Asymptotic behavior of oscillatory solutions of first order functional delay difference equations

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Abstract

In this paper, we study the asymptotic behavior of oscillatory solutions of the first order functional delay difference equation

\[ \Delta x(n) = f(n, x(n - \tau)), \quad n \geq n_0. \] (1)

A new sufficient condition is established under which every oscillatory solution of (*) tends to zero asymptotically.

Keywords: Asymptotic behavior, delay difference equation, oscillatory solution.

1. Introduction

In this paper, we consider the following first order functional delay difference equation of the form

\[ \Delta x(n) = f(n, x(n - \tau)), \quad n \in N(n_0) \] (1)

where \( \Delta \) is the forward difference operator given by \( \Delta x(n) = x(n + 1) - x(n) \), \( \tau \) is a positive integer, \( n_0 \) is a fixed integer, \( N(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots\} \), \( f : N(n_0) \times R \to R \) is a real valued function and for any \( n \in N(n_0) \), \( f(n, .) \) is a continuous function with the following properties:

\( (H_1) \) \( f(n, 0) = 0; \)
\( (H_2) \) \( uf(n, u) > 0 \) for \( u \neq 0; \) and
\( (H_3) \) there exists a sequence \( \{q(n)\} \) of positive real numbers defined on \( N(n_0) \) such that

\[ |f(n, u)| \leq q(n) |u|. \]

Qualitative theory of discrete processes has drawn considerable attention in recent years. In particular, oscillation properties of discrete analogs of delay differential equations have been studied recently by a number of authors (see e.g., [6,7,10,11]). On the other hand, relatively little is known about the asymptotic behavior of all solutions of these discrete equations, see for example [3,8,12], and the references cited therein. For the general background
of difference equations, one can refer to [1,2,5,9].

In [3], Chen et al. obtained sufficient conditions which ensure that all solutions of the first order nonlinear delay difference equation
\[
\Delta x(n) + F(n, x(n - k)) = 0, \quad n \geq n_0
\]  
(2)
tend to zero as \( n \to \infty \).

In [8], Liu et al. established sufficient conditions under which every solution of the equation
\[
\Delta x(n) = p(n)f(x(n - k)) + r(n), \quad n = 0, 1, 2, ...
\]  
(3)
converges to zero. The asymptotic behavior of the solutions of the equation
\[
\Delta x(n) + p(n)x(n - \tau) = 0, \quad n = 0, 1, 2, ...
\]  
(4)
has been extensively investigated in the literature, see for example, [4,12,13].

The purpose of this paper is to give a new sufficient condition under which every oscillatory solution of (1) tends to zero as \( n \to \infty \). By a solution of (1), we mean a nontrivial real sequence \( \{x(n)\} \) which is defined on \( N(n_0 - \tau) = \{n_0 - \tau, n_0 - \tau + 1, \ldots\} \) and which satisfies (1) for \( n \in N(n_0) \). A solution \( \{x(n)\} \) of (1) on \( N(n_0) \) is said to be oscillatory if for every positive integer \( N_0 > n_0 \), there exists \( n \geq N_0 \) such that \( x(n)x(n + 1) \leq 0 \), otherwise \( \{x(n)\} \) is said to be nonoscillatory.

Throughout this paper we use the following notations:

For any \( a, b \in N \), define
\[
N(a) = \{a, a + 1, a + 2, \ldots\},
\]
\[
N(a, b) = \{a, a + 1, a + 2, \ldots, b\},
\]
\[
Q(N) := \sup_{n \geq N} \sum_{s=n-\tau}^{n} q(s), \quad \text{for} \quad n \geq n_0 + \tau
\]
and
\[
Q_{\infty} := \lim_{N \to \infty} Q(N) = \lim_{n \to \infty} \sup_{n-\tau}^{n} q(s).
\]

2. Main Results

**Lemma 2.1** Let \( \{x(n)\} \) be a solution of (1) and \( n_0 + \tau < n_1 < n_2 - 1 \). If \( x(n) > 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \) or \( x(n) < 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \) and \( x(n_2)x(n) \leq 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \), then \( n_1 \geq n_2 - \tau \).

