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***Research paper***

# No-regret control of an ill-posed parabolic semilinear equation with an exponential nonlinearity.

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**Abstract**

Our study focuses on the optimal control of a semilinear parabolic equation featuring incomplete data and an exponential nonlinearity that may blow up in finite time. Based on the principles of no-regret control, low-regret control, and adapted low-regret control, our objective is to define the control strategy for this ill-posed problem. The ill-posed nature of the system makes formulating the optimal control directly challenging. To address this, we use a regularisation approach that incorporates incomplete information. This yields a singular optimality system that characterises the no-regret control for the original semilinear parabolic problem.

***Keywords:*** *ill-posed problem, regularization, adapted low-regret control, low-regret control, no-regret control*

## 1. Statement of the problem

We consider a bounded regular open set $\Omega \subset \mathbb{R}^N$ with boundary $\Gamma$. For a time horizon $T > 0$, we define the cylinder $Q = \Omega \times (0,T)$ and its lateral boundary $\Sigma = \Gamma \times (0,T)$. The state system is governed by the following semilinear parabolic equation:

$$(P) \quad \begin{cases} \dfrac{\partial y}{\partial t} - \Delta y - e^y = v & \text{in } Q, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \tag{1}$$

Where $y^0 \in L^2(\Omega)$ and $v \in L^2(Q)$ denotes the control. This system is a typical example of a reaction-diffusion model with control. Such models are used to describe phenomena in which spatial diffusion is coupled with a strongly nonlinear reaction. Such models are found in areas such as combustion, heat transfer, biophysics and industrial processes. The state of the system is unknown at the boundary of the domain. The exponential nonlinearity may produce solutions that diverge in finite time. This global ill-posedness necessitates an approximation strategy when studying optimal control. Even when the boundary condition $y = 0$ is imposed on $\Sigma$, problem (1) remains ill-posed. To overcome this difficulty, we proceed in two directions. First, for every positive integer $M$, let us consider the functions $f_M : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f_M(s) = \begin{cases} e^{-M}, & s < -M, \\ e^s, & -M \leq s \leq M, \\ e^M, & s > M. \end{cases} \tag{2}$$

Which is Lipschitz continuous and bounded by $e^M$. Second, for any $\varepsilon > 0$, we complete the boundary condition by prescribing a non-zero trace $\varepsilon g$, where $g$ is a sufficiently regular function on $\Sigma$.
Then, we regularize the problem $(P_{\varepsilon,M})$ as follows :

$$(P_{\varepsilon,M}) \quad \begin{cases} \dfrac{\partial y_{\varepsilon,M}}{\partial t} - \Delta y_{\varepsilon,M} - f_M(y_{\varepsilon,M}) = v & \text{in } Q, \\ y_{\varepsilon,M}(\cdot,0) = y^0 & \text{in } \Omega, \\ y_{\varepsilon,M} = \varepsilon g & \text{on } \Sigma. \end{cases} \tag{3}$$

The function $g$ takes values in $G$ and may represent a pollution or perturbation, where $G$ is a nonempty closed subspace of $L^2(\Sigma)$, equipped with the $L^2(\Sigma)$ norm.

**Remark 1.1.** *For all $\varepsilon > 0$ and $M > 0$, problem (3) has a unique solution, owing to the Lipschitz continuity of $f_M$ and the regularity of the data.*

The structure of the regularized problem (3) is not a mere mathematical artifice. It can model a variety of physical phenomena. For example, in the thermal processes, it enables the regulation of heat diffusion in media subject to exothermic reactions, with an external control serving to stabilize the temperature. A similar logic arises in reaction-diffusion systems, where the goal is to model and control fast-growing chemical or biological reactions, the regularization here acting to contain instabilities. Environmental modeling provides another natural field of application: managing the propagation of pollutants in air or water requires accounting for both external perturbations and possible depollution strategies. Finally, in industrial control systems, this framework allows for the optimization of energy or chemical processes by means of robust controls, capable of withstanding uncertainties and disturbances.
Beyond the diversity of these settings, the regularization $(\varepsilon, M)$ plays a unified mathematical role: it linearizes the asymptotic behavior of the nonlinearity $e^y$, thereby enabling the derivation of uniform a priori estimates (first in $\varepsilon$, then in $M$) and rigorously justifying the passage to the limit, $\varepsilon \to 0$ followed by $M \to \infty$, toward the original physical problem (1).
Problem (1) is well-known in the mathematical literature. It was introduced by J.-L. Lions in [5], where the author emphasised the importance of studying the control of this type of system without actually solving it. The ill-posed nature of this model was highlighted early on, in two ways. Firstly, Bodart and Demeestère (see [11]) established boundary conditions that promote finite-time blow-up of the temperature. Secondly, Brezis et al. (see [10]) proved an equivalence between the existence of a global classical solution for the evolution equation and that of a weak solution for the associated stationary problem. These results predate recent developments in optimal control and impose a fundamental constraint: any control strategy must come to terms with the inherent risk of blow-up associated with the nonlinearity. It is precisely this necessity, rather than mere technical convenience, that justifies the use of a truncation $f_M$ rather than direct treatment of the exponential.
Regarding the stability and regularity of the optimal control for well-posed semilinear parabolic equations, the work of Casas and Tröltzsch [15] on stability with respect to initial data and that of Casas and Kunisch [12] on integral constraints are major references. However, these studies rely on the assumption of global regularity of the nonlinearity, which is excluded by definition in the case of exponential growth. The exponential case only appears within the elliptic framework in the work of Casas, Kavian and Puel [13], where the problem is stationary but ill-posed. Transitioning from the elliptic setting to the parabolic setting with truncation and missing boundary data remains an obstacle that the present study specifically aims to overcome.
Classical controllability results, such as the Carleman inequalities of Coron and Guerrero [16] or the least-squares method of Fernández-Cara and Münch [17], also assume a certain tractability of the nonlinearity. These methods require sublinear or Lipschitz growth, which is not met by exponential-type dynamics. Exponential growth triggers a blow-up phenomenon that these estimates cannot accommodate, which further confirms the need for prior regularisation by truncation.
Progress has also been made in robust control under data uncertainty. Notable advances have been made by Biccari et al.(see [14]) for nonlocal equations with an integral kernel and by Gao et al. (see [18]) for hierarchical models with degeneracy. However, in these works, uncertainty is introduced additively in the state equation and does not affect the argument of the nonlinearity itself. This makes these techniques difficult to apply to our problem, where the singularity lies at the core of the nonlinearity.
The most relevant methodological framework remains that of low-regret and no-regret control, initiated by J.-L. Lions. Numerous works have enriched this approach: Nakoulima, Omrane and Velin [22] for distributed systems with incomplete data, Dorville et al. [23] for ill-posed systems, and Tindano et al. [24] for evolution cases. More recently, a variety of extensions have emerged, ranging from degenerate population models [25] to fractional parabolic systems [20], as well as the identification of diffusion coefficients [19] and the singular optimality systems of Lamien et al. [26]. However, these extensions are subject to the same restrictive assumption. For example, Nakoulima et al. in [22] and Casas [21], the nonlinear function is assumed to be continuously differentiable and strictly increasing on the real line and cancels out to zero. These properties guarantee the Gâteaux differentiability of the control-state map via the implicit function theorem. Our setting differs significantly, however, as the truncation of the exponential function yields only a Lipschitz function that is $C^1$ almost everywhere and wich does not cancel out to zero. These characteristics invalidate the standard arguments and require the introduction of a specific condition of non-saturation.
Although interest in robust control strategies under uncertainty is growing, no existing work addresses the three specific challenges of equation (1) simultaneously: an ill-posed exponential nonlinearity, regularisation inducing differentiability only almost everywhere, and missing boundary data. This work aims to address these issues by combining regularisation with an adapted low-regret control approach and no-regret control.
The rest of the paper is organised as follows: in the section 2, we analyse the regularized problem. In the section 3 we apply optimal control to the regularized problem by associating a cost function. Then, we characterize the low-regret control of the regularized problem which converges to the no-regret control of the original problem.

## 2. Analysis of the regularized problem

In this section, we consider the regularized problem (3), for which we analyze the convergence as $\varepsilon \to 0$ and as $M \to \infty$, respectively.

### 2.1. Convergence as $\varepsilon$ tends to zero

We study the behaviour of the solution $y_{\varepsilon,M}$ to problem (3) as the parameter $\varepsilon$ tends to 0. We show that $y_{\varepsilon,M}$ converges to the solution of the homogeneous problem.

