



Existence and stability results for neutral stochastic delay differential equations driven by a fractional Brownian motion

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Abstract

In this paper we investigate the existence, uniqueness, asymptotic behavior of mild solutions to neutral stochastic differential equations with delays driven by a fractional Brownian motion in a Hilbert space. The cases of finite and infinite delays are analyzed.

Keywords: Asymptotic Behaviors; Delays; Fractional Brownian Motion; Mild Solution; Wiener Integral.

1. Introduction

The theory of stochastic differential equations driven by a fractional Brownian motion (fBm) has been studied intensively in recent years [1], [2], [3], [4], and [5]. The fBm received much attention because of its huge range of potential applications in several fields like telecommunications, networks, finance markets, biology and so on [6], [7], [8], [9]. Moreover, one of the simplest stochastic processes that is Gaussian, self-similar, and has stationary increments is fBm [10]. In particular, fBm is a generalization of the classical Brownian motion, which depends on a parameter $H \in (0, 1)$ called the Hurst index [9]. It should be mentioned that when $H = \frac{1}{2}$, the stochastic process is a standard Brownian motion; when $H \neq \frac{1}{2}$, it behaves completely in a different way than the standard Brownian motion, in particular neither is a semimartingale nor a Markov process. It is a self-similar process with stationary increments and has a long-memory when $H \neq \frac{1}{2}$. These significant properties make fBm a natural candidate as a model for noise in a wide variety of physical phenomena such as mathematical finance, communication networks, hydrology and medicine. The existence and uniqueness of mild solutions for a class of stochastic differential equations in a Hilbert space with a standard, cylindrical fBm with the Hurst parameter in the interval $(\frac{1}{2}, 1)$ has been studied [11]. Maslowski and Nualart [12] have studied the existence and uniqueness of a mild solution for nonlinear stochastic evolution equations in a Hilbert space driven by a cylindrical fBm under some regularity and boundedness conditions on the coefficients. Recently, Caraballo and et al [13] investigated the existence and uniqueness of mild solutions to stochastic delay equations driven by fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. An existence and uniqueness result of mild solutions for a class of neutral stochastic differential equation with finite delay, driven by an fBm in a Hilbert space has been investigated [14] in Boufoussi and Hajji. The asymptotic behavior of solutions for stochastic differential equations with fBm has only been investigated by a few authors [13], [14], [15]. Moreover Nguyen [16] has studied the asymptotic behaviors of mild solutions to neutral stochastic differential equations driven by an fBm. Motivated by this consideration in this paper we investigate the existence and uniqueness and asymptotic behaviors of mild solutions for a neutral stochastic differential equation with finite or infinite delays driven by fBm in the following form

$$d \left[x(t) - g \left(t, x(t - r(t)) \right) \right] = \left[Ax(t) + f \left(t, x(t - \rho(t)) \right) \right] dt + h \left(t, x(t - \delta(t)) \right) dW(t) + \sigma(t) dB_Q^H(t) ; t \geq 0$$

$$x(t) = \phi(t), t \in (-\tau, 0] (0 < \tau \leq \infty) \tag{1.1}$$

where A is an infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t \geq 0}$ in a Hilbert space X with norm $\|\cdot\|$, $B_Q^H(t)$ denotes an fBm with $H > \frac{1}{2}$ on a real and separable Hilbert space Y , $r, \rho, \delta: [0, \infty) \rightarrow [0, \tau]$ are continuous, $f, g: [0, \infty) \times X \rightarrow X$, $h: [0, \infty) \times X \rightarrow \mathbb{L}_2^0$, $\sigma: [0, \infty) \rightarrow \mathcal{L}_Q^0(Y, X)$, the initial data $\phi \in C((-\tau, 0], X)$ the space of all continuous functions from $(-\tau, 0]$ to X and has finite second moments. Further we assume W and B^H are independent. The main tool of this paper is the fixed point theory which was proposed by Burton [17].

2. Preliminaries

In this section we first recall the fBm as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout the paper. Let (Ω, \mathcal{F}, P) be a complete probability space and $T > 0$ be an arbitrary fixed horizon. Let Y and X be two real, separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators from Y to X .

Let $\mathbb{L}_2^0(Y, X)$ be separable Hilbert space with respect to the Hilbert-Schmidt norm $\|\cdot\|_{HS}$. Let $\mathcal{L}_Q^0(Y, X)$ be the space of all $\psi \in \mathcal{L}(Y, X)$ such that $\psi Q^{\frac{1}{2}}$ is a Hilbert Schmidt operator. The norm is given by $\|\psi\|_{\mathcal{L}_Q^0}^2 = \|\psi Q^{\frac{1}{2}}\|_{HS}^2 = \text{tr}(\psi Q \psi^*)$. Then ψ is called a Q Hilbert –Schmidt operator from Y to X . Let $\mathcal{L}_2^0(\Omega, X)$ denote the space of \mathcal{F}_0 - measurable, X -valued and square integrable Stochastic processes. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ be a two sided one dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$ with the covariance function

$$R_H(t, s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), t, s \in \mathbb{R}.$$

It is known that $\beta^H(t)$ with $H > \frac{1}{2}$ admits the following Wiener integral representation

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s),$$

where $W = \{W(t): t \in [0, T]\}$ a Wiener is process and $K_H(t, s)$ is the kernel given by

$$K_H(t, s) = c_H \int_s^t (u - s)^{H-3/2} \left(\frac{u}{s}\right)^{H-1/2} du, t > s.$$

with $c_H = \frac{1}{\sqrt{B(\frac{H(2H-1)}{2-2H, H-\frac{1}{2}})}}$ with $\beta(\cdot)$ representing the beta function. Let $K_H(t, s) = 0$ if $t \leq s$.

We will denote by \mathcal{H} the reproducing kernel Hilbert space of the fBm. Infact \mathcal{H} is the closure of set of indicator functions $\{1_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $1_{[0,t]} \rightarrow \beta^H(t)$ can be extended to an isometry between \mathcal{H} and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ by the previous isometry. We recall for $\psi, \varphi \in \mathcal{H}$ their scalar product in \mathcal{H} is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H - 1) \int_0^T \int_0^T \psi(s)\varphi(t) |t - s|^{2H-2} ds dt$$

For the deterministic function $\varphi \in L^2([0, T])$, the fractional Weiner integral of φ with respect to β^H is defined by

$$\int_0^T \varphi(s) d\beta^H(t) = \int_0^T K_H^* \varphi(s) d\beta^H(s),$$

where $K_H^* \varphi(s) = \int_s^T \varphi(r) \frac{\partial k}{\partial r}(r, s) dr$

