



Two matrix methods for solution of nonlinear and linear Lane-Emden type equations with mixed condition by operational matrix

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Abstract

The aim of this article is to present an efficient numerical procedure for solving Lane-Emden type equations. We present two practical matrix method for solving Lane-Emden type equations with mixed conditions by Bernstein polynomials operational matrices (BPOMs) on interval $[a, b]$. This methods transforms Lane-Emden type equations and the given conditions into matrix equation which corresponds to a system of linear algebraic equations. We also give some numerical examples to demonstrate the efficiency and validity of the operational matrices for solving Lane-Emden type equations (LEEs).

Keywords: *Bernstein polynomials, Lane-Emden type equation, Operational matrices.*

1. Introduction

Lane-Emden type equation is an ordinary differential equation which describes the equilibrium density distribution in self-gravitating sphere of polytropic isothermal gas and has a singularity at the origin. Many real life phenomena in mathematical physics and astrophysics can be modeled by Lane-Emden type equations such as thermal explosions [1], stellar structure [2], and the thermal behavior of a spherical cloud of gas, isothermal gas spheres and thermionic currents [3].

In recent years, Many powerful methods have been presented for solving of Lane-Emden type equations. For instance, the homotopy perturbation method [4, 6], the Legendre wavelets [7], the variational iteration method [8, 9], the B-spline method [10], the Adomian decomposition method [11], the Bessel collocation method [12], the Pade series method [13], the rational Legendre pseudospectral method [14], the nonperturbative approximate method [15], the Hermite functions collocation method [16], and the variational approach method [17].

Continuous or piecewise polynomials are incredibly useful mathematical tools as they are precisely defined, calculated rapidly on a modern computer system and can represent a great variety of functions. They can be differentiated and integrated without difficulty. The aim of the present paper is to apply Bernstein operational matrices to propose a reliable numerical technique for solving linear and nonlinear Lane-Emden equations. Some special cases of the problem are solved to show its validity and efficiency in comparison with other existing numerical methods. The approximate solution obtained by the proposed method shows its superiority on the other existing numerical

solution.

In this work we used that operational matrices for numerical solution of nonlinear Lane-Emden equation,

$$u''(x) + \frac{\alpha}{x}u'(x) + f(x, u) = g(x), \quad \alpha, x \geq 0, \quad (1)$$

under the mixed conditions

$$\sum_{k=0}^1 \left(a_{ik}u^{(k)}(a) + b_{ik}u^{(k)}(b) \right) = \lambda_i, \quad i = 0, 1, \quad (2)$$

where $a_{ik}, b_{ik}, \lambda_i$ and α are constant. $f(x, u)$ is a nonlinear function of x and $u(x)$, whereas $u(x)$ is unknown. The aim of this study is to find the solution of the problem as the truncated Bernstein series defined by

$$u(x) = \sum_{n=0}^N u_n B_{n,N}(x), \quad a \leq x \leq b, \quad (3)$$

where $B_{n,N}(x)$, $n = 0, 1, 2, \dots, N$ denotes the Bernstein polynomials; u_n , $0 \leq n \leq N$ are unknown Bernstein coefficients, and N is any positive integer chosen such that $N \geq 2$. To find a numeric solution in the form (3) of the problem (1), we use the collocation points defined by

$$x_i = a + \frac{b-a}{2N-1}i, \quad i = 0, 1, \dots, N. \quad (4)$$

The Bernstein basis polynomials of degree N are defined by:

$$B_{n,N}(x) = \binom{N}{n} \frac{(x-a)^n (b-x)^{N-n}}{(b-a)^N}, \quad a \leq x \leq b, \quad n = 0, 1, \dots, N. \quad (5)$$

The Bernstein polynomials $B_{n,N}(x)$ in the matrix form as follows

$$\Phi(x) = [B_{0,N}(x), B_{1,N}(x), \dots, B_{N,N}(x)]. \quad (6)$$

This paper is organized as follows: first, we introduce Lane-Emden type equations and Bernstein polynomials. In Section 2 we apply Bernstein polynomials approximation and their operational matrices together with collocation method to reduce the Lane-Emden type equation to a system of nonlinear or linear equations. Section 3 exhibits the final system. Section 4 we present a special case. Section 5 illustrates some numerical examples to show the accuracy of this method. Finally Section 6 concludes the paper.