**Lemma 2.1.** *For all $\varepsilon > 0$ and $M > 0$, problem (3) has a unique solution $y_{\varepsilon,M} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Moreover, the estimates*

$$\|y_{\varepsilon,M}\|_{L^\infty(0,T;L^2(\Omega))} \leq K, \quad \|y_{\varepsilon,M}\|_{L^2(0,T;H^1(\Omega))} \leq K, \quad \left\|\frac{\partial y_{\varepsilon,M}}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} \leq K. \tag{4}$$

*are satisfied, where $K$ is a constant independent of $\varepsilon$.*

*Proof.* Let $w \in L^2(0,T;H^1(\Omega))$ be a lifting of the boundary data such that $w = g$ on $\Sigma$. we have the estimate

$$\|w\|_{L^2(0,T;H^1(\Omega))} \leq C\|g\|_{L^2(\Sigma)}. \tag{5}$$

We choose this lift such that $w(.,0) = 0$. Let us define $z_{\varepsilon,M} := y_{\varepsilon,M} - \varepsilon w$. Substituting $y_{\varepsilon,M} = z_{\varepsilon,M} + \varepsilon w$ into problem (3), we obtain:

$$\begin{cases} \frac{\partial z_{\varepsilon,M}}{\partial t} - \Delta z_{\varepsilon,M} = f_M(y_{\varepsilon,M}) + \varepsilon(-\frac{\partial w}{\partial t} + \Delta w) + v & \text{in } Q, \\ z_{\varepsilon,M}(\cdot,0) = y^0 & \text{in } \Omega, \\ z_{\varepsilon,M} = 0 & \text{on } \Sigma. \end{cases} \tag{6}$$

By multiplying (6) by $z_{\varepsilon,M}$ and integrating over $\Omega$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|z_{\varepsilon,M}\|^2_{L^2(\Omega)} + \|\nabla z_{\varepsilon,M}\|^2_{L^2(\Omega)} = \int_\Omega v z_{\varepsilon,M}\,dx + \int_\Omega f_M(y_{\varepsilon,M})\, z_{\varepsilon,M}\,dx + \varepsilon\int_\Omega \left(-\frac{\partial w}{\partial t} + \Delta w\right) z_{\varepsilon,M}\,dx. \tag{a}$$

Each term on the right-hand side is estimated using the Cauchy-Schwarz and Young inequalities. By definition of the truncation, $|f_M(s)| \leq e^M$ for all $s \in \mathbb{R}$.

$$\left|\int_\Omega f_M(y_{\varepsilon,M})\, z_{\varepsilon,M}\,dx\right| \leq e^M \int_\Omega |z_{\varepsilon,M}|\,dx \leq \frac{1}{2}\|z_{\varepsilon,M}\|^2_{L^2(\Omega)} + \frac{1}{2}C(\Omega)\,e^{2M}. \tag{b}$$

Similarly:

$$\left|\varepsilon\int_\Omega \left(-\frac{\partial w}{\partial t} + \Delta w\right) z_{\varepsilon,M}\,dx\right| \leq \frac{\varepsilon^2}{2}\left\|-\frac{\partial w}{\partial t} + \Delta w\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|z_{\varepsilon,M}\right\|^2_{L^2(\Omega)}. \tag{c}$$

And:

$$\left|\int_\Omega v\, z_{\varepsilon,M}\,dx\right| \leq \frac{1}{2}\|v\|^2_{l^2(\Omega)} + \frac{1}{2}\|z_{\varepsilon,M}\|^2_{L^2(\Omega)} \tag{d}$$

By substituting (b), (c) and (d) into (a), we get

$$\frac{1}{2}\frac{d}{dt}\|z_{\varepsilon,M}\|^2_{L^2(\Omega)} + \|\nabla z_{\varepsilon,M}\|^2_{L^2(\Omega)} \leq \frac{3}{2}\|z_{\varepsilon,M}\|^2_{L^2(\Omega)} + L, \tag{7}$$

where $L = \frac{1}{2}C(\Omega)\,e^{2M} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{2}\left\|-\frac{\partial w}{\partial t} + \Delta w\right\|^2_{L^2(\Omega)}$ ,with $\frac{\varepsilon^2}{2}\left\|-\frac{\partial w}{\partial t} + \Delta w\right\|^2_{L^2(\Omega)} = o(\varepsilon)$.

Let $t \in [0,T]$. Integrating (7) from 0 to $t$ and applying Gronwall's lemma, we obtain:

$\|z_{\varepsilon,M}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z_{\varepsilon,M}\|^2_{L^2(0,T;H^1_0(\Omega))} \leq C$. Where, $C$ is a constant depending on $\Omega$, $T$, $e^M$, $\|y^0\|_{L^2(\Omega)}$, and $\|v\|_{L^2(Q)}$.

Since $y_{\varepsilon,M} = z_{\varepsilon,M} + \varepsilon w$, by the Cauchy-Schwarz and Young inequalities, we obtain:

$$\begin{aligned} \|y_{\varepsilon,M}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_{\varepsilon,M}\|^2_{L^2(0,T;H^1(\Omega))} &= \|z_{\varepsilon,M} + \varepsilon w\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z_{\varepsilon,M} + \varepsilon w\|^2_{L^2(0,T;H^1(\Omega))} \\ &\leq 2\|z_{\varepsilon,M}\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\|z_{\varepsilon,M}\|^2_{L^2(0,T;H^1_0(\Omega))} + \\ &\quad 2\varepsilon\left(\|w\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w\|^2_{L^2(0,T;H^1_0(\Omega))}\right) \\ &\leq 2C + 2\varepsilon\left(\|w\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w\|^2_{L^2(0,T;H^1(\Omega))}\right) \\ &\leq 2C + 1 = K \end{aligned}$$

Since $\varepsilon(\|w\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w\|^2_{L^2(0,T;H^1(\Omega))}) \xrightarrow[\varepsilon\to 0]{} 0$, it follows that $\varepsilon(\|w\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w\|^2_{L^2(0,T;H^1(\Omega))}) \leq 1$ .

Where $K$ is a constant depending only on $\Omega$, $T$, $e^M$, $\|y^0\|_{L^2(\Omega)}$ and $\|v\|_{L^2(Q)}$.

Therefore: $\|y_{\varepsilon,M}\|_{L^\infty(0,T;L^2(\Omega))} \leq K$, and $\|y_{\varepsilon,M}\|_{L^2(0,T;H^1(\Omega))} \leq K$.

Estimation of $\frac{\partial y_{\varepsilon,M}}{\partial t}$ in $L^2(0,T;H^{-1}(\Omega))$. The norm of an element $\varphi \in H^{-1}(\Omega)$ is defined by:

$$\|\varphi\|_{H^{-1}(\Omega)} = \sup_{\phi \in H^1_0(\Omega), \phi \neq 0} \frac{|\langle \varphi, \phi\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}|}{\|\phi\|_{H^1_0(\Omega)}} \tag{8}$$

From the equation (3), we have $\frac{\partial y_{\varepsilon,M}}{\partial t} = \Delta y_{\varepsilon,M} + f_M(y_{\varepsilon,M}) + v \quad$ in $L^2(0,T;H^{-1}(\Omega))$.

And for all $\phi \in H^1_0(\Omega)$ with $\|\phi\|_{H^1_0(\Omega)} \leq 1$, we have:

$\left|\left\langle \frac{\partial y_{\varepsilon,M}}{\partial t}, \phi\right\rangle\right| \leq |\langle \Delta y_{\varepsilon,M}, \phi\rangle| + |\langle f_M(y_{\varepsilon,M}), \phi\rangle| + |\langle v, \phi\rangle|$

Since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ continuously, we get:

$$|\langle \Delta y_{\varepsilon,M}, \phi\rangle| = \int_\Omega \nabla y_{\varepsilon,M}\cdot\nabla\phi\, dx \le \|\nabla y_{\varepsilon,M}\|_{H^{-1}(\Omega)}\|\nabla\phi\|_{H_0^1(\Omega)} \le \|\nabla y_{\varepsilon,M}\|_{H^{-1}(\Omega)} \le C_1\|\nabla y_{\varepsilon,M}\|_{L^2(\Omega)},$$

for the non-linear term, on get:

$$|\langle f_M(y_{\varepsilon,M}),\phi\rangle| \le \|f_M(y_{\varepsilon,M})\|_{H^{-1}(\Omega)}\|\phi\|_{H_0^1(\Omega)} \le C_2 e^M\|\phi\|_{H_0^1(\Omega)} \leqslant C_2 e^M, \quad \text{and} \quad |\langle v,\phi\rangle| \le \|v\|_{H^{-1}(\Omega)}\|\phi\|_{H_0^1(\Omega)} \le C_3\|v\|_{L^2(\Omega)}.$$