Let $\{\beta_n^H(t)\}_{n \in \mathbb{N}}$ be a sequence of two-sided one dimensional standard fBm mutually independent on (Ω, \mathcal{F}, P) . When one consider the following series

$$\sum_{n=1}^{\infty} \beta_n^H(t) e_n, t \geq 0$$

where $\{e_n\}$ is a orthonormal basis in Y , this series does not necessarily converge in the space Y . Thus we consider a Y - valued stochastic process $B_Q^H(t)$ given formally by the following series

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n, t \geq 0$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space Y , that is, it holds that $B_Q^H(t) \in L^2(\Omega, Y)$ then we say that the above $B_Q^H(t)$ is a Y - valued Q - cylindrical fBm with covariance operator Q . For example if $\{\lambda_n\}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \lambda_n e_n$ assuming that Q is a nuclear operator in Y (ie., $\sum_{n=1}^{\infty} \lambda_n < \infty$) then the stochastic process

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n^H(t) e_n \quad t \geq 0$$

is well defined as a Y -valued Q - cylindrical fBm.

where $\beta_n^H(t)$ are real, independent fBm's. This process is a Y -valued Gaussian, it starts from 0, zero mean and covariance

$$E\langle \beta^H(t, x) | \beta^H(s, y) \rangle = R(s, t) \langle Q(x), y \rangle \text{ For all } x, y \in Y \text{ and } t, s \in [0, T].$$

Definition 2.1: Let $\varphi: [0, T] \rightarrow \mathcal{L}_Q^0(Y, X)$ such that $\sum_{n=1}^{\infty} \left\| K_H^* \varphi Q^{1/2} e_n \right\|_{L^2([0, T]; X)} < \infty$ (2.1)

Then its stochastic integral with respect to the fBm B_Q^H is defined, for $t \geq 0$, as follows

$$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) \sqrt{\lambda_n} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} k_H^*(\varphi e_n)(s) d\beta_n(s)$$

Notice that if

$$\sum_{n=1}^{\infty} \left\| \varphi \sqrt{\lambda_n} e_n \right\|_{L_H^1([0, T]; H)} < \infty \quad (2.2)$$

Lemma 2.2: For any $\varphi: [0, T] \rightarrow \mathcal{L}_Q^0(Y, X)$ satisfies $\int_0^t \|\varphi(s)\|_{\mathcal{L}_Q^0}^2 ds < \infty$ such that (2.2) holds, and for any $\alpha, \beta \in [0, T]$ with $\alpha > \beta$,

$$E \left\| \int_{\beta}^{\alpha} \varphi(s) dB_Q^H(s) \right\|^2 \leq cH(2H - 1)(\alpha - \beta)^{2H-1} \sum_{n=1}^{\infty} \int_{\beta}^{\alpha} \left\| \varphi(s) Q^{1/2} e_n \right\|^2 ds$$

where $c = c(H)$. If in addition, $\sum_{n=1}^{\infty} \left\| \varphi(t) Q^{1/2} e_n \right\|$ is uniformly convergent for $t \in [0, T]$. Then

$$E \left\| \int_{\beta}^{\alpha} \varphi(s) dB_Q^H(s) \right\|^2 \leq cH(2H - 1)(\alpha - \beta)^{2H-1} \int_{\beta}^{\alpha} \|\varphi(s)\|_{\mathcal{L}_Q^0}^2 ds$$

We now suppose that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A , and the semigroup $S(t)$ is uniformly bounded that is to say $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$. Then, for $0 < \alpha \leq 1$, it is possible to define the fractional power operator $(-A)^\alpha$ as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore the subspace $D(-A)^\alpha$ is dense in X and the expression $\|X\|_\alpha = \|(-A)^\alpha X\|$, $x \in D(-A)^\alpha$ defines a norm on $X_\alpha = D(-A)^\alpha$. The following properties are well known in [18].

Lemma 2.3:[18] Under the above conditions the following properties hold.

- i) X_α is a Banach space for $0 < \alpha \leq 1$
- ii) If the resolvent operator of A is compact, then embedding $X_\beta \subset X_\alpha$ is continuous and compact for $0 < \alpha \leq \beta$.
- iii) For every $0 < \alpha \leq 1$, there exists M_α such that $\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}$, $\lambda > 0$, $t \geq 0$.

Lemma 2.4:[19] For any $r \geq 1$ and for arbitrary L_2^0 valued predictable process $\varphi(\cdot)$,

$$\sup_{s \in [0, t]} E \left\| \int_0^s \varphi(u) dw(u) \right\|_X^{2r} \leq (r(2r - 1))^r \left(\int_0^t (E \|\varphi(s)\|_{L_2^0}^{2r})^{\frac{1}{r}} ds \right)^r$$

3. Existence and uniqueness of a solution

In this section we study the existence and uniqueness of mild solutions for equation (1.1). For this equation we assume that the following conditions hold.

(H₁) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X . Further we suppose that $0 \in \rho(A)$ and that $\|S(t)\| \leq M$ and

$$\|(-A)^{1-\beta} S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}.$$

For some constants M , $M_{1-\beta}$ and every $t \in [0, T]$

(H₂) There exists a positive constants C_1, C_2 and $L_1, L_2 > 0$ such that, for all $t \geq 0, x, y \in X$

- i) $\|f(t, x) - f(t, y)\|^2 \leq L_1 \|x - y\|^2$
- ii) $\|f(t, x)\|^2 \leq C_1(1 + \|x\|^2)$
- iii) $\|h(t, x) - h(t, y)\|^2 \leq L_2 \|x - y\|^2$
- iv) $\|h(t, x)\|^2 \leq C_2(1 + \|x\|^2)$

(H₃) There exists a constant $0 < \beta < 1, L_3 > 0$ such that the function g is X_β -valued and satisfies for all $t \geq 0, x, y \in X$ such that

- i) $\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq L_3 \|x - y\|^2$
- ii) $\|(-A)^\beta g(t, x)\|^2 \leq C_3(1 + \|x\|^2)$
- iii) The constants L_3 and $\beta, \|(-A)^{-\beta}\|_{L_3} < 1$

(H₄) The function $(-A)^\beta g$ is continuous in the quadratic mean sense square. For all functions X

$$\lim_{t \rightarrow \infty} \|(-A)^\beta g(t, x(t)) - (-A)^\beta g(s, x(s))\|^2 = 0$$

(H₅) The function $\sigma: [0, \infty) \rightarrow \mathcal{L}_Q^0(Y, X)$ satisfies $\int_0^T \|\sigma(s)\|_{\mathcal{L}_Q^0}^2 ds < \infty$, for all $T > 0$.

Moreover we assume that $\phi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$.