**Proof.** Assume the contrary, that is, \( n_1 < n_2 - \tau \). Without loss of generality, we may suppose that \( x(n) > 0 \) for \( n \in N(n_1 + 1, n_2 - 1) \). Then \( x(n_2) \leq 0 \) and there exists an integer \( n^* \) satisfying \( n_1 \leq n^* - \tau < n^* < n_2 \). Then
\[
\Delta x(n) = f(n, x(n - \tau)) > 0
\]
for \( n \in N(n^*, n_2) \), which implies \( x(n_2) > x(n^*) > 0 \). This is a contradiction.

The proof is complete.

**Lemma 2.2** Given \( \delta > 0 \), there exists an increasing sequence \( \{h(n)\} \) of nonnegative real numbers such that
\[
h(n) - h(n - \tau) = \frac{\delta}{2}, \quad n \geq n_0 + \tau.
\]  
(5)
Proof. Choose a sequence \( \{N_k\} \) of integers such that \( N_0 = n_0 \) and for \( k = 0, 1, 2, ..., N_{k+1} = N_k + \tau \). Then \( \lim_{k \to \infty} N_k = \infty \). Let us define

\[
h(n) = \frac{\delta}{2} \left( \frac{n-N_k}{\tau} + k \right), \quad \text{for} \quad n \in N(N_k, N_{k+1} - 1)
\]

for \( k = 0, 1, 2, ... \). We see that \( h(N_k) = \frac{k\delta}{2} \) for all \( k \) and \( \{h(n)\} \) is an increasing sequence on \( N(n_0) \). For any \( n \in N(N_k, N_{k+1} - 1), k = 1, 2, 3, ... \),

\[
h(n) < h(N_{k+1}) \quad \text{and} \quad h(n-\tau) \geq h(N_k - \tau),
\]

which implies

\[
h(n) - h(n-\tau) = \frac{\delta}{2} \left( \frac{n-N_k}{\tau} + k \right) - \frac{\delta}{2} \left( \frac{n-\tau-N_{k-1}}{\tau} + k - 1 \right) = \frac{\delta}{2}.
\]

Therefore (5) holds for \( n \geq N_1 \).

**Theorem 2.3** Let \( \{h(n)\} \) be an increasing sequence of positive real numbers satisfying (5) for some \( \delta > 0 \). If \( Q(N) \leq \frac{\sqrt{11}-1}{2} \) for some \( N \geq n_0+\tau \), then for any oscillatory solution \( \{x(n)\} \) of (1), there exists a \( K = K(\beta, h, x) > 0 \) such that

\[
|x(n)| < K\bar{e}^{\beta h(n)}, \quad n \in N(n_0),
\]

where

\[
\beta = \begin{cases} 
\frac{2}{3\delta} \log \left( \frac{4}{Q(N)+1} \right), \quad & Q(N) < 1 \\
\frac{2}{3\delta} \log \left( \frac{2}{Q(N)+\frac{1}{2}} \right), \quad & 1 \leq Q(N) \leq \frac{\sqrt{11}-1}{2}.
\end{cases}
\]

**Proof.** Since \( \{x(n)\} \) is an oscillatory solution of (1), there exists a sufficiently large \( n^* > N_0 + \tau \) such that \( x(n^*) \leq 0 \). We will show that (6) holds for a positive constant \( K \) such that

\[
K > \max_{n_0 \leq n \leq n^*} e^{\beta h(n)} |x(n)|.
\]

Assume that (6) does not hold. Then there exists an integer \( \xi > n^* \) such that

\[
|x(n)| < K\bar{e}^{\beta h(n)} \quad \text{for} \quad n \in N(n_0, \xi - 1) \quad \text{and} \quad |x(\xi)| \geq K\bar{e}^{\beta h(\xi)}.
\]

Then \( x(\xi) \neq 0 \). Since \( \{x(n)\} \) is oscillatory and \( \xi > n^* \), we can define two integers \( n_1, n_2 \in N(n_0) \) by

\[
n_1 = \sup \{ n : n < \xi, \quad x(n)x(\xi) \leq 0 \}
\]

and

\[
n_2 = \inf \{ n : n > \xi, \quad x(n)x(\xi) \leq 0 \}.
\]