2. Fundamental relations

2.1. Function approximation

2.1.1. Approximate of $u(x)$

A function $u(x)$, square integrable in $[a, b]$, may be expressed in terms of Bernstein basis [18]. In practice, only the first $(N+1)$ -terms Bernstein polynomials are considered. Hence, if we write

$$u(x) \simeq \sum_{i=0}^N u_i B_{i,N}(x) = \Phi(x)U, \quad (7)$$

where

$$U = [u_0, u_1, \dots, u_N]^T, \quad (8)$$

then

$$U = \mathbf{Q}^{-1}(\Phi(x), u(x)), \quad (9)$$

where \mathbf{Q} is said dual matrix of $\Phi(x)$ and is given in [18]. Now, we can write the unknown function $u(x)$ in the matrix form as follows

$$u(x) = \Phi(x)U, \quad (10)$$

Using the expression (5) and taking $n = 0, 1, \dots, N$, we find the corresponding matrix relation as

$$\Phi(x) = \Delta(x)A, \quad (11)$$

where

$$\Delta(x) = [1, x, \dots, x^N], \quad (12)$$

and A is the $(N + 1) \times (N + 1)$ matrix. For example, on the interval $[0, 1]$ and $[-1, 1]$, we have

$$A_{[0,1]} = \begin{bmatrix} (-1)^0 \binom{N}{0} & 0 & \dots & 0 \\ \vdots & \ddots & & \\ (-1)^1 \binom{N}{0} \binom{N-0}{1} & (-1)^0 \binom{N}{n} & & \vdots \\ \vdots & & \ddots & 0 \\ (-1)^{N-0} \binom{N}{0} \binom{N-0}{N-0} & \dots & (-1)^{N-n} \binom{N}{n} \binom{N-n}{N-n} & \dots & (-1)^0 \binom{N}{N} \end{bmatrix},$$

$$A_{[-1,1]} = \begin{bmatrix} (-1)^0 \binom{N}{0} & (-1)^0 \binom{N-1}{0} & \dots & \binom{N}{0} \\ (-1)^1 \binom{N}{1} & (-1)^1 \binom{N-1}{1} + (-1)^0 \binom{N-1}{0} & \dots & \binom{N}{1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^N \binom{N}{N} & (-1)^{N-1} \binom{N-1}{N-1} & \dots & \binom{N}{N} \end{bmatrix}.$$

From relations (10), (6) and (11), we obtain the following matrix forms

$$u(x) = \Phi(x)U = \Delta(x)AU. \quad (13)$$

2.1.2. Approximate of $f(x, u)$

We can also approximate the function $f(x, u)$ by the Bernstein polynomials as

$$f(x, u) \simeq f(x, \Phi(x)U) = \Phi(x)H, \quad (14)$$

where the unknown is

$$H = [h_0, h_1, \dots, h_N]^T, \quad (15)$$

similarly (9), we have

$$H = \mathbf{Q}^{-1}(\Phi(x), f(x, \Phi(x)U)). \quad (16)$$

2.2. Matrix relations for higher order derivatives of $u(x)$ based on $\Delta(x)$

We consider the solution $u(x)$ and its k th derivative $u^{(k)}(x)$ defined by the truncated Bernstein series (3). Then we can write its k th derivative in the matrix form

$$u^{(k)}(x) = \Phi^{(k)}(x)U, \quad k = 1, 2, \dots \quad (17)$$

and

$$u^{(k)}(x) = \Delta^{(k)}(x)AU, \tag{18}$$

To obtain the matrix $\Delta^{(k)}(x)$ in terms of the matrix $\Delta(x)$, we can use the following procedure:

$$\begin{aligned} \Delta^{(1)}(x) &= \Delta(x)M^1, \\ \Delta^{(2)}(x) &= \Delta^{(1)}(x)M = \Delta(x)M^2, \\ &\vdots \\ \Delta^{(k)}(x) &= \Delta^{(k-1)}(x)M = \Delta(x)M^k, \end{aligned} \tag{19}$$

where

$$M = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{20}$$

Consequently, by substituting the matrix form (18) and (19) into (17), we have the matrix relation

$$u^{(k)}(x) = \Delta(x)M^kAU. \tag{21}$$

2.3. Matrix relations for higher order derivatives of $u(x)$ based on $\Phi(x)$

The differentiation of vector $\Phi(x)$ in Eq. (6) can be expressed as

$$\Phi'(x) = \Phi(x)\mathbf{D}, \tag{22}$$

where \mathbf{D} is the $(N + 1) \times (N + 1)$ operational matrix of derivatives for Bernstein polynomials. From (11), we have $\Phi(x) = \Delta(x)A$ and then

$$\Phi'(x) = [0, 1, 2x, \dots, Nx^{N-1}] A, \tag{23}$$

Defining $(N) \times (N + 1)$ matrix V and vector $\Delta^*(x)$ as

$$V = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & N \end{bmatrix}, \quad \Delta^*(x) = [1, x, x^2, \dots, x^{N-1}], \tag{24}$$

equation (23) may then be restated as

$$\Phi'(x) = \Delta^*(x)VA. \tag{25}$$

We now expand vector $\Delta^*(x)$ in terms of $\Phi(x)$. We get $\Delta^*(x) = \Phi(x)B^*$ where

$$B^* = [A_{[1]}^{-1}, A_{[2]}^{-1}, A_{[3]}^{-1}, \dots, A_{[N]}^{-1}], \tag{26}$$

where $A_{[k]}^{-1}$ is the $(k + 1)$ th row of A^{-1} for $k = 1, 2, \dots, N$. so

$$\Phi'(x) = \Phi(x)B^*VA. \tag{27}$$

Therefore, we have the operational matrix of derivative as

$$\mathbf{D} = B^*VA. \tag{28}$$

If we approximate $u(x) \simeq \Phi(x)U$, then for $k \geq 2$ (k is the order of derivatives), we get

$$u^{(k)}(x) \simeq \Phi^{(k)}(x)U = \Phi(x)\mathbf{D}^kU. \tag{29}$$

2.4. Matrix relations for the mixed conditions

We obtain the corresponding matrix forms for the conditions (2) by means of the relations (21) as

$$\sum_{k=0}^1 [a_{ik}\Delta(a) + b_{ik}\Delta(b)]M^k AU = \lambda_i, \quad i = 0, 1. \quad (30)$$

Briefly, the matrix form for conditions (2) is

$$\mathcal{I}_i U = [\lambda_i], \quad (31)$$

where

$$\begin{aligned} \mathcal{I}_i &= \sum_{k=0}^1 [a_{ik}\Delta(a) + b_{ik}\Delta(b)]M^k A \\ &= [z_{i0}, z_{i1}, \dots, z_{iN}]. \end{aligned} \quad (32)$$

3. Methods of solution

We are now ready to construct the fundamental matrix equation corresponding to Eq. (1).

3.1. First method

In this method, we substitute and simplify the matrix relations (21) into Eq. (1), and obtain the fundamental matrix equation

$$\Delta(x)M^2 AU + \frac{\alpha}{x}\Delta(x)MAU + \Delta(x)AH = g(x). \quad (33)$$

By plugging the collocation points x_i defined by (4), we get the system of matrix equations

$$\Delta(x_i)M^2 AU + \frac{\alpha}{x_i}\Delta(x_i)MAU + \Delta(x_i)AH = g(x_i), \quad i = 0, 1, \dots, N, \quad (34)$$

or briefly the fundamental matrix equation

$$\left\{ \Delta M^2 A + \Gamma \Delta M A \right\} U = G - \Delta A H. \quad (35)$$

where

$$\Delta = \begin{bmatrix} \Delta(x_0) \\ \Delta(x_1) \\ \vdots \\ \Delta(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}, \quad (36)$$

and

$$\Gamma = \begin{bmatrix} \frac{\alpha}{x_0} & 0 & 0 & \dots & 0 \\ 0 & \frac{\alpha}{x_1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\alpha}{x_N} \end{bmatrix}, \quad G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \quad (37)$$