Consequently: $\left\|\dfrac{\partial y_{\varepsilon,M}}{\partial t}\right\|_{H^{-1}(\Omega)} \le C_1\|\nabla y_{\varepsilon,M}\|_{L^2(\Omega)} + C_2 e^M + C_3\|v\|_{L^2(\Omega)}$

Integrating with respect to time, we get:

$$\begin{aligned}\left\|\frac{\partial y_{\varepsilon,M}}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} &\le C_1\int_0^T \|\nabla y_{\varepsilon,M}\|_{L^2(\Omega)}dt + C_2Te^M + C_3\|v\|_{L^2(Q)}\\ &\le C_1\|y_{\varepsilon,M}\|_{L^2(0,T,H^1(\Omega))} + C_2Te^M + C_3\|v\|_{L^2(Q)}\\ &\le C_1K + C_2Te^M + C_3\|v\|_{L^2(Q)} = K'.\end{aligned}$$

□

**Theorem 2.1.** *When $\varepsilon\to 0$, the solution $y_{\varepsilon,M}$ of (3) converges to the solution $y_M$ of the following problem:*

$$(P_M)\quad \begin{cases}\dfrac{\partial y_M}{\partial t} - \Delta y_M - f_M(y_M) = v & \text{in } Q,\\ y_M(\cdot,0) = y^0 & \text{in } \Omega,\\ y_M = 0 & \text{on } \Sigma,\end{cases} \tag{9}$$

*with $y^0 \in L^\infty(\Omega)$, $v \in L^\infty(Q)$.*

*Proof.* According to Lemma 2.1, the sequence $y_{\varepsilon,M}$ is bounded. Therefore, a subsequence can be extracted, still denoted by $y_{\varepsilon,M}$, which converges weakly to $y_M$, the solution to equation (9).
Setting $w_\varepsilon = y_{\varepsilon,M} - y_M$, we obtain the following system:

$$\begin{cases}\dfrac{\partial w_\varepsilon}{\partial t} - \Delta w_\varepsilon = f_M(y_{\varepsilon,M}) - f_M(y_M) & \text{in } Q,\\ w_\varepsilon(\cdot,0) = 0 & \text{in } \Omega,\\ w_\varepsilon = \varepsilon g & \text{on } \Sigma.\end{cases} \tag{10}$$

Let us therefore set $\tilde w_\varepsilon = z_{\varepsilon,M} - y_M \Longrightarrow \tilde w_\varepsilon = y_{\varepsilon,M} - \varepsilon w - y_M \Longrightarrow \tilde w_\varepsilon = w_\varepsilon - \varepsilon w.$ Then, $\tilde w_\varepsilon = 0$ on $\Sigma$ and $\tilde w_\varepsilon(\cdot,0) = 0$ in $\Omega$. Hence,

$$\begin{cases}\dfrac{\partial \tilde w_\varepsilon}{\partial t} - \Delta \tilde w_\varepsilon = f_M(y_{\varepsilon,M}) - f_M(y_M) + \varepsilon\left(-\dfrac{\partial w}{\partial t} + \Delta w\right) & \text{in } Q,\\ \tilde w_\varepsilon(\cdot,0) = 0 & \text{in } \Omega,\\ \tilde w_\varepsilon = 0 & \text{on } \Sigma.\end{cases} \tag{11}$$

By multiplying the state equation of (11) by $\tilde w_\varepsilon$ and integrating over $\Omega$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\tilde w_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\tilde w_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega \left[f_M(y_{\varepsilon,M}) - f_M(y_M)\right]\tilde w_\varepsilon\, dx + \varepsilon\int_\Omega\left(-\frac{\partial w}{\partial t} + \Delta w\right)\tilde w_\varepsilon\, dx \tag{12}$$

Let us estimate the terms on the right-hand side:

$$\begin{aligned}\int_\Omega \left|f_M(y_{\varepsilon,M}) - f_M(y_M)\right|\tilde w_\varepsilon\, dx &\le e^M\int_\Omega \left|y_{\varepsilon,M} - y_M\right|\tilde w_\varepsilon\, dx\\ &\le e^M\left(\int_\Omega |\tilde w_\varepsilon|\tilde w_\varepsilon\, dx + \varepsilon\int_\Omega |w|\,\tilde w_\varepsilon\, dx\right)\\ &\le e^M\left(\int_\Omega |\tilde w_\varepsilon|^2 dx + \frac{\varepsilon^2}{2}\int_\Omega |w|^2 dx + \frac{1}{2}\int_\Omega |\tilde w_\varepsilon|^2 dx\right)\\ &\le \frac{3}{2}e^M\|\tilde w_\varepsilon\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{2}e^M\|w\|^2_{L^2(\Omega)}.\end{aligned}$$

And the second term yields: $\varepsilon\displaystyle\int_\Omega\left(-\frac{\partial w}{\partial t} + \Delta w\right)\tilde w_\varepsilon\, dx \le \frac{\varepsilon^2}{2}\left\|-\frac{\partial w}{\partial t} + \Delta w\right\|^2_{L^2(\Omega)} + \frac{1}{2}\|\tilde w_\varepsilon\|^2_{L^2(\Omega)}$

Consequently, relation (12) becomes: $\dfrac{1}{2}\dfrac{d}{dt}\|\tilde w_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\tilde w_\varepsilon\|^2_{L^2(\Omega)} \le C\|\tilde w_\varepsilon\|^2_{L^2(\Omega)} + \dfrac{\varepsilon^2}{2}e^M\|w\|^2_{L^2(\Omega)} + \dfrac{\varepsilon^2}{2}\left\|-\dfrac{\partial w}{\partial t} + \Delta w\right\|^2_{L^2(\Omega)}$

Integrating this inequality from 0 to $t$.

$$\frac{1}{2}\|\tilde{w}_\varepsilon\|^2_{L^2(\Omega)} + \|\tilde{w}_\varepsilon\|^2_{L^2(0,T;H^1_0(\Omega))} \leq C\int_0^t \|\tilde{w}_\varepsilon\|^2_{L^2(\Omega)}\,ds + \frac{\varepsilon^2}{2}e^M\|w\|^2_{L^2(Q)} + \frac{\varepsilon^2}{2}\left\|-\frac{\partial w}{\partial t}+\Delta w\right\|^2_{L^2(Q)}$$

The application of Gronwall's lemma yields: $\|\tilde{w}_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\tilde{w}_\varepsilon\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \varepsilon^2 C$.

Since $y_{\varepsilon,M} - y_M = \tilde{w}_\varepsilon + \varepsilon w$, by the Cauchy-Schwarz and Young inequalities, we obtain:

$\|y_{\varepsilon,M} - y_M\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_{\varepsilon,M} - y_M\|^2_{L^2(0,T;H^1(\Omega))} \leq \varepsilon^2 C \underset{\varepsilon\to 0}{\longrightarrow} 0$. It just comes: $y_{\varepsilon,M} \longrightarrow y_M \quad \text{in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$.

We now prove that $f_M(y_{\varepsilon,M})$ converges strongly to $f_M(y_M)$.

As $y_{\varepsilon,M} \longrightarrow y_M \quad \text{in } L^2(Q)$, we have $\|f_M(y_{\varepsilon,M}) - f_M(y_M)\|_{L^2(Q)} \leq e^M\|y_{\varepsilon,M} - y_M\|_{L^2(Q)} \underset{\varepsilon\to 0}{\longrightarrow} 0$. Therefore, $f_M(y_{\varepsilon,M}) \longrightarrow f_M(y_M)$ in $L^2(Q)$. □

### 2.2. Convergence of $y_M$ to the solution of $(P)$

We study the limit of the solution $y_M$ of problem (9) as the truncation threshold $M$ tends to infinity. For this purpose, we need some assumptions.

**Theorem 2.2.** *Let $y^0 \in L^\infty(\Omega)$ and $v \in L^\infty(Q)$. For all $M > 0$, let $y_M$ be the solution of problem (9).*

*Set $A = \left\|y^0\right\|_{L^\infty(\Omega)}, \quad B = \left\|v^+\right\|_{L^\infty(Q)}$. Let $z$ be the maximal solution of $z'(t) = e^{z(t)} + B, \quad z(0) = A$.*

*If $z$ is defined on $[0,T]$, then $y_M(x,t) \leq z(t) \quad$ a.e. in $Q$, for all $M > 0$. Moreover, $y_M(x,t) \geq m_0 - \left\|v^-\right\|_{L^\infty(Q)} t, \quad m_0 = \min\left(0, \operatorname{essinf}_\Omega y^0\right)$.*

*Consequently, there exists a constant $C_T > 0$, independent of $M$, such that $\|y_M\|_{L^\infty(Q)} \leq C_T$.*

*Proof.* • **Uniform lower bound in $M$**

Consider $\tilde{z}(t) := m_0 - \|v^-\|_{L^\infty(Q)} t$. Then $\tilde{z}(t) \leq 0$ for all $t \in [0,T]$, so on the boundary $\tilde{z}(t) \leq 0 = y_M\big|_\Sigma$.