We see that \( n^* \leq n_1 < \xi < n_2 \) and \( x(\xi) > 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \) or \( x(\xi) < 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \). Also, we have \( x(n_1)x(n) \leq 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \) and \( x(n_2)x(n) \leq 0 \) for all \( n \in N(n_1 + 1, n_2 - 1) \). Lemma 2.1 leads to \( n_1 \geq n_2 - \tau \) and hence \( n_1 \geq u - \tau \) for \( u \in N(n_1, n_2) \). Then

\[
x(u - \tau) = |x(u - \tau) - x(n_1) + x(n_1)|
\]

\[
\leq |x(n_1) - x(u - \tau)| + |x(n_1)|
\]

\[
= \left| \sum_{s=u-\tau}^{n_1-1} \Delta x(s) \right| + |x(n_1)|
\]

\[
= \left| \sum_{s=u-\tau}^{n_1-1} f(s, x(s - \tau)) \right| + |x(n_1)|
\]
which implies that for $u \in N(n_1, n_2)$,
\[
|x(u - \tau)| \leq \sum_{s = u - \tau}^{n_1 - 1} q(s) |x(s - \tau)| + |x(n_1)|. \tag{9}
\]

Moreover, because of $x(n_1)x(\xi) \leq 0$, we have
\[
|x(\xi)| \leq |x(\xi) - x(n_1)|
= \left| \sum_{u = n_1}^{\xi - 1} \Delta x(u) \right|
= \left| \sum_{u = n_1}^{\xi - 1} f(u, x(u - \tau)) \right|
\leq \sum_{u = n_2}^{\xi - 1} q(u) |x(u - \tau)|,
\]
and
\[
|x(\xi)| \leq |x(n_2) - x(\xi)|
= \left| \sum_{u = \xi}^{n_2 - 1} \Delta x(u) \right|
\leq \left| \sum_{u = \xi}^{n_2 - 1} f(u, x(u - \tau)) \right|
\leq \sum_{u = \xi}^{n_2 - 1} q(u) |x(u - \tau)|.
\]

Thus we obtain
\[
|x(\xi)| \leq \frac{1}{2} \sum_{u = n_1}^{n_2 - 1} q(u) |x(u - \tau)|. \tag{10}
\]

Here we consider two cases.

Case 1: $Q(N) < 1$. We note that $\beta > 0$. By (9) and (10),
\[
|x(\xi)| \leq \frac{1}{2} \sum_{u = n_1}^{n_2 - 1} q(u) \left( \sum_{s = u - \tau}^{n_1 - 1} q(s) |x(s - \tau)| + |x(n_1)| \right)
\]
\[
\text{or}
|x(\xi)| \leq \frac{1}{2} \sum_{u = n_1}^{n_2 - 1} q(u) \sum_{s = u - \tau}^{n_1 - 1} q(s) |x(s - \tau)| + \frac{1}{2} |x(n_1)| \sum_{u = n_1}^{n_2 - 1} q(u) \tag{11}
\]

By (8), we see that $|x(s - \tau)| < K e^{\beta h(s - \tau)}$ for $s \in N(n_0 + \tau, n_1)$. Then by (11),
\[
|x(\xi)| < \frac{1}{2} \sum_{u = n_1}^{n_2 - 1} q(u) \sum_{s = u - \tau}^{n_1 - 1} q(s) K e^{\beta h(s - \tau)} + \frac{1}{2} |x(n_1)| \sum_{u = n_1}^{n_2 - 1} q(u).
\]

Since $\{h(n)\}$ is increasing and $\beta > 0$,
\[
|x(\xi)| < K \frac{1}{2} \sum_{u = n_1}^{n_2 - 1} q(u) e^{\beta h(u - 2 \tau)} \sum_{s = u - \tau}^{n_1 - 1} q(s) + \frac{1}{2} |x(n_1)| \sum_{u = n_1}^{n_2 - 1} q(u)
\]
Then we have a contradiction to the assumption that

\[ |x(s)| < \frac{K}{2} e^{\beta h(n_1 - 2\tau)} \sum_{u=n_1}^{n_2-1} q(u) \sum_{s=u}^{n_1-1} q(s) + \frac{1}{2} |x(n_1)| \sum_{u=n_1}^{n_2-1} q(u). \]

