (2w + 3)u' = c, \tag{10}$$

where c is an integration constant that is to be determined later.
Suppose that the solution of ODE Eq. (10) can be expressed by a polynomial in G'/G as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{11}$$

where $G = G(\xi)$ satisfies the second order LODE (7). Balancing the terms u''' and u'^2 in Eq. (10), yields to $m = 1$. So we can write (11) as the following simple form

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{12}$$

By substituting (12) into Eq. (10) and collecting all terms with the same power of G'/G together, and equating each coefficient of the terms to zero, we derive a set of algebraic equations for determining $\alpha_1, \alpha_0, c, w, \lambda,$ and μ as follows

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^0 : \ 3\alpha_1\mu\text{-}\alpha_1\lambda^2\mu\text{+}3\alpha_1^2\mu^2\text{+}2\mathrm{w}\,\alpha_1\mu\text{-c} = 0, \\
&\left(\frac{G'}{G}\right)^1 : \ \text{-}8\alpha_1\lambda\mu\text{-}\alpha_1\lambda^3\text{+}6\alpha_1^2\lambda\mu\text{+}2\mathrm{w}\alpha_1\lambda\text{+}3\alpha_1\lambda = 0, \\
&\left(\frac{G'}{G}\right)^2 : \ \text{-}8\alpha_1\mu\text{-}7\alpha_1\lambda^2\text{+}6\alpha_1^2\mu\text{+}3\alpha_1^2\lambda^2\text{+}2\mathrm{w}\alpha_1\text{+}3\alpha_1 = 0, \\
&\left(\frac{G'}{G}\right)^3 : \ \text{-}12\alpha_1\lambda\text{+}6\alpha_1^2\lambda = 0, \\
&\left(\frac{G'}{G}\right)^4 : \ \text{-}6\alpha_1 + 3\alpha_1^2 = 0.
\end{aligned} \tag{13}$$

By the solution of these algebraic equations, the following results are obtained

$$\alpha_1 = 2, \ w = -2\mu + \frac{1}{2}\lambda^2 - \frac{3}{2}, \ c = 4\mu. \tag{14}$$

Where $\lambda, \mu,$ and α_0 are arbitrary constants.
By substituting (14) into (12), we drive

$$u(\xi) = 2\left(\frac{G'}{G}\right) + \alpha_0, \tag{15}$$

where

$$\xi = x + y + z - \left(-2\mu + \frac{1}{2}\lambda^2 - \frac{3}{2}\right)t. \tag{16}$$

By substituting the general solutions of Eq. (7) into (15) we would have three types of traveling wave solutions of the JM equation as follows:
When $\lambda^2 - 4\mu > 0,$

$$u_1(\xi)=\sqrt{\lambda^2-4\mu}\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)-\lambda+\alpha_0. \quad (17)$$

While $\lambda^2-4\mu<0$,

$$u_2(\xi)=\sqrt{4\mu-\lambda^2}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right)-\lambda+\alpha_0. \quad (18)$$

For $\lambda^2-4\mu=0$,

$$u_3(\xi)=\frac{2B}{A+B\xi}-\lambda+\alpha_0. \quad (19)$$

Where A and B are arbitrary constants and

$$\xi=x+y+z-\left(-2\mu+\frac{1}{2}\lambda^2-\frac{3}{2}\right)t. \quad (20)$$

In particular case of $A\neq 0$, and $B=0$, Eqs. (17) and (18) turns to

$$u(\xi)=\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)-\lambda+\alpha_0.$$

$$u(\xi)=-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi\right)-\lambda+\alpha_0.$$

And for $A=0$, and $B\neq 0$, Eqs. (17) and (18) turns to

$$u(\xi)=\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)-\lambda+\alpha_0.$$

$$u(\xi)=\sqrt{4\mu-\lambda^2}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi\right)-\lambda+\alpha_0.$$