Moreover, $\tilde{z}(0) = m_0 \leq y^0 \quad$ a.e. in $\Omega$. We compute: $\partial_t\tilde{z} - \Delta\tilde{z} - f_M(\tilde{z}) - v = -\|v^-\|_{L^\infty(Q)} - f_M(\tilde{z}) - v$.

Since $f_M(\tilde{z}) \geq 0$, we have $-\|v^-\|_{L^\infty(Q)} - f_M(\tilde{z}) - v \leq -\|v^-\|_{L^\infty(Q)} - v$.

Since $v = v^+ - v^-$, we get $-\|v^-\|_{L^\infty(Q)} - v = -\|v^-\|_{L^\infty(Q)} - v^+ + v^- \leq 0$. Thus $\partial_t\tilde{z} - \Delta\tilde{z} - f_M(\tilde{z}) - v \leq 0$.

Hence $\tilde{z}$ is a subsolution of $(P_M)$. By the parabolic comparison principle, $\tilde{z} \leq y_M \quad$ in $Q$, from which $y_M(x,t) \geq m_0 - \|v^-\|_{L^\infty(Q)} t$.

• **Uniform upper bound in $M$**

Consider the solution $z$ of $z'(t) = e^{z(t)} + B, \quad z(0) = A$. Since $A \geq 0$ and $z' = e^z + B \geq 0$, we have $z(t) \geq 0 \quad \forall t \in [0,T]$.

Therefore, for all $M > 0$, $f_M(z(t)) \leq e^{z(t)}$. Indeed: if $0 \leq z(t) \leq M$, then $f_M(z(t)) = e^{z(t)}$; if $z(t) > M$, then $f_M(z(t)) = e^M \leq e^{z(t)}$.

Furthermore, since $B = \|v^+\|_{L^\infty(Q)}$, we have $v(x,t) \leq B \quad$ a.e. in $Q$.

We then compute: $\partial_t z - \Delta z - f_M(z) - v = z'(t) - f_M(z(t)) - v \geq e^{z(t)} + B - f_M(z) - v \geq 0$.

Hence $z$ is a supersolution. Moreover, $z(0) = A \geq y^0 \quad$ a.e. in $\Omega$, $\quad z(t) \geq 0 = y_M\big|_\Sigma$.

The comparison principle gives $y_M(x,t) \leq z(t) \quad$ in $Q$. Thus$\|y_M^+\|_{L^\infty(Q)} \leq \|z\|_{L^\infty(0,T)} \quad$ uniformly in $M$.

Combining both estimates: $\quad m_0 - T\|v^-\|_{L^\infty(Q)} \leq y_M(x,t) \leq \|z\|_{L^\infty(0,T)}$, and therefore $\|y_M\|_{L^\infty(Q)} \leq \max\left(\|z\|_{L^\infty(0,T)}, \left|m_0 - T\|v^-\|_{L^\infty(Q)}\right|\right)$. □

**Remark 2.1.** *The majorant ODE is defined for a given time interval. The precise condition is $T < T^*(A,B) := \displaystyle\int_A^{+\infty} \frac{ds}{e^s + B}$.*

*Under this hypothesis, $z$ does not blow up before $T$, so the uniform bound in $M$ is valid.*

*Special case $B = 0$. If $v \leq 0$, then $B = 0$ and $z'(t) = e^{z(t)}, \quad z(0) = A$.*

*This can be solved explicitly: $z(t) = -\ln\left(e^{-A} - t\right)$, defined as long as $t < e^{-A}$. Thus a sufficient condition is $T < e^{-A}$.*

In the following analysis, we consider the interval $[0,T]$ with $T<T^*(A,B)$. On this interval, there exists a constant $C_T>0$, independent of $M$, such that

$$\|y_M\|_{L^\infty(Q)}\le C_T. \tag{13}$$

Thanks to (13), we have

$$|f_M(y_M)|\le e^{C_T}\quad \text{a.e. in } Q. \tag{14}$$

**Lemma 2.2.** *The solution $y_M$ of problem (9) satisfies:*

$$\|y_M\|_{L^\infty(0,T;L^2(\Omega))}\le C,\qquad \|y_M\|_{L^2(0,T;H_0^1(\Omega))}\le C,\qquad \left\|\frac{\partial y_M}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))}\le C. \tag{15}$$

*Here $C$ denotes various strictly positive constants, independent of $M$.*

*Proof.* By multiplying the state equation of (9) by $y_M$ and integrating over $\Omega$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |y_M|^2\,dx+\int_\Omega |\nabla y_M|^2\,dx=\int_\Omega f_M(y_M)y_M\,dx+\int_\Omega v y_M\,dx.$$

By estimation, we obtain: $\left|\int_\Omega f_M(y_M)y_M\,dx\right|\le e^{C_T}\int_\Omega |y_M|\,dx\le \frac{1}{2}e^{C_T}\|y_M\|_{L^2(\Omega)}+\frac{1}{2}C^2(\Omega)e^{C_T}$, and $\left|\int_\Omega v y_M\,dx\right|\le \frac{1}{2}\|v\|^2_{L^2(\Omega)}+\frac{1}{2}\|y_M\|^2_{L^2(\Omega)}$.

Let us $\lambda=\frac{1}{2}C^2(\Omega)e^{C_T}+\frac{1}{2}\|v\|^2_{L^2(\Omega)}$, and $K_T=\frac{1}{2}(e^{C_T}+1)$. Then, $\frac{1}{2}\frac{d}{dt}\|y_M\|^2_{L^2(\Omega)}+\|\nabla y_M\|^2_{L^2(\Omega)}\le K_T\|y_M\|^2_{L^2(\Omega)}+\lambda$.

By integrating from 0 to $t$ $(0\le t\le T)$ and applying Gronwall's lemma, we obtain:

$$\|y_M(t)\|^2_{L^\infty(0,T;L^2(\Omega))}+\|y_M\|^2_{L^2(0,T;H_0^1(\Omega))}\le C\quad \forall t\in[0,T]. \tag{16}$$

Where $C=\|y^0\|^2_{L^2(\Omega)}+\lambda T$.

For the time derivative: for all $\varphi\in H_0^1(\Omega)$, we have the equality in the sense of distributions:

$$\langle\frac{\partial y_M}{\partial t},\varphi\rangle=-\int_\Omega \nabla y_M\cdot\nabla\varphi\,dx+\int_\Omega f_M(y_M)\varphi\,dx+\int_\Omega v\varphi\,dx.$$

Using relation (8), we obtain:

$$\left\|\frac{\partial y_M}{\partial t}\right\|_{H^{-1}(\Omega)}\le \|\nabla y_M\|_{L^2(\Omega)}+c_1\|f_M(y_M)\|_{L^2(\Omega)}+c_2\|v\|_{L^2(\Omega)}.$$

By integrating this inequality from 0 to $t$ $(0\le t\le T)$ and according to (13) and (14), we deduce that $\left(\frac{\partial y_M}{\partial t}\right)$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. □

**Theorem 2.3.** *When $M\to\infty$, the solution of the problem (9) converges to the solution $y$ of the problem (1) in the following spaces:*

$$\begin{aligned} y_M&\to y\quad \text{a.e. and in } L^2(Q),\\ f_M(y_M)&\to e^y\quad \text{a.e. and in } L^2(Q),\\ y_M&\longrightarrow y\quad \text{in } L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)).\end{aligned} \tag{17}$$

*Proof.* The estimates obtained in Lemma 2.2 show that the sequence $(y_M)$ is bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, and that $\left(\frac{\partial y_M}{\partial t}\right)$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.