Since

\[ \sum_{u=n_1}^{n_2-1} q(u) \sum_{s=u}^{n_1-1} q(s) \geq \frac{1}{2} \sum_{u=n_1}^{n_2-1} q(u) \sum_{s=n_1}^{n_2-1} q(s) = \frac{1}{2} \left( \sum_{u=n_1}^{n_2-1} q(u) \right)^2, \]

(12)

Then

\[ |x(s)| < \frac{K}{2} e^{\beta h(n_1 - 2\tau)} \left\{ Q(N) \sum_{u=n_1}^{n_2-1} q(u) - \frac{1}{2} \left( \sum_{u=n_1}^{n_2-1} q(u) \right)^2 \right\} + \frac{1}{2} |x(n_1)| Q(N). \]

(13)

The right side of (13) is a quadratic function of

\[ \sum_{u=n_1}^{n_2-1} q(u) \quad \text{and} \quad 0 < \sum_{u=n_1}^{n_2-1} q(u) \leq \sum_{u=n_1}^{n_2-1} q(u) \leq Q(N). \]

Then

\[ |x(s)| < \frac{K}{2} e^{\beta h(n_1 - 2\tau)} \left\{ Q^2(N) - \frac{Q^2(N)}{2} \right\} + \frac{K}{2} e^{\beta h(n_1 - 2\tau)} Q(N) \]

\[ = \frac{K}{4} e^{\beta h(n_1 - 2\tau)} \left\{ 2Q^2(N) \right\} \]

\[ = \frac{K}{4} e^{\beta h(n_1 - 2\tau)} \left\{ Q^2(N) + 2Q(N) \right\} \]

\[ \leq \frac{K}{4} e^{\beta h(\xi - 3\tau)} \left\{ (Q(N) + 1)^2 - 1 \right\} \]

\[ \leq \frac{K}{4} \left\{ (Q(N) + 1)^2 - 1 \right\} e^{\beta h(\xi - 3\tau)} \]

\[ = \frac{K}{4} \left\{ (Q(N) + 1)^2 - 1 \right\} e^{\beta h(\xi - 3\tau)} e^{-\beta h(\xi)} . \]

By Lemma 2.2, \( h(\xi) - h(\xi - 3\tau) = \frac{3\beta}{2} \).

Then, we have

\[ |x(s)| \leq \frac{1}{4} \left\{ (Q(N) + 1)^2 - 1 \right\} e^{\frac{3\beta}{2}} \left( Ke^{-\beta h(\xi)} \right) . \]

Thus, (7) implies \( |x(s)| < Ke^{\beta h(\xi)} \).

Then we have a contradiction to the assumption that \( |x(s)| \geq Ke^{\beta h(\xi)} \).

Case 2: \( 1 \leq Q(N) \leq \frac{\sqrt{n_1} - 1}{2} \). We note that \( \beta \geq 0 \). There are two possibilities.

Case 2.1: \( 1 \leq \sum_{u=n_1}^{n_2-1} q(u) \leq \frac{\sqrt{n_1} - 1}{2} \). There exists an integer \( \eta \) such that \( n_1 \leq \eta \leq n_2 - 1 \) and \( \sum_{u=\eta}^{n_2-1} q(u) \geq 1 \).

By (9) and (10), we have

\[ |x(s)| \leq \frac{1}{2} \sum_{u=n_1}^{n_2-1} q(u) |x(u - \tau)| \]
\[
\begin{align*}
\sum_{n=1}^{\eta-1} q(n) \geq 1,
\end{align*}
\]

\[
\sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) |x(s - \tau)| = \left( \sum_{u=\eta}^{n_2-1} q(u) \right) \left( \sum_{s=\eta}^{n_2-1} q(s) |x(s - \tau)| \right) \geq \sum_{s=\eta}^{n_2-1} q(s) |x(s - \tau)|
\]

which implies
\[
|x(\xi)| \leq \frac{1}{2} \sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) |x(s - \tau)| + \frac{1}{2} \sum_{u=\eta}^{n_2-1} q(u) |x(n_1)|.
\]