4 Application to the TDB equation

By using the transformations

$$v(x,t)=e^{-u(x,t)},\ \ \xi=x-wt,$$

Eq. (2) turns to an ordinary differential equation, as the following

$$w(vv''-v'^2)-v^3-v^4=0, \quad (21)$$

and

$$u=\operatorname{arcsin}h\left[\frac{v^{-1}-v}{2}\right].$$

Suppose that the solution of ODE Eq. (21) can be expressed by a polynomial in G'/G as shown in (11), where $G=G(\xi)$ satisfies the second order LODE (7). By considering the homogeneous balance between vv'' and v^4 in Eq. (21), we required that

$$2m+2=4m, \quad (22)$$

and

$$m = 1. \tag{23}$$

So we can write (11) as the following simple form

$$v(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{24}$$

By substituting (24) into Eq. (21) and collecting all terms with the same power of G'/G together, the left-hand side of Eq. (21) is converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero yields to a set of simultaneous algebraic equations for determining $\alpha_1, \alpha_0, w, \lambda,$ and μ as follows:

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^0 : -\alpha_0^3 + \mathrm{w}\alpha_0\alpha_1\lambda\mu - \mathrm{w}\alpha_1^2\mu^2 - \alpha_0^4 = 0, \\
&\left(\frac{G'}{G}\right)^1 : 2\mathrm{w}\alpha_0\alpha_1\mu + \mathrm{w}\alpha_0\alpha_1\lambda^2 - \mathrm{w}\alpha_1^2\lambda\mu - 3\alpha_0^2\alpha_1 - 4\alpha_0^3\alpha_1 = 0, \\
&\left(\frac{G'}{G}\right)^2 : 3\mathrm{w}\alpha_0\alpha_1\lambda - 3\alpha_0\alpha_1^2 - 6\alpha_0^2\alpha_1^2 = 0, \\
&\left(\frac{G'}{G}\right)^3 : 2\mathrm{w}\alpha_0\alpha_1 + \mathrm{w}\alpha_1^2\lambda - \alpha_1^3 - 4\alpha_0\alpha_1^3 = 0, \\
&\left(\frac{G'}{G}\right)^4 : \mathrm{w}\alpha_1^2 - \alpha_1^4 = 0.
\end{aligned} \tag{25}$$

Solving this algebraic equations above, yields to

$$\alpha_0 = \pm\frac{1}{2}\frac{1}{\sqrt{\lambda^2 - 4\mu}}\lambda - \frac{1}{2}, \quad \alpha_1 = \pm\frac{1}{\sqrt{\lambda^2 - 4\mu}}, w = \frac{1}{\lambda^2 - 4\mu}. \tag{26}$$

Where λ and μ are arbitrary constants.
By substituting (26) into (24), we drive

$$v(\xi) = \pm\frac{1}{\sqrt{\lambda^2 - 4\mu}}\left(\frac{G'}{G}\right) \pm \frac{1}{2}\frac{1}{\sqrt{\lambda^2 - 4\mu}}\lambda - \frac{1}{2}, \tag{27}$$

Where

$$\xi = x - \frac{1}{\lambda^2 - 4\mu}t. \tag{28}$$

By substituting the general solutions of Eq. (7) into (27), three types of traveling wave solutions of the TDB equation have been obtained:
When $\lambda^2 - 4\mu > 0$,

$$v_{1,2}(\xi) = \pm\frac{1}{2}\left(\frac{A \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + B \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{A \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + B \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right) - \frac{1}{2}. \tag{29}$$

While $\lambda^2 - 4\mu < 0$,

$$v_{3,4}(\xi) = \pm\frac{1}{2}\left(\frac{-A \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + B \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{A \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + B \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right) - \frac{1}{2}. \tag{30}$$

In particular case of $A \neq 0, B = 0,$ and $c = \frac{1}{-4\mu}$, Eq. (29) turns to

$$v(x,t)=-\frac{1}{2}\left(1\pm\tanh\left(\frac{1}{2\sqrt{c}}(x-ct)\right)\right).$$

This is the same as the result obtained by wazwaz [21].

5 Conclusion

In this article, G'/G -expansion method is used to obtain exact solutions of the Jimbo-Miwa and TDB equations. The solutions being determined in this paper are more general, and it is not difficult to arrive at some known analytic solutions for certain choices of the parameters A and B. The results show that G'/G -expansion method is a powerful tool for obtaining exact solution. Applications of G'/G -expansion method for some equations are under study in our research group. The computations associated in this work were performed by using Maple 13.

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