Definition 3.1: A X valued stochastic process $\{x(t), t \in (-\tau, T)\}$ is called a mild solution of Eqn. (1.1) if

- i) $x(\cdot) \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$
- ii) $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$
- iii) For arbitrary $t \in [0, T]$ we have

$$\begin{aligned} x(t) = & S(t) \left(\varphi(0) + g \left(0, \varphi(-r(0)) \right) \right) - g \left(t, x(t - r(t)) \right) \\ & - \int_0^t AS(t-s) g \left(s, x(s - r(s)) \right) ds + \int_0^t S(t-s) f \left(s, x(s - \rho(s)) \right) ds \\ & + \int_0^t S(t-s) h \left(s, x(s - \delta(s)) \right) dW(s) + \int_0^t S(t-s) \sigma(s) dB^H(s) \end{aligned}$$

Theorem 3.1: Suppose (H_1) to (H_5) hold. Then for all $T > 0$, then Eqn. (1.1) has a unique mild solution on $[-\tau, T]$.

Proof: Fix $T > 0$ and let $\mathcal{B} := \mathcal{C}[-\tau, T], \mathbb{L}^2(\Omega, X)$, be the Banach space of all continuous functions from $[-\tau, T]$ into $\mathbb{L}^2(\Omega, X)$, equipped with the supremum norm $\|\xi\|_{\mathcal{B}_T} = \sup_{u \in [-\tau, T]} (E \|\xi(u)\|^2)^{1/2}$ and let us consider the set $S_T = \{x \in \mathcal{B}_T: x(s) = \phi(s) \text{ for } s \in [-\tau, 0]\}$.

S_T is a closed subset of \mathcal{B}_T provided with the norm $\|\cdot\|_{\mathcal{B}_T}$. Define the operator ψ on S_T by $\psi(x)(t) = \phi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0, T]$

$$\begin{aligned} \psi(x)(t) &= S(t) \left(\phi(0) + g \left(0, \phi(-r(0)) \right) \right) - g \left(t, x(t-r(t)) \right) \\ &\quad - \int_0^t AS(t-s)g \left(s, x(s-r(s)) \right) ds + \int_0^t S(t-s) f \left(s, x(s-\rho(s)) \right) ds \\ &\quad + \int_0^t S(t-s) h \left(s, x(s-\delta(s)) \right) dW(s) + \int_0^t S(t-s) \sigma(s) dB^H(s) \end{aligned}$$

Then it is clear to prove the existence of mild solutions to Eqn. (1.1) is equivalent to find a fixed point for the operator ψ . Next we will show by Banach fixed point theorem that ψ has a unique fixed point. We divide the proof into two steps. Step 1. For arbitrary $x \in S_T$, let us prove that $t \rightarrow \psi(x)(t)$ is continuous on the interval $[0, T]$ in the $L^2(\Omega, X)$ sense. Let $0 < t < T$ and $|h|$ be sufficiently small. Then for any $x \in S_T$, we have

$$\begin{aligned} \|\psi(x)(t+h) - \psi(x)(t)\| &\leq \left\| (S(t+h) - S(t)) \left(\phi(0) + g \left(0, \phi(-r(0)) \right) \right) \right\| \\ &\quad + \left\| g \left(t+h, x(t+h-r(t+h)) \right) - g \left(t, x(t-r(t)) \right) \right\| \\ &\quad + \left\| \int_0^{t+h} AS(t+h-s)g \left(s, x(s-r(s)) \right) ds - \int_0^t AS(t-s)g \left(s, x(s-r(s)) \right) ds \right\| \\ &\quad + \left\| \int_0^{t+h} S(t+h-s) f \left(s, x(s-\rho(s)) \right) ds - \int_0^t S(t-s) f \left(s, x(s-\rho(s)) \right) ds \right\| \\ &\quad + \left\| \int_0^{t+h} S(t+h-s) h \left(s, x(s-\delta(s)) \right) dw(s) - \int_0^t S(t-s) h \left(s, x(s-\delta(s)) \right) dw(s) \right\| \\ &\quad + \left\| \int_0^{t+h} S(t+h-s) \sigma(s) dB^H(s) - \int_0^t S(t-s) \sigma(s) dB^H(s) \right\| \\ &= \sum_{1 \leq i \leq 6} I_i(h) \end{aligned}$$

By the strong continuity of $S(t)$, we have

$$\lim_{h \rightarrow 0} \left\| (S(t+h) - S(t)) \left(\phi(0) + g \left(0, \phi(-r(0)) \right) \right) \right\| = 0$$

The condition (H_1) assure that

$$\left\| (S(t+h) - S(t)) \left(\phi(0) + g \left(0, \phi(-r(0)) \right) \right) \right\| \leq 2M \left\| \phi(0) + g \left(0, \phi(-r(0)) \right) \right\| \in L^2(\Omega)$$

Then we conclude by the Lebesgue dominated theorem that $\lim_{h \rightarrow 0} \mathbb{E}|I_1(h)| = 0$

By using the fact that the operator $(-A)^{-\beta}$ is bounded, we obtained that

$$\mathbb{E}|I_2(h)|^2 \leq \|(-A)^{-\beta}\|^2 \mathbb{E} \left\| (-A)^\beta g \left(t+h, x(t+h-r(t+h)) \right) - (-A)^\beta g \left(t, x(t-r(t)) \right) \right\|^2$$

Then we conclude by condition (H_4) that $\lim_{h \rightarrow 0} \mathbb{E}|I_2(h)| = 0$

For $I_3(h)$, we suppose that $h > 0$, then we have

$$\begin{aligned} I_3(h) &\leq \left\| \int_0^t S((h)-I)(-A)^{1-\beta} S(t-s)(-A)^\beta g \left(s, x(s-r(s)) \right) ds \right\| \\ &\quad + \left\| \int_t^{t+h} (-A)^{1-\beta} S(t+h-s)(-A)^\beta g \left(s, x(s-r(s)) \right) ds \right\| \end{aligned}$$

By Holder's inequality

$$\begin{aligned} \mathbb{E} \left\| \int_0^t S((h)-I)(-A)^{1-\beta} S(t-s)(-A)^\beta g \left(s, x(s-r(s)) \right) ds \right\|^2 \\ \leq t \int_0^t \left\| S((h)-I)(-A)^{1-\beta} S(t-s)(-A)^\beta g \left(s, x(s-r(s)) \right) \right\|^2 ds \end{aligned}$$

By using the strong continuity of $S(t)$, we have for each $s \in [0, t]$,

$$\lim_{h \rightarrow 0} S((h) - I)(-A)^{1-\beta} S(t-s)(-A)^\beta g(s, x(s-r(s))) = 0$$

By using condition (H_1) , condition (ii) in (H_3) and the fact that $0 < \beta < 1$, we obtain

$$\begin{aligned} & \left\| S((h) - I)(-A)^{1-\beta} S(t-s)(-A)^\beta g(s, x(s-r(s))) \right\| \\ & \leq \frac{(M+1)M_{1-\beta}}{(t-s)^{1-\beta}} \left\| (-A)^\beta g(s, x(s-r(s))) \right\| \in \mathbb{L}^2([0, t] \times \Omega) \end{aligned}$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_0^t S((h) - I)(-A)^{1-\beta} S(t-s)(-A)^\beta g(s, x(s-r(s))) ds \right\|^2 = 0$$