(14)

By (8) and the fact that \( s - \tau \leq n_1 < \xi \) for \( s \in N(n_0 + \tau, n_2) \), \(|x(s - \tau)| < Ke^{-\beta h(s - \tau)}\) for \( s \in N(n_0 + \tau, n) \). Then by (14),
\[
|x(\xi)| \leq \frac{1}{2} \sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s)Ke^{\beta h(s - \tau)} + \frac{K}{2} e^{-\beta h(n_1 - \tau)} \sum_{u=\eta}^{n_2-1} q(u)
\]

\[
\leq K \frac{n_2-1}{2} \sum_{u=\eta}^{n_2-1} q(u)e^{\beta(u-2\tau)} \sum_{s=\eta}^{n_2-1} q(s) + \frac{K}{2} e^{-\beta h(n_1 - \tau)} \sum_{s=\eta}^{n_2-1} q(s)
\]

\[
\leq K \frac{n_2-1}{2} e^{\beta h(n_1 - 2\tau)} \left\{ \sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) + \sum_{s=\eta}^{n_2-1} q(s) \right\}
\]

\[
\leq K \frac{n_2-1}{2} e^{\beta h(n_1 - 2\tau)} \left\{ \sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) + Q(N) \right\}
\]

\[
= K \frac{n_2-1}{2} e^{\beta h(n_1 - 2\tau)} \left\{ \sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) - \sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) + Q(N) \right\}.
\]

Since
\[
\sum_{u=\eta}^{n_2-1} q(u) \sum_{s=\eta}^{n_2-1} q(s) \geq \frac{1}{2} \left( \sum_{u=\eta}^{n_2-1} q(u) \right)^2,
\]

\[
|x(\xi)| < K \left\{ Q(N) \sum_{u=\eta}^{n_2-1} q(u) - \frac{1}{2} \left( \sum_{u=\eta}^{n_2-1} q(u) \right)^2 + Q(N) \right\}
\]

\[
\leq K \left\{ Q^2(N) - \frac{1}{2} + Q(N) \right\}.
\]
By (7),
\[
\frac{K}{2} e^{\beta h(n_1 - 2\tau)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}
\]
\[
\leq K e^{\beta h(n_1 - 2\tau)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}
\]
\[
= K e^{\beta h(\xi - 3\tau)} e^{-\beta h(\xi)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}
\]
\[
= K e^{3\beta h} e^{-\beta h(\xi)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}.
\]
Thus, (7) implies $|x(\xi)| < K e^{\beta h(\xi)}$. Then we have a contradiction to the assumption that $|x(\xi)| \geq K e^{\beta h(\xi)}$.

**Case 2.2:** $\sum_{n=n_1}^{n_2-1} q(n) < 1$. In the same way as a Case 1, we have
\[
|x(\xi)| < K e^{\beta h(n_1 - 2\tau)} \left\{ Q(N) \sum_{u=n_1}^{n_2-1} q(u) - \frac{1}{2} \left( \sum_{u=n_1}^{n_2-1} q(u) \right)^2 + Q(N) \right\}.
\]
Since $Q(N) \sum_{u=n_1}^{n_2-1} q(u) - \frac{1}{2} \left( \sum_{u=n_1}^{n_2-1} q(u) \right)^2$ is a quadratic function of $\sum_{u=n_1}^{n_2-1} q(u)$ and $0 < \sum_{u=n_1}^{n_2-1} q(u) < 1 \leq Q(N)$, we have
\[
|x(\xi)| < K e^{\beta h(n_1 - 2\tau)} \left\{ Q(N) \cdot 1 - \frac{1}{2} \cdot 1^2 + Q(N) \right\}
\]
\[
\leq K e^{\beta h(n_1 - 2\tau)} \left\{ Q^2(N) + Q(N) - \frac{1}{2} \right\}
\]
\[
\leq K e^{\beta h(\xi - 3\tau)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}
\]
\[
= K e^{\beta h(\xi - 3\tau)} e^{-\beta h(\xi)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}
\]
\[
= K e^{\frac{3\beta h}{2}} e^{-\beta h(\xi)} \left\{ (Q(N) + \frac{1}{2})^2 - \frac{3}{4} \right\}.
\]
By (7), $|x(\xi)| < K e^{\beta h(\xi)}$. Then we have a contradiction to the assumption that $|x(\xi)| \geq K e^{\beta h(\xi)}$. Hence, by virtue of the Case 1 and 2, we obtain (6). The proof is complete.