Hence, Eq. (35) can be written in the form

$$\mathbf{W}U = \mathcal{F} \quad \text{or} \quad [\mathbf{W}; \mathcal{F}], \quad \mathbf{W} = [w_{pq}], \quad p, q = 0, 1, \dots, N, \quad (38)$$

where

$$\mathbf{W} = [w_{pq}] = \Delta M^2 A + \Gamma \Delta M A, \quad \mathcal{F} = G - \Delta A H, \tag{39}$$

Finally, to obtain the solution of Eq. (1) under the conditions (2), we replace the row matrices (32) by the last 2 rows of the matrix (38), and get the new augmented matrix

$$[\overline{\mathbf{W}}; \overline{\mathcal{F}}] = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & g(x_0) - \Delta(x_0)AH \\ w_{10} & w_{11} & \dots & w_{1N} & ; & g(x_1) - \Delta(x_1)AH \\ \dots & \dots & & \dots & ; & \dots \\ w_{N,0} & w_{N,1} & \dots & w_{N,N} & ; & g(x_N) - \Delta(x_N)AH \\ \\ z_{00} & z_{01} & \dots & z_{0N} & ; & \lambda_0 \\ z_{10} & z_{11} & \dots & z_{1N} & ; & \lambda_1 \end{bmatrix}. \tag{40}$$

If $rank \overline{\mathbf{W}} = rank [\overline{\mathbf{W}}; \overline{\mathcal{F}}] = N + 1$, then we can write $U = (\overline{\mathbf{W}})^{-1} \overline{\mathcal{F}}$; thus, the matrix U (thereby the coefficients u_0, u_1, \dots, u_N) is uniquely determined. Also the Eq. (1) under the conditions (2) has a unique solution. This solution is given by truncated Bernstein series (3). On the other hand, when $|\overline{\mathbf{W}}| = 0$, if $rank \overline{\mathbf{W}} = rank [\overline{\mathbf{W}}; \overline{\mathcal{F}}] < N + 1$, then we may find a particular solution. Otherwise if $rank \overline{\mathbf{W}} \neq rank [\overline{\mathbf{W}}; \overline{\mathcal{F}}] = N + 1$, then there is no solution.

3.2. Second method

In this method, we substitute and simplify the matrix relations (11) and (29) into Eq. (1), and obtain the fundamental matrix equation

$$\Delta(x)AD^2U + \frac{\alpha}{x}\Delta(x)ADU + \Delta(x)AH = g(x), \tag{41}$$

By plugging the collocation points x_i defined by (4), we get the system of matrix equations

$$\Delta(x_i)AD^2U + \frac{\alpha}{x_i}\Delta(x_i)ADU + \Delta(x_i)AH = g(x_i), \quad i = 0, 1, \dots, N, \tag{42}$$

or briefly the fundamental matrix equation

$$\left\{ \Delta AD^2 + \Gamma \Delta AD \right\} U = G - \Delta AH. \tag{43}$$

4. Special case

4.1. Linear Lane–Emden equation

In Eq. (1), if $f(x, u) = p(x)u(x)$ then we have a linear Lane–Emden equation as

$$u''(x) + \frac{\alpha}{x}u'(x) + p(x)u(x) = g(x), \quad \alpha, x \geq 0. \tag{44}$$

Similarly nonlinear Lane-Emden equation, we have the fundamental matrix equation as follows

$$\left\{ \Delta M^2 A + \Gamma \Delta M A + P \Delta A \right\} U = G, \tag{45}$$

where

$$P = \begin{bmatrix} p(x_0) & 0 & 0 & \dots & 0 \\ 0 & p(x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p(x_N) \end{bmatrix}. \tag{46}$$