Consequently, we can extract subsequences, still denoted in the same way, that converge weakly as follows:

$$y_M\rightharpoonup y\quad \text{in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)),$$

$$\text{and}\quad \frac{\partial y_M}{\partial t}\rightharpoonup\frac{\partial y}{\partial t}\quad \text{in } L^2(0,T;H^{-1}(\Omega)).$$

In addition, as $\Omega$ is bounded and regular, the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact, and the embedding $L^2(\Omega)\hookrightarrow H^{-1}(\Omega)$ is continuous.
By the Aubin-Lions theorem, the sequence $(y_M)$ is relatively compact in $L^2(Q)$.
Thus, one can extract a subsequence (still denoted $(y_M)$) such that

$$y_M\longrightarrow y\quad \text{strongly in } L^2(Q),$$

and therefore, up to a subsequence, $y_M\to y$ almost everywhere in $Q$.
Since (13) gives $|y_M|\le C_T$ for all $y_M$ solution of (9) then, $f_M(y_M)=e^{y_M}$ for $M>C_T$. As $y_M$ converges strongly to $y$ and $f_M$ is continuous, consequently $f_M(y_M)\longrightarrow e^y$ a.e. Moreover, $|f_M(y_M)|\le e^{C_T}$ is integrable over $Q$. The Lebesgue dominated convergence theorem applies and yields

$$f_M(y_M)\longrightarrow e^y\quad \text{strongly in } L^2(Q).$$

Now set $\beta = y_M - y$. Then, $\beta$ satisfies:

$$\begin{cases} \dfrac{\partial \beta}{\partial t} - \Delta\beta - f_M(y_M) + e^y = 0 & \text{in } Q, \\ \beta(\cdot,0) = 0 & \text{in } \Omega, \\ \beta = 0 & \text{on } \Sigma. \end{cases}$$

By multiplying the state equation by $\beta$ and integrating over $\Omega$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\beta\|^2_{L^2(\Omega)} + \|\nabla\beta\|^2_{L^2(\Omega)} = \int_\Omega \big(f_M(y_M) - e^y\big)\beta\, dx.$$

By the Cauchy-Schwarz and Young inequalities, we have:

$$\frac{1}{2}\frac{d}{dt}\|\beta\|^2_{L^2(\Omega)} + \|\nabla\beta\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|f_M(y_M) - e^y\|^2_{L^2(\Omega)} + \frac{1}{2}\|\beta\|^2_{L^2(\Omega)}.$$

Integrating this inequality from 0 to $t$ and applying Gronwall's lemma, we obtain:

$$\|\beta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\beta\|^2_{L^2(0,T;H^1_0(\Omega))} \leq C\|f_M(y_M) - e^y\|^2_{L^2(Q)}.$$

Consequently, $\|y_M - y\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_M - y\|^2_{L^2(0,T;H^1_0(\Omega))} \leq C\|f_M(y_M) - e^y\|^2_{L^2(Q)}$.

From, $(17)_2$ we have $\|f_M(y_M) - e^y\|^2_{L^2(Q)} \to 0 \quad$ as $M \to \infty$. Hence, $y_M \longrightarrow y$ in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. □

## 3. Application of optimal control

In this section, we focus on controlling the system given by equation (3). To this end, we employ the concepts of low-regret and no-regret control. Let $y_{\varepsilon,M} := y_{\varepsilon,M}(v,g)$ denote the state of system (3), where $g$ belongs to $L^2(\Sigma)$, and $v$ belongs to $L^2(Q)$. The control v is chosen so that the state $y_{\varepsilon,M}(v,g)$ is close to the desired state $y_d$ belonging to $L^2(Q)$. We associate problem (3) with the cost functional defined as follows:

$$J_\varepsilon(v,g) = \frac{1}{2}\|y_{\varepsilon,M}(v,g) - y_d\|^2_{L^2(Q)} + \frac{\alpha}{2}\|v\|^2_{L^2(Q)}. \tag{18}$$

Where $\alpha > 0$. We are interested in the following problem:

$$\inf_{v\in L^2(Q)} \sup_{g\in L^2(\Sigma)} J_\varepsilon(v,g). \tag{19}$$

First of all, let us show that the mapping $(v,g) \longmapsto y_{\varepsilon,M}$ is differentiable.

**Theorem 3.1.** *The mapping $\xi : L^2(Q) \times L^2(\Sigma) \longrightarrow L^2(0,T;H^1(\Omega))$, $\quad (v,g) \longmapsto y_{\varepsilon,M}$ is Gâteaux differentiable on $L^2(Q) \times L^2(\Sigma)$. Its Gâteaux differential $D_G\xi(v,g)(\delta v,\delta g) = z_{\varepsilon,M}$ is the unique solution of*

$$\begin{cases} \dfrac{\partial z_{\varepsilon,M}}{\partial t} - \Delta z_{\varepsilon,M} - f'_M(y_{\varepsilon,M})z_{\varepsilon,M} = \delta v & \text{in } Q, \\ z_{\varepsilon,M}(\cdot,0) = 0 & \text{in } \Omega, \\ z_{\varepsilon,M} = \varepsilon\delta g & \text{on } \Sigma. \end{cases} \tag{20}$$

*Proof.* Fix $(v,g) \in L^2(Q) \times L^2(\Sigma)$ and $(\delta v,\delta g) \in L^2(Q) \times L^2(\Sigma)$. For $\lambda$ sufficiently small, set

$$y_\lambda := \xi(v + \lambda\delta v, g + \lambda\delta g), \quad y_{\varepsilon,M} := \xi(v,g) \quad \text{and} \quad w_\lambda := \frac{y_\lambda - y_{\varepsilon,M}}{\lambda}.$$

$w_\lambda$ satisfies:

$$\begin{cases} \dfrac{\partial w_\lambda}{\partial t} - \Delta w_\lambda = \dfrac{f_M(y_\lambda) - f_M(y_{\varepsilon,M})}{\lambda} + \delta v & \text{in } Q, \\ w_\lambda \cdot,0) = 0 & \text{in } \Omega, \\ w_\lambda = \varepsilon\delta g & \text{on } \Sigma. \end{cases} \tag{21}$$

Define $z_{\varepsilon,M}$ as the solution of the linearized equation (20). It suffices to show that $w_\lambda \to z_{\varepsilon,M} \quad$ in $L^2(0,T;H^1(\Omega)) \quad$ as $\lambda \to 0$.

Set $r_\lambda := w_\lambda - z_{\varepsilon,M}$ solution of :

$$\begin{cases} \dfrac{\partial r_\lambda}{\partial t} - \Delta r_\lambda - f'_M(y_{\varepsilon,M})r_\lambda = \eta_\lambda & \text{in } Q, \\ r_\lambda(\cdot,0) = 0 & \text{in } \Omega, \\ r_\lambda = 0 & \text{on } \Sigma, \end{cases} \tag{22}$$

where $\eta_\lambda = \dfrac{f_M(y_\lambda) - f_M(y_{\varepsilon,M}) - f'_M(y_{\varepsilon,M})(y_\lambda - y_{\varepsilon,M})}{\lambda}$.

Since $f_M$ is differentiable almost everywhere, we have for almost every $(x,\tau)\in Q$,

$$\frac{f_M(y_\lambda(x,\tau))-f_M(y_{\varepsilon,M}(x,\tau))-f'_M(y_{\varepsilon,M}(x,\tau))(y_\lambda(x,\tau)-y_{\varepsilon,M}(x,\tau))}{y_\lambda(x,\tau)-y_{\varepsilon,M}(x,\tau)}\longrightarrow 0$$

as $\lambda\to 0$, whenever $y_{\varepsilon,M}(x,\tau)\neq \pm M$. Since the points $\pm M$ form a negligible set, it follows that

$$\eta_\lambda(x,\tau)\to 0 \quad \text{for almost every } (x,\tau)\in Q.$$

Moreover, the Lipschitz continuity of $f_M$ yields $|\eta_\lambda|\leq C|w_\lambda|$, for some constant $C>0$ independent of $\lambda$. Since $(w_\lambda)$ is bounded in $L^2(0,T;H^1(\Omega))$, hence in $L^2(Q)$, the dominated convergence theorem applies and gives $\eta_\lambda\to 0$ in $L^2(Q)$.
Applying standard energy estimates to (22), we obtain $\|r_\lambda\|_{L^2(0,T;H^1(\Omega))}\leq C\|\eta_\lambda\|_{L^2(Q)}$.
As $\eta_\lambda\to 0$ in $L^2(Q)$, it follows that $\|r_\lambda\|_{L^2(0,T;H^1(\Omega))}\longrightarrow 0$.
In other words,

$$\left\|\frac{\xi(v+\lambda\delta v,g+\lambda\delta g)-\xi(v,g)}{\lambda}-z_{\varepsilon,M}\right\|_{L^2(0,T;H^1(\Omega))}\longrightarrow 0 \quad \text{as } \lambda\to 0.$$

Thus, $\xi$ is Gâteaux differentiable at $(v,g)$, and its differential in the direction $(\delta v,\delta g)$ is given by the solution $z_{\varepsilon,M}$ of (20). □