By conditions (H_1) , condition (ii) in (H_3) and Holder's inequality we get

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} (-A)^{1-\beta} S(t+h-s)(-A)^\beta g(s, x(s-r(s))) ds \right\|^2 \\ & \leq \frac{M^2_{1-\beta}}{2\beta-1} h^{2\beta-1} \int_0^T C_3^2 (\mathbb{E} \|x(s) - r(s)\|^2 + 1) ds \end{aligned}$$

Then $\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_t^{t+h} (-A)^{1-\beta} S(t+h-s)(-A)^\beta g(s, x(s-r(s))) ds \right\|^2 = 0$

Hence that $\lim_{h \rightarrow 0} \mathbb{E} |I_3(h)| = 0$

$$\begin{aligned} I_4(h) & \leq \left\| \int_0^t S((h) - I) S(t-s) f(s, x(s-\rho(s))) ds \right\| \\ & \quad + \left\| \int_t^{t+h} S(t+h-s) f(s, x(s-\rho(s))) ds \right\| \end{aligned}$$

$$\begin{aligned} \text{Now } \mathbb{E} & \left\| \int_0^t S((h) - I) S(t-s) f(s, x(s-\rho(s))) ds \right\|^2 \\ & \leq \int_0^t S((h) - I) S(t-s) C_1^2 (\mathbb{E} \|x(s-\rho(s))\|^2 + 1) ds \\ & \leq M \int_0^T S((h) - I) C_1^2 \mathbb{E} (\|x(s-\rho(s))\|^2 + 1) ds \end{aligned}$$

Since $\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_0^t S((h) - I) S(t-s) f(s, x(s-\rho(s))) ds \right\|^2 = 0$

$$\begin{aligned} \text{and } \mathbb{E} & \left\| \int_t^{t+h} S(t+h-s) f(s, x(s-\rho(s))) ds \right\|^2 \\ & \leq M^2 \mathbb{E} \left\| \int_t^{t+h} f(s, x(s-\rho(s))) ds \right\|^2 \\ & \leq M^2 \int_t^{t+h} C_1^2 \mathbb{E} (\|x(s-\rho(s))\|^2 + 1) ds \end{aligned}$$

Then $\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_t^{t+h} S(t+h-s) f(s, x(s-\rho(s))) ds \right\|^2 = 0$

Hence $\lim_{h \rightarrow 0} \mathbb{E} |I_4(h)| = 0$

$$\text{Now } I_5(h) \leq \left\| \int_0^t S((h) - I) S(t-s) h(s, x(s-\delta(s))) dW(s) \right\|$$

$$+ \left\| \int_t^{t+h} S(t+h-s) h(s, x(s-\delta(s))) dW(s) \right\|$$

Now $\mathbb{E} \left\| \int_0^t S((h)-I)S(t-s) h(s, x(s-\delta(s))) dW(s) \right\|^2$

$$\leq \int_0^t S((h)-I)S(t-s) C_2^2 \left(\mathbb{E} \|x(s-\delta(s))\|^2 + 1 \right) ds$$

$$\leq M^2 \int_0^t S((h)-I) C_2^2 \left(\mathbb{E} \|x(s-\delta(s))\|^2 + 1 \right) ds$$

Therefore $\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_0^t S((h)-I)S(t-s) h(s, x(s-\delta(s))) dW(s) \right\|^2 = 0$

and $\mathbb{E} \left\| \int_t^{t+h} S(t+h-s) h(s, x(s-\delta(s))) dW(s) \right\|^2$

$$\leq M^2 \int_t^{t+h} C_2^2 \mathbb{E} \left(\|x(s-\delta(s))\|^2 + 1 \right) ds$$

Then $\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_t^{t+h} S(t+h-s) h(s, x(s-\delta(s))) dW(s) \right\|^2 = 0$

Hence $\lim_{h \rightarrow 0} \mathbb{E}|I_5(h)| = 0$
 For the term $I_6(h)$, we have

$$I_6(h) \leq \left\| \int_0^t S((h)-I)S(t-s) \sigma(s) dB^H(s) \right\| + \left\| \int_t^{t+h} S(t+h-s) \sigma(s) dB^H(s) \right\|$$

By condition (H_1) and lemma 2.2 we get

$$\mathbb{E} \left\| \int_0^t S((h)-I)S(t-s) \sigma(s) dB^H(s) \right\|^2$$

$$\leq cH(2H-1)(t)^{2H-1} \int_0^t \|S((h)-I)S(t-s)\sigma(s)\|_{L^0_Q}^2 ds$$

$$\leq cH(2H-1)(t)^{2H-1} M \int_0^t \|S((h)-I)\sigma(s)\|_{L^0_Q}^2 ds$$

Since $\lim_{h \rightarrow 0} \|S((h)-I)\sigma(s)\|_{L^0_Q}^2 = 0$ and

$$\|S((h)-I)\sigma(s)\|_{L^0_Q}^2 \leq 4M^2 \|\sigma(s)\|_{L^0_Q}^2 \in \mathcal{L}^1([0, T], ds)$$

we conclude by the dominated convergence theorem that

$$\lim_{h \rightarrow 0} \mathbb{E} \left\| \int_0^t S((h)-I)S(t-s) \sigma(s) dB^H(s) \right\|^2 = 0$$

Applying lemma 2.2, we get

$$\mathbb{E} \left\| \int_t^{t+h} S(t+h-s) \sigma(s) dB^H(s) \right\| \leq cH(h)^{2H-1} M^2 \int_t^{t+h} \|\sigma(s)\|_{L^0_Q}^2 ds \rightarrow 0$$

Hence that $\lim_{h \rightarrow 0} \mathbb{E}|I_6(h)| = 0$

The above arguments show that $\lim_{h \rightarrow 0} \mathbb{E} \|\psi(x)(t+h) - \psi(x)(t)\|^2 = 0$. Hence we conclude that the function $t \rightarrow \psi(x)(t)$ is continuous on $[0, T]$ in the \mathbb{L}^2 - sense.

Step 2. Now we are going to show that ψ is a contraction mapping in S_{T_1} with some $T_1 \leq T$.