Using Theorem 2.3, we have two corollaries.

**Corollary 2.4** If $Q(N) \leq \frac{\sqrt{11} - 1}{2}$ for some $N \geq n_0 + \tau$, then every oscillatory solution of (1) is bounded.

**Corollary 2.5** If $Q_{\infty} < \frac{\sqrt{11} - 1}{2}$, then every oscillatory solution of (1) tends to zero as $n \to \infty$.

In the case where $\frac{\sqrt{11} - 1}{2} < Q(N) < \infty$ for some $N \geq n_0 + \tau$ we can prove the following theorem in the same way as case 1 in the proof of Theorem 2.3:

**Theorem 2.6** If $\{h(n)\}$ is an increasing sequence of positive real numbers defined on $N(n_0)$ and $Q(N) > \frac{\sqrt{11} - 1}{2}$ for some $N \geq n_0 + \tau$, then for any oscillatory solution $\{x(n)\}$ of (1), there exists a constant $K = K(h, x) > 0$ such that
\[
|x(n)| < K e^{h(n)}, \quad n \geq n_0.
\]
Lemma 2.7 Let \( f(n, x(n) - \tau) = q(n)x(n - \tau) \), where \( \{q(n)\} \) is a sequence of nonnegative and real numbers defined on \( N(n_0) \), \( \{x(n)\} \) be a solution of (1) and \( Q_\infty < \infty \). If \( \{x(n)\} \) is not oscillatory, then there exist a \( N_0 = N_0(x) > n_0 \) and a positive constant \( C = C(x) \) such that

\[
|x(n)| \geq C \exp \left\{ \frac{1}{Q_\infty + 2} \sum_{s=n_0}^{n-1} q(s) \right\}, \quad n > N_0.
\]

Proof. Without loss of generality, we may assume that \( \{x(n)\} \) is eventually positive, i.e., there exists \( n_1 > n_0 \) such that \( x(n) > 0 \) for any \( n > n_1 \). Choose \( N_0 > n_0 \), such that \( N_0 - 3\tau > n_1 \) and \( \sum_{s=n_1}^{n} q(s) < Q_\infty + 1, \quad n > N_0 \).

If a solution \( x(n) \) satisfies (15). Choose \( \sum_{s=n_1}^{n} q(s) < Q_\infty + 1 \), \( n > N_0 \).

Then

\[
\frac{x(n)}{x(n+1)} \geq \frac{1}{Q_\infty + 2},
\]

which leads to that

\[
x(n) \geq x(N_0) \exp \left\{ \frac{1}{Q_\infty + 2} \sum_{s=N_0}^{n-1} q(s) \right\}
\]

\[
= x(N_0) \exp \left\{ \frac{-1}{Q_\infty + 2} \sum_{s=N_0}^{n-1} q(s) \right\} \exp \left\{ \frac{1}{Q_\infty + 2} \sum_{s=n_0}^{n-1} q(s) \right\}, \quad n > N_0.
\]

The proof is complete. By this lemma we obtain the following theorem:

Theorem 2.8 Assume that \( Q_\infty < \frac{\sqrt{11} - 1}{2} \).

(i) If a solution \( \{x(n)\} \) of (1) satisfies

\[
\lim_{n \to \infty} \frac{x(n)}{\exp \left\{ \frac{1}{Q_\infty + 2} \sum_{s=n_0}^{n-1} q(s) \right\}} = 0,
\]

then \( x(n) \) tends to zero as \( n \to \infty \).

(ii) If

\[
\sum_{s=n_0}^{\infty} q(s) = \infty,
\]

then every bounded solution of (1) tends to zero as \( n \to \infty \).