**Definition 3.1.** *A no-regret control for problem (3), if it exists, is a function $v\in L^2(Q)$ that solves*

$$\inf_{v\in L^2(Q)}\left(\sup_{g\in L^2(\Sigma)}\left(J_\varepsilon(v,g)-J_\varepsilon(0,g)\right)\right). \tag{23}$$

As $y_{\varepsilon,M}(v,g)$ is nonlinear and inherits the regularity of the solution of (3), in (18) we replace $y_{\varepsilon,M}(v,g)$ by its linearized form $y_{\varepsilon,M}(v,0)+\dfrac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)$. Thus, we define a new cost functional $J_{1\varepsilon}$ as follows:

$$J_{1\varepsilon}(v,g)=J_\varepsilon(v,0)+\frac{\partial J_\varepsilon}{\partial g}(v,0)(g). \tag{24}$$

Then, we are led to the following linearized problem:

$$\inf_{v\in L^2(Q)}\left(\sup_{g\in L^2(\Sigma)}\left(J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)\right)\right). \tag{25}$$

Recall that the solution of (25), if it exists, is called the no-regret control of the linearized problem. We now state the following lemma:

**Lemma 3.1.** *Let $J_\varepsilon$ be given by (18). For any $(v,g)\in L^2(Q)\times L^2(\Sigma)$, we have:*

$$J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+\left\langle y_\varepsilon(v,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle-\left\langle y_{\varepsilon,M}(0,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g)\right\rangle \tag{26}$$

*Proof.* The mapping $(v,g)\longmapsto y_{\varepsilon,M}(v,g)$ is differentiable on $L^2(Q)\times L^2(\Sigma)$, and its differential at the point $(v,0)$ in the direction $(0,g)$ is:

$$\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)=\lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)}{\theta}.$$

We have:

$$J_\varepsilon(v,\theta g)=J_\varepsilon(v,0)+\frac{1}{2}\|y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)\|^2_{L^2(Q)}+\langle y_{\varepsilon,M}(v,0)-y_d,\, y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)\rangle.$$

Consequently,

$$\begin{aligned}\lim_{\theta\to 0}\frac{J_\varepsilon(v,\theta g)-J_\varepsilon(v,0)}{\theta} &= \left\langle y_{\varepsilon,M}(v,0)-y_d,\lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)}{\theta}\right\rangle+\lim_{\theta\to 0}\frac{\theta}{2}\left\|\frac{y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)}{\theta}\right\|^2_{L^2(Q)}\\ &= \left\langle y_{\varepsilon,M}(v,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle\\ &= \frac{\partial J_\varepsilon}{\partial g}(v,0)(g).\end{aligned} \tag{27}$$

Substituting (27) into (24) gives:

$$J_{1\varepsilon}(v,g)=J_\varepsilon(v,0)+\left\langle y_{\varepsilon,M}(v,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle. \tag{28}$$

Setting $v=0$ in (28) we obtain:

$$J_{1\varepsilon}(0,g)=J_\varepsilon(0,0)+\left\langle y_{\varepsilon,M}(0,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g)\right\rangle. \tag{29}$$

Consequently,

$$J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+\left\langle y_{\varepsilon,M}(v,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle-\left\langle y_\varepsilon(0,0)-y_d,\frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g)\right\rangle.$$

□

The following results clarify the terms.

$$\left\langle y_{\varepsilon,M}(v,0)-y_d, \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle \quad \text{and} \quad \left\langle y_{\varepsilon,M}(0,0)-y_d, \frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g)\right\rangle \quad \text{in (28) and (29).}$$

**Proposition 3.1.** *The partial derivative* $\frac{\partial y_{\varepsilon,M}}{\partial g} := \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g) = \lim_{\theta\to 0} \frac{y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)}{\theta}$ *is the solution to the following system:*

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial y_{\varepsilon,M}}{\partial g}\right) - \Delta\left(\frac{\partial y_{\varepsilon,M}}{\partial g}\right) - \frac{\partial y_{\varepsilon,M}}{\partial g} f'_M(y_{\varepsilon,M}) = 0 & \text{in } Q, \\ \left.\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g) = \varepsilon g & \text{on } \Sigma. \end{cases} \tag{30}$$

*Proof.* Let $y_{\varepsilon,M}(v,0)$ be the solution to problem (3) for $g=0$, so:

• In $Q$, we have:

$$\frac{\partial}{\partial g}\left[\frac{\partial y_{\varepsilon,M}(v,0)}{\partial t} - \Delta y_{\varepsilon,M}(v,0) - f_M(y_{\varepsilon,M}(v,0))\right] = 0 \Longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial y_{\varepsilon,M}}{\partial g}\right) - \Delta\left(\frac{\partial y_{\varepsilon,M}}{\partial g}\right) - \frac{\partial y_{\varepsilon,M}}{\partial g} f'_M(y_{\varepsilon,M}) = 0$$

because $v$ does not depend on $g$.

• For the initial condition in $\Omega$, we have:

$$\left.\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right|_{t=0} = \lim_{\theta\to 0} \frac{y_{\varepsilon,M}(v,\theta g)|_{t=0} - y_{\varepsilon,M}(v,0)|_{t=0}}{\theta} = 0 \quad \text{because } y_{\varepsilon,M}(v,g)|_{t=0} = y^0 \text{ in } \Omega.$$

• For the boundary condition, we have:

$$\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g) = \lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v,\theta g)-y_{\varepsilon,M}(v,0)}{\theta} = \lim_{\theta\to 0}\frac{\theta\varepsilon g - 0}{\theta} = \varepsilon g \text{ on } \Sigma.$$

□

**Proposition 3.2.** *Similarly, we define the partial derivative* $\frac{\partial y_{\varepsilon,M}}{\partial g} := \frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g) = \lim_{\theta\to 0}\frac{y_{\varepsilon,M}(0,\theta g)-y_{\varepsilon,M}(0,0)}{\theta}$ *solution to the following problem:*

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial y_{\varepsilon,M}}{\partial g}\right) - \Delta\left(\frac{\partial y_{\varepsilon,M}}{\partial g}\right) - \frac{\partial y_{\varepsilon,M}}{\partial g} f'_M(y_{\varepsilon,M}) = 0 & \text{in } Q, \\ \left.\frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g)\right|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g) = \varepsilon g & \text{on } \Sigma. \end{cases} \tag{31}$$

$y_{\varepsilon,M}(0,0)$ *is the solution to problem (3) for* $(v,g)=(0,0)$.

*Proof.* The Proposition 3.2 is obtained by using an argument similar to that of Proposition 3.1, and by setting $v=0$. □

We have the following lemma:

**Lemma 3.2.** *For all* $(v,g)\in L^2(Q)\times L^2(\Sigma)$, *we have:*

$$J_{1\varepsilon}(v,g) - J_{1\varepsilon}(0,g) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + \left\langle \frac{\partial(\zeta_{\varepsilon,M}(v)-\zeta_{\varepsilon,M}(0))}{\partial \nu}, \varepsilon g\right\rangle_{L^2(\Sigma)}. \tag{32}$$

*Where* $\zeta_{\varepsilon,M}(v) = \zeta_{\varepsilon,M}(x,t;v)$ *is the solution of the adjoint problem:*

$$\begin{cases} -\frac{\partial \zeta_{\varepsilon,M}}{\partial t}(v) - \Delta\zeta_{\varepsilon,M}(v) - \zeta_{\varepsilon,M}(v) f'_M(y_{\varepsilon,M}) = -y_\varepsilon(v,0)+y_d & \text{in } Q, \\ \zeta_{\varepsilon,M}(T,v) = 0 & \text{in } \Omega, \\ \zeta_{\varepsilon,M}(v) = 0 & \text{on } \Sigma. \end{cases} \tag{33}$$

*Proof.* Let us multiply the equation of state in (33) by $\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)$. By integrating by parts and using the boundary conditions, $\zeta_{\varepsilon,M}(T,v)=0$ and $\left.\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right|_{t=0} = 0$ in $\Omega$, we have:

$$\left\langle -\frac{\partial \zeta_{\varepsilon,M}}{\partial t}(v), \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial t}\left(\frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right), \zeta_{\varepsilon,M}(v)\right\rangle_{L^2(Q)}.$$

Let us now set $\zeta_{\varepsilon,M}(v)=0$ and $\dfrac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)=\varepsilon g$ over $\Sigma$ and apply Green's formula, we obtain:

$$\left\langle -\Delta\zeta_{\varepsilon,M)}(v),\, \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial g}\big(-\Delta y_{\varepsilon,M}(v,0)\big)(g),\, \zeta_{\varepsilon,M}(v)\right\rangle_{L^2(Q)} - \left\langle \frac{\partial \zeta_{\varepsilon,M}}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)}.$$