Let $x, y \in S_T$ by using the inequality $(a+b+c+d)^2 \leq \frac{1}{k}a^2 + \frac{3}{1-k}b^2 + \frac{3}{1-k}c^2 + \frac{3}{1-k}d^2$ where $k = L_3 \|(-A)^{-\beta}\| < 1$, we obtain any fixed $t \in [0, T]$

$$\mathbb{E} \|\psi(x)(t) - \psi(y)(t)\|^2 \leq \frac{1}{k} \left\| g(t, x(t-r(t))) - g(t, y(t-r(t))) \right\|^2$$

$$\begin{aligned}
 & + \frac{3}{1-k} \left\| \int_0^t AS(t-s) \left(g(s, x(s-r(s))) - g(s, y(s-r(s))) \right) ds \right\|^2 \\
 & + \frac{3}{1-k} \left\| \int_0^t S(t-s) \left(f(s, x(s-\rho(s))) - f(s, y(s-\rho(s))) \right) ds \right\|^2 \\
 & + \frac{3}{1-k} \left\| \int_0^t S(t-s) \left(h(s, x(s-\delta(s))) - h(s, y(s-\delta(s))) \right) dW(s) \right\|^2 \\
 \leq & \frac{1}{k} \|(-A)^{-\beta}\|^2 \left\| (-A)^\beta g(t, x(t-r(t))) - (-A)^\beta g(t, y(t-r(t))) \right\|^2 \\
 & + \frac{3}{1-k} \left\| \int_0^t (-A)^{1-\beta} S(t-s) (-A)^\beta \left(g(s, x(s-r(s))) - g(s, y(s-r(s))) \right) ds \right\|^2 \\
 & + \frac{3}{1-k} \left\| \int_0^t S(t-s) (-A)^\beta \left(f(s, x(s-\rho(s))) - f(s, y(s-\rho(s))) \right) ds \right\|^2 \\
 & + \frac{3}{1-k} \left\| \int_0^t S(t-s) (-A)^\beta \left(h(s, x(s-\delta(s))) - h(s, y(s-\delta(s))) \right) dW(s) \right\|^2
 \end{aligned}$$

By Lipchitz property of $(-A)^\beta g$ and f, h combined with Holder’s inequality, we obtain

$$\begin{aligned}
 \mathbb{E} \|\psi(x)(t) - \psi(y)(t)\|^2 & \leq k \mathbb{E} \|x(t-r) - y(t-r)\|^2 \\
 & + \frac{3}{1-k} C_3^2 M_{1-\beta}^2 \left(\frac{t^{2\beta-1}}{2\beta-1} \right) \int_0^t \mathbb{E} \|x(s-r) - y(s-r)\|^2 ds \\
 & + \frac{3}{1-k} t M^2 C_1^2 \int_0^t \mathbb{E} \|x(s-\rho(s)) - y(s-\rho(s))\|^2 ds \\
 & + \frac{3}{1-k} t M^2 C_2^2 \int_0^t \mathbb{E} \|x(s-\delta(s)) - y(s-\delta(s))\|^2 ds
 \end{aligned}$$

Hence $\sup_{s \in [-\tau, t]} \mathbb{E} \|\psi(x)(s) - \psi(y)(s)\|^2 \leq \gamma(t) \sup_{s \in [-\tau, t]} \mathbb{E} \|x(s) - y(s)\|^2$

where $\gamma(t) = k + \frac{3C_3^2 M_{1-\beta}^2}{(1-k)(2\beta-1)} t^{2\beta} + \frac{3M^2 C_1^2 t^2}{1-k} + \frac{3M^2 C_2^2 t^2}{1-k}$

By condition (iii) in (H_3) we have, $\gamma(0) = k = \|(-A)^{-\beta}\|_{L_3} < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of Eqn.(1.1) on $[-\tau, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau, T]$ in finitely many steps. This completes the proof.

4. Stability analysis

In this section we establish the results for the case of finite delays, then the case of infinite delays can be proved. Our method is based on the contraction mapping principle.

In order to prove the required results, we assume the following additional conditions.

(H_6) The semigroup $S(t)$ satisfies $\exists \lambda > 0, \exists M > 0$ such that $\|S(t)\| \leq Me^{-\lambda t}, t \geq 0$

(H_7) The function $\sigma: [0, \infty) \rightarrow L^0_Q$ satisfies $\int_0^\infty e^{\gamma s} \|\sigma(s)\|_{L^0_Q}^2 ds < \infty$.

We first consider the case of finite delays, $\tau < \infty$.

Theorem 4.1: (Finite delays) Assume that $f(t, 0) = g(t, 0) = h(t, 0) = 0, t \geq 0$, the assumptions $(H_1) - (H_7)$ holds and that

$$4 \left(L_3 \|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_3 \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} + M^2 L_1 \lambda^{-2} + M^2 L_2 (2\lambda)^{-2} \right) < 1 \tag{4.1}$$

where $\Gamma(\cdot)$ is the gamma function, $M_{1-\beta}$ is the corresponding constant in lemma (2.3). Then the mild solution to (1.1) exists uniquely and is exponential decay to zero in mean square. i.e., there exists a pair of positive constants $a > 0$ and $M^* = M^*(\varphi, a)$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M^*e^{-at}, \forall t \geq 0.$$

Proof: Denote by \mathcal{s} the space of all stochastic process $x(t, \omega): (-\tau, \infty) \times \Omega \rightarrow X$ satisfying $x(t) = \phi(t), t \in (-\tau, 0]$ and there exists some constants $a > 0$ and $M^* = M^*(\varphi, a) > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M^*e^{-at}, \forall t \geq 0. \tag{4.2}$$

It is to check that \mathcal{s} is a Banach space endowed with a norm $\|x\|_{\mathcal{s}}^2 = \sup_{t \geq 0} \mathbb{E}|x(t)|^2$. Without loss of generality, we assume that $a < \lambda$. We denote the operator Φ on \mathcal{s} by

$$(\Phi x)(t) = \phi(t), t \in (-\tau, 0] \text{ and}$$

$$\begin{aligned} (\Phi x)(t) &= S(t) \left(\phi(0) + g(0, \phi(-r(0))) \right) - g(t, x(t-r(t))) \\ &\quad - \int_0^t AS(t-s)g(s, x(s-r(s))) ds + \int_0^t S(t-s) f(s, x(s-\rho(s))) ds \\ &\quad + \int_0^t S(t-s) h(s, x(s-\delta(s))) dW(s) + \int_0^t S(t-s) \sigma(s) dB^H(s) \end{aligned}$$

To get desired results, it is enough to show that operator Φ has a unique fixed point in \mathcal{s} . For this purpose, we use the contraction mapping principle.