Proof. (i) By Lemma 2.7 and (15), \( \{x(n)\} \) is oscillatory. Therefore, by Corollary 2.5, \( x(n) \) tends to zero as \( n \to \infty \).

(ii) Let \( \{x(n)\} \) be a bounded solution of (1). By (16), \( \{x(n)\} \) satisfies (15).

Hence \( x(n) \) tends to zero as \( n \to \infty \).

The proof is complete.
3. Equations with special forcing term

Let $A_0$ be the set of all real sequences $\{a(n)\}$ defined on $N(n_0 - \tau)$ such that

$$\lim_{n \to \infty} a(n) = 0.$$  

Consider the following equation:

$$\Delta x(n) = q(n)x(n - \tau) + r(n), \quad n \geq n_0,$$

where the sequence $\{r(n)\}$ is given by $r(n) = q(n)a(n - \tau) - \Delta a(n)$ with some $\{a(n)\} \in A_0$. We compare the asymptotic behavior of the oscillatory solution of (17) with that of the equation (1).

**Lemma 3.1** Let $\{y(n)\}$ be a sequence of real numbers defined on $N(n_0)$ and $\{a(n)\} \in A_0$. If $z(n) = y(n) + a(n)$ is a solution of (1) and $\{y(n)\}$ is oscillatory, then $\{z(n)\}$ is also oscillatory.

**Proof.** Assume that $\{z(n)\}$ is not oscillatory. Then there exists $n_1 > n_0 + \tau$ such that $|z(n)| > 0$ for $n > n_1$. Without loss of generality, we may assume that $z(n) > 0$. Since $\{z(n)\}$ is a solution of (1) and $q(n) \geq 0$, $\Delta z(n) = q(n)z(n - \tau) \geq 0$ for $n > n_2$ for some $n_2 > n_1 + \tau$. Then $z(n) \geq z(n_2) > 0$ for $n > n_2$. Since $\{a(n)\}$ tends to zero as $n \to \infty$, we have

$$\liminf_{n \to \infty} y(n) = \liminf_{n \to \infty} \{z(n) - a(n)\} \geq z(n_2) > 0.$$  

This is a contradiction to the assumption that $\{y(n)\}$ is oscillatory. The proof is complete.

**Theorem 3.2** If every oscillatory solution of (1) tends to zero as $n \to \infty$, then every oscillatory solution of (17) tends to zero as $n \to \infty$.

**Proof.** Let $\{x(n)\}$ be an oscillatory solution of (17). Then it follows that

$$\Delta x(n) = q(n)x(n - \tau) + q(n)a(n - \tau) - \Delta a(n),$$

which implies that

$$\Delta(x(n) + a(n)) = q(n)(x(n - \tau) + a(n - \tau)).$$

Set

$$z(n) = x(n) + a(n).$$

Then $\{z(n)\}$ is a solution of (1), we see from Lemma 3.1 that $\{z(n)\}$ is oscillatory. Therefore $z(n)$ tends to zero as $n \to \infty$ by assumption. Hence $x(n) = z(n) - a(n)$ tends to zero as $n \to \infty$.

The proof is complete.

**Corollary 3.3** If $Q_\infty < \frac{\sqrt{\pi} - 1}{2}$, then every oscillatory solution of (17) tends to zero as $n \to \infty$.

**Proof.** By Corollary 2.5, every oscillatory solution of (1) tends to zero as $n \to \infty$. By Theorem 3.2, we obtain that every oscillatory solution of (17) tends to zero as $n \to \infty$.

The proof is complete. Now, let us consider the equations:

$$\Delta x(n) = qx(n - \tau) + \mu \lambda^{-n}, \quad n \geq n_0$$  \hspace{1cm} (18)

and

$$\Delta x(n) = qx(n - \tau)$$  \hspace{1cm} (19)

where $\mu$ is a constant, $\tau$ is a positive integer and $q$, $\lambda$ are positive real numbers with $\lambda > 1$.

**Theorem 3.4** Every oscillatory solution of (18) tends to zero as $n \to \infty$, if and only if every oscillatory solution of (19) tends to zero as $n \to \infty$.  