So,

$$\left\langle -(y_{\varepsilon,M}(v,0)-y_d),\, \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle_{L^2(Q)} = -\left\langle \frac{\partial \zeta_{\varepsilon,M}}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)}$$

Indeed,

$$\frac{\partial}{\partial t}\left(\frac{\partial z_\varepsilon}{\partial g}(v,0)(g)\right)-\Delta\left(\frac{\partial z_\varepsilon}{\partial g}(v,0)(g)\right)-\zeta_{\varepsilon,M}(v)f_M(y_{\varepsilon,M})=0 \text{ in } Q.$$

Then,

$$\left\langle y_{\varepsilon,M}(v,0)-y_d,\, \frac{\partial y_{\varepsilon,M}}{\partial g}(v,0)(g)\right\rangle_{L^2(Q)} = \left\langle \frac{\partial \zeta_{\varepsilon,M}}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)} \tag{34}$$

If $v=0$, system (33) becomes:

$$\begin{cases} -\dfrac{\partial \zeta_{\varepsilon,M}}{\partial t}(0)-\Delta\zeta_{\varepsilon,M}(0)-\zeta_{\varepsilon,M}(0)\,f'_M(y_{\varepsilon,M})=-y_{\varepsilon,M}(0,0)+y_d & \text{in } Q,\\ \zeta_{\varepsilon,M}(T,0)=0 & \text{in } \Omega,\\ \zeta_{\varepsilon,M}(0)=0 & \text{on } \Sigma. \end{cases} \tag{35}$$

Let us multiply the equation of state in (35) by $\dfrac{\partial y_{\varepsilon,M}}{\partial g}(0,0)(g)$, and by analogy with the previous results, we obtain:

$$\left\langle -(y_{\varepsilon,M}(0,0)-y_d),\, \frac{\partial y_{\varepsilon,M}}{\partial g}(0,0)\right\rangle_{L^2(Q)} = -\left\langle \frac{\partial \zeta_{\varepsilon,M}}{\partial \nu}(0),\, \varepsilon g\right\rangle_{L^2(\Sigma)}. \tag{36}$$

Substituting (34) and (36) into equation (26) gives:

$$J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+\left\langle \frac{\partial(\zeta_{\varepsilon,M}(v)-\zeta_{\varepsilon,M}(0))}{\partial \nu},\, \varepsilon g\right\rangle_{L^2(\Sigma)}.$$

□

We now define the application $S$ acting on $L^2(Q)$ by $S(v)=\zeta_{\varepsilon,M}(v)-\zeta_{\varepsilon,M}(0)$. We obtained the following result:

$$\sup_{g\in L^2(\Sigma)}\big(J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)\big)=J_\varepsilon(v,0)-J_\varepsilon(0,0)+\sup_{g\in L^2(\Sigma)}\left\langle \frac{\partial S}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)} \tag{37}$$

with

$$\sup_{g\in L^2(\Sigma)}\left\langle \frac{\partial S}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)} = \begin{cases} +\infty & \text{if } \left\langle \dfrac{\partial S}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)} \neq 0,\\ 0 & \text{si } \dfrac{\partial S}{\partial \nu}(v)=0. \end{cases}$$

The value of $\sup_{g\in L^2(\Sigma)}\left\langle \dfrac{\partial S}{\partial \nu}(v), \varepsilon g\right\rangle_{L^2(\Sigma)}$ does not allow us to characterize the no-regret control of the minimisation problem:

$$\inf_{v\in L^2(Q)}\left[\sup_{g\in L^2(\Sigma)}\big(J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)\big)\right] \tag{38}$$

Hence, the use of the low-regret control approach to seek overall control:

$$\mathscr{K}_1=\left\{v\in L^2(Q),\, \left\langle \frac{\partial S}{\partial \nu}(v),\, \varepsilon g\right\rangle_{L^2(\Sigma)}=0,\ \forall g\in L^2(\Sigma)\right\}.$$

### 3.1. Low-regret control

In this section, we focus on the low-regret control problem, which depends on the pollution $g$ and its norm. We proceed by relaxing the problem (38). Then, we consider the following definition:

**Definition 3.2.** *Let $\gamma>0$ be the relaxation parameter. Then, the problem (38) becomes:*

$$\inf_{v\in L^2(Q)}\left[\sup_{g\in L^2(\Sigma)}\Big(J_{1\varepsilon}(v,g)-J_{1\varepsilon}(0,g)-\gamma\|g\|^2_{L^2(\Sigma)}\Big)\right]. \tag{39}$$

*The solution to the problem (39), if it exists, is called the low-regret control.*

**Lemma 3.3.** *Let $J_\varepsilon$ be the function defined by (18). Then, for any $v \in L^2(Q)$, the problem (39) becomes:*

$$\inf_{v\in L^2(Q)} \left[ J_\varepsilon(v,0) - J_\varepsilon(0,0) + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)} \right]. \tag{40}$$

*Proof.* According to Lemma 3.2, we have:

$$\sup_{g\in L^2(\Sigma)} \left( J_{1\varepsilon}(v,g) - J_{1\varepsilon}(0,g) - \gamma \|g\|^2_{L^2(\Sigma)} \right) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + \sup_{g\in L^2(\Sigma)} \left( -\gamma\|g\|^2_{L^2(\Sigma)} + \left\langle \frac{\partial S}{\partial v}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \right).$$

According to Fenchel's transformation, we have:

$$2 \sup_{g\in L^2(\Sigma)} \left( \left\langle \frac{1}{2}\frac{\partial S}{\partial v}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} - \frac{\gamma}{2}\|g\|^2_{L^2(\Sigma)} \right) = \frac{\varepsilon^2}{2\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)} - \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)} = \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)}.$$

Thus, problem (39) becomes:

$$\inf_{v\in L^2(Q)} \mathscr{J}^\gamma_\varepsilon(v). \tag{41}$$

Where $\mathscr{J}^\gamma_\varepsilon(v) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + \dfrac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)}$. □

We reformulate the problem (39) as follows: For any $\gamma > 0$ , find $u^\gamma_\varepsilon \in L^2(Q)$ such that:

$$\mathscr{J}^\gamma_\varepsilon(u^\gamma_\varepsilon) = \inf_{v\in L^2(Q)} \mathscr{J}^\gamma_\varepsilon(v). \tag{42}$$

The problem (42) is a low-regret problem, and its solution is called a low-regret control.

**Remark 3.1.** *Unlike in the linear case, the mapping $v \longmapsto J_\varepsilon(v,0) - J_\varepsilon(0,0) + \dfrac{\varepsilon^2}{4\gamma} \left\| \dfrac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)}$ is not convex, so $u^\gamma_\varepsilon$ is not necessarily unique. Furthermore, it is not certain that $u^\gamma_\varepsilon$ converges in the set $\mathscr{K}_1$. We shall adapt the cost function $\mathscr{J}^\gamma_\varepsilon(v)$ to a given no-regret optimal control, as in the work of Lions [5] for the penalisation method.*

#### 3.1.1. Adapted low-regret control

Let $\tilde{u}$ be a no-regret optimal control, and define the function:

$$v \longmapsto J_\varepsilon(v,0) - J_\varepsilon(0,0) + \frac{1}{2}\|v-\tilde{u}\|^2_{L^2(Q)} + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)}.$$

We consider the following problem:

$$\inf_{v\in L^2(Q)} \mathscr{J}^\gamma_{\varepsilon a}(v). \tag{43}$$

Where

$$\mathscr{J}^\gamma_{\varepsilon a}(v) = J_\varepsilon(v,0) - J_\varepsilon(0,0) + \frac{1}{2}\|v-\tilde{u}\|^2_{L^2(Q)} + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)}. \tag{44}$$

So, we have the following proposition:

**Proposition 3.3.** *Problem (43)-(44) has at least one solution, $u^\gamma_\varepsilon$, in $L^2(Q)$. This solution is known as an adapted low-regret control.*

*Proof.* By the definition of $\mathscr{J}^\gamma_{\varepsilon a}$ and for all $v \in L^2(Q)$, we have: $\mathscr{J}^\gamma_{\varepsilon a}(v) \geq -J_\varepsilon(0,0)$.

As a result $-J_\varepsilon(0,0) \leq J_\varepsilon(v,0) - J_\varepsilon(0,0) + \dfrac{1}{2}\|v-\tilde{u}\|^2_{L^2(Q)} + \dfrac{\varepsilon^2}{4\gamma} \left\| \dfrac{\partial S}{\partial v}(v) \right\|^2_{L^2(\Sigma)}$.