Step 1. We first verify that $\Phi(\mathcal{s}) \subset \mathcal{s}$. For convenience of notation, we denote by $M_i^*, i = 1, 2, \dots$ the finite positive constants depending on φ, a . By assumption (H_6) we have

$$\mathbb{E}\|p_1(t)\|^2 \leq M^2 \mathbb{E} \left\| \phi(0) + g(0, \phi(-r(0))) \right\|^2 e^{-\lambda t} \leq M_1^* e^{-\lambda t} \tag{4.3}$$

To estimate $p_i(t), i = 2, \dots, 6$. we observe that for $x \in \mathcal{s}$ and $u(t) = r(t)$ or $\rho(t)$ or $\delta(t)$ the following useful estimate holds

$$\begin{aligned} \mathbb{E}\|x(t-u(t))\|^2 &\leq \left(M^* e^{-a(t-u(t))} + \mathbb{E}\|\phi(t-u(t))\|^2 \right) \\ &\leq \left(M^* e^{-a(t-u(t))} + \mathbb{E}\|\phi\|_c^2 e^{-a(t-u(t))} \right) \\ &\leq (M^* + \mathbb{E}\|\phi\|_c^2) e^{a\tau} e^{-at} \end{aligned}$$

where $\|\phi\|_c = \sup_{-\tau < s \leq 0} \|\phi(s)\| < \infty$. Then by assumption (i) in (H_3) we have

$$\begin{aligned} \mathbb{E}\|p_2(t)\|^2 &\leq \|(-A)^{-\beta}\|^2 \mathbb{E} \left\| (-A)^\beta g(t, x(t-r(t))) - (-A)^\beta g(t, 0) \right\|^2 \\ &\leq L_3 \|(-A)^{-\beta}\|^2 \mathbb{E}\|x(t-r(t))\|^2 \\ &\leq L_3 \|(-A)^{-\beta}\|^2 (M^* + \mathbb{E}\|\phi\|_c^2) e^{a\tau} e^{-at} \\ &\leq M_2^* e^{-at} \end{aligned} \tag{4.4}$$

Using lemma 2.3, Holder's inequality and assumption (i) in (H_3) obtain that

$$\begin{aligned} \mathbb{E}\|p_3(t)\|^2 &\leq \mathbb{E} \left\| \int_0^t AS(t-s)g(s, x(s-r(s))) ds \right\|^2 \\ &\leq \|(-A)^{1-\beta} S(t-s)\| ds \int_0^t \|(-A)^{1-\beta} S(t-s)\| \mathbb{E} \left\| (-A)^\beta g(s, x(s-r(s))) \right\|^2 ds \end{aligned}$$

$$\begin{aligned}
 &\leq M_{1-\beta}^2 L_3 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E} \|x(s-r(s))\|^2 ds \\
 &\leq M_{1-\beta}^2 L_3 \frac{\Gamma(\beta)}{\lambda^\beta} \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-as} ds \\
 &\leq M_{1-\beta}^2 L_3 \frac{\Gamma(\beta)}{\lambda^\beta} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-at} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds \\
 &\leq M_{1-\beta}^2 L_3 \frac{\Gamma^2(\beta)}{\lambda^\beta (\lambda-a)^\beta} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-at}
 \end{aligned}$$

We therefore have $\mathbb{E} \|p_3(t)\|^2 \leq M_3^* e^{-at}$ (4.5)

Similarly, we obtain by assumption(i) in (H_2) that

$$\begin{aligned}
 \mathbb{E} \|p_4(t)\|^2 &= \mathbb{E} \left\| \int_0^t S(t-s) f(s, x(s-\rho(s))) ds \right\|^2 \\
 &\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E} \|x(s-\rho(s))\|^2 ds \\
 &\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-as} ds \\
 &\leq M^2 L_1 \lambda^{-1} (\lambda-a)^{-1} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-at} \\
 &\leq M_4^* e^{-at}
 \end{aligned}$$
(4.6)

$$\begin{aligned}
 \mathbb{E} \|p_5(t)\|^2 &= \mathbb{E} \left\| \int_0^t S(t-s) h(s, x(s-\delta(s))) dw(s) \right\|^2 \\
 &\leq M^2 L_2 \int_0^t e^{-2\lambda(t-s)} \mathbb{E} \|x(s-\delta(s))\|^2 ds \\
 &\leq M^2 L_2 \int_0^t e^{-2\lambda(t-s)} ds \int_0^t e^{-2\lambda(t-s)} \mathbb{E} \|x(s-\delta(s))\|^2 ds \\
 &\leq \frac{1}{2} M^2 L_2 \lambda^{-1} \int_0^t e^{-2\lambda(t-s)} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-as} ds \\
 &\leq \frac{1}{2} M^2 L_2 \lambda^{-1} (2\lambda-a)^{-1} (M^* + \mathbb{E} \|\phi\|_c^2) e^{a\tau} e^{-at} \\
 &\leq M_5^* e^{-at}
 \end{aligned}$$
(4.7)

By using lemma 2.2 we get that

$$\begin{aligned}
 \mathbb{E} \|p_6(t)\|^2 &= \mathbb{E} \left\| \int_0^t S(t-s) \sigma(s) dB^H(s) \right\|^2 \\
 &\leq M^2 cH(2H-1) t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|_{L^2_Q}^2 ds
 \end{aligned}$$

From this inequality we can infer that

$$\mathbb{E} \|p_6(t)\|^2 \leq M^2 cH(2H-1) t^{2H-1} e^{-2\lambda't} \int_0^t e^{2\gamma s} \|\sigma(s)\|_{L^2_Q}^2 ds$$
(4.8)

where $\lambda' = \lambda \wedge \gamma$. Indeed, if $\lambda' < \gamma$, then $\lambda' = \lambda$ and we have

$$\begin{aligned}
 \mathbb{E} \|p_6(t)\|^2 &\leq M^2 cH(2H-1) t^{2H-1} e^{-2\lambda t} \int_0^\infty e^{2\lambda s} \|\sigma(s)\|_{L^2_Q}^2 ds \\
 &\leq M^2 cH(2H-1) t^{2H-1} e^{-2\lambda't} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|_{L^2_Q}^2 ds
 \end{aligned}$$

If $\gamma < \lambda$ then $\lambda' = \gamma$ and we have

$$\mathbb{E}\|p_6(t)\|^2 \leq M^2 c_H(2H - 1)t^{2H-1}e^{-2\lambda't} \int_0^t e^{2\gamma s} \|\sigma(s)\|_{L^2_Q}^2 ds$$

since $\sup_{t \geq 0} (t^{2H-1}e^{-\lambda't}) < \infty$, this together with (4.8) we have

$$\mathbb{E}\|p_6(t)\|^2 \leq M_6^* e^{-at} \tag{4.9}$$

Combining (4.3)-(4.7), (4.9) we see that there exists $\bar{M}^* > 0$ and $\bar{a} > 0$ such that $\mathbb{E}\|(\Phi x)(t)\|^2 \leq \bar{M}^* e^{-\bar{a}t}, t \geq 0$.

Hence we conclude that $\Phi(\mathcal{S}) \subset \mathcal{S}$.