Proof. Sufficiency. Suppose that every oscillatory solution of (19) tends to zero as $n \to \infty$. Define a sequence \( \{a(n)\} \) on \( N(n_0 - \tau) \) by
\[
a(n) = \frac{\mu \lambda^{-n}}{1 - \frac{\lambda}{\chi}}.
\]
Then we see that \( \{a(n)\} \in A_0 \) and \( \mu \lambda^{-n} = qa(n - \tau) - \Delta a(n) \). By Theorem 3.2 every oscillatory solution of (18) tends to zero as $n \to \infty$.

Necessity. Suppose that every oscillatory solution of (18) tends to zero as $n \to \infty$ for some $\mu \neq 0$. Let \( \{y(n)\} \) be an oscillatory solution of (19) and let \( z(n) = y(n) - a(n) \). Then
\[
\Delta z(n) = qz(n - \tau) + qa(n - \tau) - \Delta a(n)
\]
which means that \( \{z(n)\} \) is a solution of (18). We will prove that $z(n)$ tends to zero as $n \to \infty$. If \( \{z(n)\} \) is oscillatory, then $z(n)$ tends to zero as $n \to \infty$ by assumption. Therefore it is enough to consider the case that \( \{z(n)\} \) is not oscillatory. We will show that for some $N^* > n_0$,
\[
0 < |z(n)| \leq \frac{|\mu|}{1 - \frac{\lambda}{\chi}} \lambda^{-n}, \quad n > N^*.
\]
(20)

Let $\mu > 0$. Assume that $z(n) > 0$ for $n > N_1$ for some $N_1 > n_0$. Since $qz(n - \tau) > 0$ for $n > N_1 + \tau$, $\Delta z(n) > 0$ for $n > N_1 + \tau$. Then \( \{z(n)\} \) is monotonic increasing as $N(N_1 + \tau)$, which implies that for $N_2 > N_1 + \tau$,
\[
\liminf_{n \to \infty} y(n) \geq \liminf_{n \to \infty} z(n) + \liminf_{n \to \infty} a(n)
\]
\[
\geq z(N_2) > 0.
\]
Since \( \{y(n)\} \) is oscillatory, we have a contradiction. Hence $z(n) < 0$ for $n > N_3$ for some $N_3 > n_0$. Then
\[
\Delta \left( z(n) + \frac{\mu \lambda^{-n}}{1 - \frac{\lambda}{\chi}} \right) = qz(n - \tau) < 0, \quad n > N_3 + \tau,
\]
which implies that \( \left\{ z(n) + \frac{\mu \lambda^{-n}}{1 - \frac{\lambda}{\chi}} \right\} \) is monotone decreasing on $N(N_3 + \tau)$. Then we have that for any $m > N_3 + \tau$,
\[
\limsup_{n \to \infty} y(n) = \limsup_{n \to \infty} (z(n) + a(n))
\]
\[
\leq \left( \limsup_{n \to \infty} z(n) + \frac{\mu \lambda^{-n}}{1 - \frac{\lambda}{\chi}} \right) + \limsup_{n \to \infty} \left( a(n) - \frac{\mu \lambda^{-n}}{1 - \frac{\lambda}{\chi}} \right)
\]
\[
\leq z(m) + \frac{\mu \lambda^{-m}}{1 - \frac{\lambda}{\chi}}.
\]
Since $\limsup_{n \to \infty} y(n) \geq 0$, $z(m) + \frac{\mu \lambda^{-m}}{1 - \frac{\lambda}{\chi}} \geq 0$ for $m > N_3 + \tau$. Therefore $0 > z(n) \geq \frac{-\mu \lambda^{-m}}{1 - \frac{\lambda}{\chi}}$, $n > N_3 + \tau$. In case $\mu < 0$, we see in the same way that $0 < z(n) \leq \frac{-\mu \lambda^{-m}}{1 - \frac{\lambda}{\chi}}$, $n > N_4$, for some $N_4 > n_0$. Then we have (20), and hence $z(n)$ tends to zero as $n \to \infty$ when $z(n)$ is not oscillatory. Therefore $y(n) = z(n) + a(n)$ tends to zero as $n \to \infty$, which implies that every oscillatory solution of (19) tends to zero as $n \to \infty$.

The proof is complete.

References