Table 1: Approximate and exact solutions for Example 1.

x	Present method with $N = 2$	Present method with $N = 4$	Present method with $N = 6$	Exact solution
0.1	2.55744	3.27765	3.27573	3.27562
0.2	2.52077	3.33314	3.33143	3.33132
0.3	2.53269	3.42746	3.42576	3.42566
0.4	2.59322	3.56270	3.56098	3.56086
0.5	2.70234	3.74213	3.74040	3.74027
0.6	2.86006	3.97022	3.96838	3.96824
0.7	3.06637	4.25260	4.25054	4.25039
0.8	3.32129	4.59613	4.59386	4.59371
0.9	3.62480	5.00883	5.00690	5.00676
1.0	3.97691	5.49992	5.50000	5.5

5. Numerical examples

To illustrate the effectiveness of the proposed method in the present paper, several examples are presented in this section.

Example 1. Consider the linear Lane–Emden equation [19]

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) - 4u(x) = -2, \\ u(1) = 5.5, \quad u'(0) = 0 \end{cases} \quad (47)$$

where the exact solution is $u(x) = 0.5 + 5 \frac{\sinh(2x)}{x \sinh(2)}$.

Numerical results for $N = 2$, $N = 4$ and $N = 6$ in the presented method by Bernstein operational matrices are shown in Table 1.

As it can be observed, we have almost obtained the exact solution for $N = 6$ through Bernstein operational matrices. The approximated solution for $N = 2$, $N = 4$ and $N = 6$ are as follows:

$$u_2(x) = 2.642709 - 1.095698x + 2.429902x^2,$$

$$u_4(x) = 3.260101 - 0.014183x + 1.906748x^2 - 0.148239x^3 + 0.495493x^4,$$

$$u_6(x) = 3.257315 - 0.000020x + 1.838468x^2 - 0.001711x^3 + 0.373546x^4 - 0.010367x^5 + 0.042769x^6$$

Since the exact solution is $u(x) = 0.5 + 5 \frac{\sinh(2x)}{x \sinh(2)}$, if we write Taylor expansion, we will have:

$$u(x) = 0.5 + 5 \frac{\sinh(2x)}{x \sinh(2)} \simeq 3.257206 + 1.838137x^2 + 0.367627x^4 + 0.035012x^6 + O(x^8)$$

As it can be observed, as N increases, the approximate solution gets closer to the exact solution.

Example 2. Consider the Linear Lane–Emden equation [19]

$$\begin{cases} u''(x) + \frac{1}{x}u'(x) = \frac{64}{(x^2 - 8)^2 (\ln(2) + \ln(5))}, \\ u(1) = 0, \quad u'(0) = 0 \end{cases} \quad (48)$$

where the exact solution is $u(x) = 2 \log_{10}\left(\frac{7}{8 - x^2}\right)$. Numerical results for $N = 3$, $N = 5$ and $N = 7$ in the presented method by Bernstein operational matrices are shown in Table 2.

Table 2: Approximate and exact solutions for Example 2.

x	Present method with $N = 3$	Present method with $N = 5$	Present method with $N = 7$	Exact solution
0.1	-0.113213	-0.113751	-0.114463	-0.114897
0.2	-0.109783	-0.110485	-0.111196	-0.111630
0.3	-0.104065	-0.105010	-0.105723	-0.106157
0.4	-0.096060	-0.097288	-0.098002	-0.098436
0.5	-0.085768	-0.087264	-0.087976	-0.088407
0.6	-0.073188	-0.074875	-0.075569	-0.075991
0.7	-0.058322	-0.060045	-0.060688	-0.061084
0.8	-0.041168	-0.042687	-0.043221	-0.043559
0.9	-0.021728	-0.022706	-0.023039	-0.023262
1.0	4.441×10^{-16}	-1.437×10^{-15}	-7.936×10^{-16}	0