Step 2. We now show that Φ is a contraction mapping. For any $x, y \in \mathcal{S}$, we have

$$\mathbb{E}\|(\Phi x)(t) - (\Phi y)(t)\|^2 \leq 4 \sum_{i=1}^n I_i$$

Since $x(t) = y(t) = \varphi(t), t \in (-\tau, 0]$, this implies that

$$\mathbb{E}\|x(t - r(t)) - y(t - r(t))\|^2 \leq \sup_{t \geq 0} \mathbb{E}\|x(t) - y(t)\|^2$$

Then we have by assumption(i) in (H_3)

$$\begin{aligned} I_1 &= \mathbb{E} \left\| g(t, x(t - r(t))) - g(t, y(t - r(t))) \right\|^2 \\ &\leq L_3 \|(-A)^{-\beta}\|^2 \mathbb{E}\|x(t - r(t)) - y(t - r(t))\|^2 \\ &\leq L_3 \|(-A)^{-\beta}\|^2 \sup_{t \geq 0} \mathbb{E}\|x(t) - y(t)\|^2 \\ I_2 &= \mathbb{E} \left\| \int_0^t AS(t-s) \left[g(s, x(s - r(s))) - g(s, y(s - r(s))) \right] ds \right\|^2 \\ &\leq M_{1-\beta}^2 L_3 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E}\|x(s - r(s)) - y(s - r(s))\|^2 ds \\ &\leq M_{1-\beta}^2 L_3 \frac{\Gamma(\beta)}{\lambda^\beta} \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E}\|x(s - r(s)) - y(s - r(s))\|^2 ds \\ &\leq M_{1-\beta}^2 L_3 \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} \sup_{t \geq 0} \mathbb{E}\|x(t) - y(t)\|^2 \end{aligned}$$

By assumption (i) in (H_2)

$$\begin{aligned} I_3 &= \mathbb{E} \left\| \int_0^t S(t-s) \left[f(s, x(s - \rho(s))) - f(s, y(s - \rho(s))) \right] ds \right\|^2 \\ &\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E}\|x(s - \rho(s)) - y(s - \rho(s))\|^2 ds \\ &\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)} \mathbb{E}\|x(s - \rho(s)) - y(s - \rho(s))\|^2 ds \\ &\leq M^2 L_1 \lambda^{-2} \sup_{t \geq 0} \mathbb{E}\|x(t) - y(t)\|^2 \end{aligned}$$

By assumption(iii) in (H_2) , we have

$$\begin{aligned} I_4 &= \mathbb{E} \left\| \int_0^t S(t-s) \left[h(s, x(s - \delta(s))) - h(s, y(s - \delta(s))) \right] dw(s) \right\|^2 \\ &\leq M^2 L_2 \int_0^t e^{-2\lambda(t-s)} ds \int_0^t e^{-2\lambda(t-s)} \mathbb{E}\|x(s - \delta(s)) - y(s - \delta(s))\|^2 ds \\ &\leq M^2 L_2 (2\lambda)^{-1} \int_0^t e^{-2\lambda(t-s)} \mathbb{E}\|x(s - \delta(s)) - y(s - \delta(s))\|^2 ds \\ &\leq M^2 L_2 (2\lambda)^{-2} \sup_{t \geq 0} \mathbb{E}\|x(t) - y(t)\|^2 \end{aligned}$$

Thus $\mathbb{E}\|(\Phi x)(t) - (\Phi y)(t)\|^2$

$$\leq 4 \left(L_3 \|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_3 \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} + M^2 L_1 \lambda^{-2} + M^2 L_2 (2\lambda)^{-2} \right) \sup_{t \geq 0} \mathbb{E}\|x(t) - y(t)\|^2$$

By the condition (4.1), we claim that Φ is a contractive. So, applying the Banach fixed point principle, the proof is complete.

Now we consider the case of infinite delays. We assume that $t - r(t) \rightarrow \infty, t - \rho(t) \rightarrow \infty, t - \delta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Theorem 4.2: [Infinite delays] Under the conditions of theorem 4.1, the mild solution to (1.1) exists uniquely and converges to zero in mean square i.e., $\lim_{t \rightarrow \infty} \mathbb{E}\|x(t)\|^2 = 0$

Proof: Denote by \mathcal{S}' the space of all stochastic processes $x(t, \omega): (-\infty, \infty) \times \Omega \rightarrow X$ satisfying

$$x(t) = \phi(t), t \in (-\infty, 0] \text{ and}$$

$$\lim_{t \rightarrow \infty} \mathbb{E}\|x(t)\|^2 = 0 \tag{4.10}$$

We define the operator Ψ on \mathcal{S}' by $(\Psi x)(t) = \phi(t), t \in (-\infty, 0]$ and

$$\begin{aligned} (\Psi x)(t) &= S(t) \left(\phi(0) + g(0, \phi(-r(0))) \right) - g(t, x(t - r(t))) \\ &\quad - \int_0^t AS(t-s)g(s, x(s - r(s))) ds + \int_0^t S(t-s) f(s, x(s - \rho(s))) ds \\ &\quad + \int_0^t S(t-s) h(s, x(s - \delta(s))) dw(s) + \int_0^t S(t-s) \sigma(s) dB^H(s) \end{aligned}$$

Since $(\Psi x)(t) = (\Phi x)(t)$ on $[0, \infty)$, this implies that Ψ is contractive. Hence it remains to check $\Phi(\mathcal{S}') \subset \mathcal{S}'$. In order to obtain this claim, we need to show that $\lim_{t \rightarrow \infty} \mathbb{E}\|\Psi x(t)\|^2 = 0$ for all $x \in \mathcal{S}'$.

By the definition of \mathcal{S}' , assumption (H₇) and the fact $t - r(t) \rightarrow \infty, t \rightarrow \infty$. We get

$$\lim_{t \rightarrow \infty} \mathbb{E}\|P_1(t)\|^2 = \lim_{t \rightarrow \infty} \mathbb{E}\|P_2(t)\|^2 = \lim_{t \rightarrow \infty} \mathbb{E}\|P_6(t)\|^2 = 0$$

$$\begin{aligned} \text{We further have } \mathbb{E}\|p_3(t)\|^2 &\leq \mathbb{E} \left\| \int_0^t AS(t-s)g(s, x(s - r(s))) ds \right\|^2 \\ &\leq M_{1-\beta}^2 L_3 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E}\|x(s - r(s))\|^2 ds \\ &\leq M_{1-\beta}^2 L_3 \Gamma(\beta) \lambda^{-\beta} \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E}\|x(s - r(s))\|^2 ds \end{aligned}$$

For any $x \in \mathcal{S}'$ and $\epsilon > 0$ it follows from (4.10) that there exists $s_1 > 0$ such that $\mathbb{E}\|x(s - r(s))\|^2 < \epsilon$ for all $s \geq s_1$. Thus we obtain

$$\mathbb{E}\|p_3(t)\|^2 \leq M_{1-\beta}^2 L_3 \Gamma(\beta) \lambda^{-\beta} \int_0^{s_1} (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E}\|x(s - r(s))\|^2 ds + M_{1-\beta}^2 L_3 \Gamma^2(\beta) \lambda^{-2\beta} \epsilon$$

which proves that

$$\lim_{t \rightarrow \infty} \mathbb{E}\|P_3(t)\|^2 \leq M_{1-\beta}^2 L_3 \Gamma^2(\beta) \lambda^{-2\beta}, \forall \epsilon > 0 \text{ and hence, } \lim_{t \rightarrow \infty} \mathbb{E}\|P_3(t)\|^2 = 0$$

$$\begin{aligned} \mathbb{E}\|p_4(t)\|^2 &= \mathbb{E} \left\| \int_0^t S(t-s) f(s, x(s - \rho(s))) ds \right\|^2 \\ &\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E}\|x(s - \rho(s))\|^2 ds \\ &\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)} \mathbb{E}\|x(s - \rho(s))\|^2 ds \end{aligned}$$

For any $x \in \mathcal{S}'$ and $\epsilon > 0$ there exists a $s_1 > 0$ such that $\mathbb{E}\|x(s - r(s))\|^2 < \epsilon$ for all $s \geq s_1$.

Thus we obtain $\mathbb{E}\|p_4(t)\|^2 \leq M^2 L_1 \lambda^{-1} \int_0^{s_1} e^{-\lambda(t-s)} \mathbb{E}\|x(s - \rho(s))\|^2 ds + M^2 L_1 \lambda^{-2} \epsilon$
 which proves that $\lim_{t \rightarrow \infty} \mathbb{E}\|p_4(t)\|^2 \leq M^2 L_1 \lambda^{-2} \epsilon, \forall \epsilon > 0$ and hence, $\lim_{t \rightarrow \infty} \mathbb{E}\|p_4(t)\|^2 = 0$

$$\begin{aligned} \mathbb{E}\|p_5(t)\|^2 &= \mathbb{E} \left\| \int_0^t S(t-s) h(s, x(s - \delta(s))) dw(s) \right\|^2 \\ &\leq M^2 L_2 \int_0^t e^{-2\lambda(t-s)} \mathbb{E}\|x(s - \delta(s))\|^2 ds \end{aligned}$$

For any $x \in \mathcal{S}'$ and $\epsilon > 0$ there exists a $s_1 > 0$ such that $\mathbb{E}\|x(s - r(s))\|^2 < \epsilon$ for all $s \geq s_1$.

Thus we obtain $\mathbb{E}\|p_5(t)\|^2 \leq M^2 L_2 \int_0^{s_1} e^{-2\lambda(t-s)} \mathbb{E}\|x(s - \delta(s))\|^2 ds + M^2 L_2 (2\lambda)^{-1} \epsilon$
 which proves that $\lim_{t \rightarrow \infty} \mathbb{E}\|p_5(t)\|^2 \leq M^2 L_2 (2\lambda)^{-1} \epsilon, \forall \epsilon > 0$ and hence, $\lim_{t \rightarrow \infty} \mathbb{E}\|p_5(t)\|^2 = 0$
 Once again, by applying the Banach fixed point principle, we complete the proof of the theorem.

To illustrate the obtained theory, let us end this section with an example.

5. Example

Consider the following neutral stochastic partial differential equations with delays driven by a fBm in the following form

$$\frac{\partial}{\partial t} [z(t, \xi) + \alpha_1 G(t, z(t - \tau, \xi))] = \frac{\partial^2}{\partial \xi^2} z(t, \xi) + \alpha_2 F(t, z(t - \tau, \xi)) \partial t + \alpha_3 \phi(t, z(t - \tau, \xi)) dw(t) + \Theta(t) dB_Q^H(t)$$

$$z(t, 0) = z(t, \pi) = 0 \tag{5.1}$$

$$z(t, \xi) = \phi(t, \xi), t \in (-\infty, 0], 0 \leq \xi \leq \pi$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ are constants.

Let $X = L^2(0, \pi)$ with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Define $A: X \rightarrow X$ by $Ax = x''$ with domain $D(A) = \{x \in X: x, x' \text{ are absolutely continuous } x'' \in X, x(0) = x(\pi) = 0\}$. Then Eqn. (5.1) can be written in the form of equation (1.1) with the co-efficient $g(t, x) = \alpha_1 G(t, z(t - \tau, \xi)), f(t, x) = \alpha_2 F(t, z(t - \tau, \xi))$ and $h(t, x) = \alpha_3 \phi(t, z(t - \tau, \xi)), \sigma(t) = \Theta(t)$.

For the operator A , it is known from Pazy [18] that the following properties hold.

* $Ax = \sum_{n=1}^{\infty} A^2 \langle x, e_n \rangle e_n, x \in D(A)$, where $e_n(t) = \sqrt{\frac{2}{\pi}} \sin(nt)$, $n = 1, 2, 3, \dots$ is the orthogonal set of eigenvectors.

* A is the infinitesimal generator of an analytic semigroup $(S(t))_{t \geq 0}$ in X .

$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n$, For all $x \in X$ and for every $t > 0$.

Furthermore $\|S(t)\| \leq e^{-\pi^2 t}, t \geq 0$.

* The bounded linear operator $(-A)^{3/4}$ is well defined and is given by

$$(-A)^{3/4} x = \sum_{n=1}^{\infty} (n)^{3/2} \langle x, e_n \rangle e_n$$

with domain $D((-A)^{3/4}) = \{x \in X: \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n \in X\}$. Furthermore $\|(-A)^{3/4}\| = 1$ and $\|(-A)^{3/4}\| \leq \frac{1}{\Gamma(3/4)} \int_0^{\infty} t^{-(1/4)} \|S(t)\| dt < \frac{1}{\pi^{3/2}}$. Thus (H_1) holds with $M = 1, \lambda = \pi^2, (H_2)$ holds with $L_1 = \alpha_2^2, (H_3)$ and (H_4) holds with $\beta = 3/4, L_2 = \|(-A)^{3/4}\|^2 \alpha_1^2 = \alpha_1^2$ and (H_6) holds with $\gamma = \frac{1}{2}$. Consequently we conclude by theorem 4.2, that the stochastic partial equation (5.1) has a unique mild solution and that this solution converges to zero in mean square if the parameters $\alpha_1, \alpha_2, \alpha_3$ satisfy the following conditions.

$$\frac{\alpha_1^2}{\pi^3} + \frac{M_1^2}{4} \Gamma^2\left(\frac{3}{4}\right) \frac{\alpha_1^2}{\pi^3} + \frac{\alpha_2^2}{\pi^4} + \frac{\alpha_3^2}{\pi^4} < \frac{1}{4}$$

6. Conclusion

In this paper we derived existence, conditions ensuring the exponential decay to zero in mean square of a neutral stochastic delay differential equations driven by fBm. In addition we also established the case of infinite delays which has not yet been discussed in the context of neutral stochastic delay differential equations driven by fBm.

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