As it can be observed, we have almost obtained the exact solution for $N = 7$ through Bernstein operational matrices. The approximated solution for $N = 3, N = 5$ and $N = 7$ are as follows:

$$\begin{aligned}
 u_3(x) &= -0.114357 - 6.93889 \times 10^{-18} x + 0.114357 x^2, \\
 u_5(x) &= -0.114835 - 0.000037 x + 0.108589 x^2 + 0.000348 x^3 + 0.006810 x^4 - 0.000876 x^5, \\
 u_7(x) &= -0.115549 - 0.0000005 x + 0.108573 x^2 + 0.000011 x^3 + 0.006809 x^4 - 0.000043 x^5 + 0.000384 x^6 \\
 &\quad - 0.000184 x^7.
 \end{aligned}$$

Since the exact solution is $u(x) = 2 \log_{10}(\frac{7}{8-x^2})$, if we write Taylor expansion, we will have:

$$2 \log_{10}(\frac{7}{8-x^2}) \simeq -0.115984 + 0.108574 x^2 + 0.006786 x^4 + 0.000565 x^6 + O(x^8)$$

As it can be observed, as N increases, the approximate solution gets closer to the exact solution.

Example 3. Consider the nonlinear Lane–Emden equation

$$\begin{cases}
 u''(x) + \frac{2}{x}u'(x) + e^{u(x)} = \frac{e^{\sin(x)}x - \sin(x)x + 2 \cos(x)}{x}, \\
 u(-1) = \sin(-1), \quad u'(1) = \cos(1),
 \end{cases} \tag{49}$$

where the exact solution is $u(x) = \sin(x)$. Numerical results for $N = 4, N = 6$ and $N = 8$ in the presented method by Bernstein operational matrices are shown in Table 3.

As it can be observed, we have almost obtained the exact solution for $N = 8$ through Bernstein operational matrices. The approximated solution for $N = 4, N = 6$ and $N = 8$ are as follows:

$$\begin{aligned}
 u_4(x) &= -0.000118 + 0.998664 x + 0.000389 x^2 - 0.158528 x^3 - 0.001607 x^4, \\
 u_6(x) &= 0.000005 + 0.999994 x + 0.000026 x^2 - 0.166624 x^3 - 0.000224 x^4 + 0.008044 x^5 + 0.000137 x^6, \\
 u_8(x) &= 0.00000001 + 0.999999 x + 0.0000002 x^2 - 0.166667 x^3 - 0.000003 x^4 + 0.0083331 x^5 + 0.000007 x^6 \\
 &\quad - 0.000194 x^7 - 0.000004 x^8,
 \end{aligned}$$

Since the exact solution is $u(x) = \sin(x)$, if we write Taylor expansion, we will have:

$$\sin(x) \simeq 1.0 x - 0.166667 x^3 + 0.008333 x^5 - 0.000198 x^7 + O(x^9)$$

As it can be observed, as N increases, the approximate solution gets closer to the exact solution.

Table 3: Approximate and exact solutions for Example 3.

x	Present method with $N = 4$	Present method with $N = 6$	Present method with $N = 8$	Exact solution
0.1	0.099594	0.099838	0.099833	0.099833
0.2	0.198359	0.198674	0.198669	0.198669
0.3	0.295223	0.295524	0.295520	0.295520
0.4	0.389223	0.389419	0.389418	0.389418
0.5	0.479395	0.479419	0.479425	0.479425
0.6	0.564771	0.564623	0.564642	0.564642
0.7	0.644377	0.644176	0.644218	0.644218
0.8	0.717238	0.717285	0.717356	0.717356
0.9	0.782374	0.783228	0.783328	0.783327
1.0	0.838801	0.841358	0.841472	0.841471

6. Conclusion

In this paper, we have proposed a numerical solution to solve Lane–Emden equation with mixed conditions by Bernstein polynomials operational matrices. We use formula for numerical examples and it is obvious that the numerical solution coincides with the exact solution even with a few Bernstein polynomials used in the approximation. Finally, errors show that the approximation becomes more accurate when N is increased. Therefore, for better results, it is recommended to use a larger N .

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