The set $\mathscr{K}_2 = \left\{ v \in L^2(Q),\ \mathscr{J}^\gamma_{\varepsilon a}(v) \geq -J_\varepsilon(0,0) \right\}$ is non-empty.

Indeed, by setting $v = 0$ we have: $\mathscr{J}^\gamma_{\varepsilon a}(0) = \dfrac{1}{2}\|\tilde{u}\|^2_{L^2(Q)} \geq -J_\varepsilon(0,0)$. There therefore exists a constant $C(\varepsilon,\gamma)$ such that :

$C(\varepsilon,\gamma) = \inf\limits_{v\in L^2(Q)} \mathscr{J}^\gamma_{\varepsilon a}(v)$.

To establish this result, consider a minimising sequence $(v_n)_{n\in\mathbb{N}}$ which converges to $d^\gamma_\varepsilon$. Then, $d^\gamma_\varepsilon = \lim\limits_{n\to+\infty} \mathscr{J}^\gamma_{\varepsilon a}(v_n)$. And we have:

$$-J_\varepsilon(0,0) \leq \frac{1}{2}\|y_{\varepsilon,M}(v_n,0) - y_d\|^2_{L^2(Q)} + \frac{\alpha}{2}\|v_n\|^2_{L^2(Q)} - \frac{1}{2}\|y_{\varepsilon,M}(0,0) - y_d\|^2_{L^2(Q)} + \frac{1}{2}\|v_n - \tilde{u}\|^2_{L^2(Q)} + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial v}(v_n) \right\|^2_{L^2(\Sigma)} \leq d^\gamma_\varepsilon + 1.$$

We conclude that there exist constants $C_i(\varepsilon,\gamma)$; $i=\overline{1,3}$ independent of $n$ such that:

$$\begin{cases} \|v_n\|_{L^2(Q)} \leq C_1(\varepsilon,\gamma), \\ \left\|y_{\varepsilon,M}(v_n,0)\right\|_{L^2(Q)} \leq C_2(\varepsilon,\gamma), \\ \dfrac{\varepsilon}{\sqrt{\gamma}}\left\|\dfrac{\partial S}{\partial v}(v_n)\right\|^2_{L^2(\Sigma)} \leq C_3(\varepsilon,\gamma). \end{cases}$$

We can therefore extract a subsequence $(v_n)$ such that $v_n \rightharpoonup u_\varepsilon^\gamma$ in $L^2(Q)$. □

The following propositions constitute fundamental tools for the analysis and characterization of optimal control.

**Proposition 3.4.** *Let $w\in L^2(Q)$ and $\theta\in ]0,1[$.*
*The partial derivative $\dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w) = \lim\limits_{\theta\to 0}\dfrac{y_{\varepsilon,M}(v+\theta w,0)-y_{\varepsilon,M}(v,0)}{\theta}$ is a solution to the following problem:*

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}\right) - \Delta\left(\dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}\right) - \dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v} f'_M(y^\gamma_{\varepsilon,M}) = w & \text{in } Q, \\ \dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w)\Big|_{t=0} = 0 & \text{in } \Omega, \\ \dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w) = 0 & \text{on } \Sigma. \end{cases} \tag{45}$$

*Proof.* Let us show that $\dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w)$ is a solution to (45). For all $w\in L^2(Q)$ and all $\theta\in ]0;1[$. We have:

$$\frac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0) = \lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v+\theta w,0)-y_{\varepsilon,M}(v,0)}{\theta}.$$

• In $Q$, On the one hand, we have:

$$\frac{\partial y^\gamma_{\varepsilon,M}}{\partial t} - \Delta(y^\gamma_{\varepsilon,M}) - f_M(y^\gamma_{\varepsilon,M}) = v \implies \frac{\partial}{\partial v}\left(\frac{\partial y^\gamma_{\varepsilon,M}}{\partial t} - \Delta(y^\gamma_{\varepsilon,M}) - f_M(y^\gamma_{\varepsilon,M})\right)(v,0)(w) = \lim_{\theta\to 0}\frac{v+\theta w - v}{\theta} = w \quad \text{in } Q.$$

We also have:

$$\frac{\partial}{\partial v}\left(\frac{\partial y^\gamma_{\varepsilon,M}}{\partial t} - \Delta(y^\gamma_{\varepsilon,M}) - f_M(y^\gamma_{\varepsilon,M})\right)(v,0)(w) = \frac{\partial}{\partial t}\left(\frac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w)\right) - \Delta\left(\frac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w)\right) - f'_M(y^\gamma_{\varepsilon,M})\frac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w)$$

• For the initial condition in $\Omega$.

$$\frac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w)\Big|_{t=0} = \lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v+\theta w,0)|_{t=0} - y_{\varepsilon,M}(v,0)|_{t=0}}{\theta} = 0 \text{ in } \Omega.$$

• On $\Sigma$, we have:

$$\frac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w) = \lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v+\theta w,0)-y_{\varepsilon,M}(v,0)}{\theta} = 0 \text{ on } \Sigma.$$

□

**Proposition 3.5.** *For all $w\in L^2(Q)$, the partial derivative $\dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v}(v)(w) = \lim\limits_{\theta\to 0}\dfrac{\zeta_\varepsilon(v+\theta w)-\zeta_\varepsilon(v)}{\theta}$ is a solution of*

$$\begin{cases} -\dfrac{\partial}{\partial t}\left(\dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v}\right) - \Delta\left(\dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v}\right) - \dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v} f'_M(y^\gamma_{\varepsilon,M}) = -\dfrac{\partial y^\gamma_{\varepsilon,M}}{\partial v}(v,0)(w) & \text{in } Q, \\ \dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v}(v)(w)\Big|_{t=T} = 0 & \text{in } \Omega, \\ \dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v}(v)(w) = 0 & \text{on } \Sigma. \end{cases} \tag{46}$$

*Proof.* Let us show that $\dfrac{\partial\zeta^\gamma_\varepsilon}{\partial v}(v)(w)$ is a solution to (46).

• In $Q$, we have:

$$-\frac{\partial}{\partial t}\left(\frac{\partial\zeta_\varepsilon^\gamma}{\partial v}(v)(w)\right)-\Delta\left(\frac{\partial\zeta_\varepsilon^\gamma}{\partial v}(v)(w)\right)-\frac{\partial\zeta_\varepsilon^\gamma}{\partial v}(v)(w)f_M'(y_{\varepsilon,M}^\gamma)$$

$$= \lim_{\theta\to 0}\left(\frac{\frac{\partial\zeta_\varepsilon}{\partial t}(v+\theta w)-\Delta\zeta_\varepsilon(v+\theta w)-\zeta_\varepsilon(v+\theta w)f_M(y_{\varepsilon,M})}{\theta}\right)-\lim_{\theta\to 0}\left(\frac{\frac{\partial\zeta_\varepsilon}{\partial t}(v,0)-\Delta\zeta_\varepsilon(v,0)-\zeta_\varepsilon(v,0)f_M(y_{\varepsilon,M})}{\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{-(y_{\varepsilon,M}(v+\theta w,0)-y_d)+y_{\varepsilon,M}(v,0)-y_d}{\theta}\quad \text{according to (33)}$$

$$= -\lim_{\theta\to 0}\frac{y_{\varepsilon,M}(v+\theta w,0)-y_{\varepsilon,M}(v,0)}{\theta}$$

$$= -\frac{\partial y_{\varepsilon,M}^\gamma}{\partial v}(v,0)(w)\quad \text{in } Q,$$

• In $\Omega$, we have:

$$\left.\frac{\partial\zeta_{\varepsilon,M}^\gamma}{\partial v}(v)(w)\right|_{t=T}=\lim_{\theta\to 0}\frac{\zeta_{\varepsilon,M}(v+\theta w)|_{t=T}-\zeta_{\varepsilon,M}(v)|_{t=T}}{\theta}=0\quad \text{because } \zeta_{\varepsilon(v),M}|_{t=T}=0 \text{ in } \Omega.$$

• On $\Sigma$, we have:

$$\frac{\partial\zeta_{\varepsilon,M}^\gamma}{\partial v}(v)(w)=\lim_{\theta\to 0}\frac{\zeta_{\varepsilon,M}(v+\theta w)-\zeta_{\varepsilon,M}(v)}{\theta}=0\quad \text{because } \zeta_{\varepsilon,M}(v)=0 \text{ on } \Sigma.$$

□

#### 3.1.2. Characterization of adapted low-regret control.

**Proposition 3.6.** *Let $u_\varepsilon^\gamma\in L^2(Q)$ be the adapted low-regret control. Then, there exist $\{y_{\varepsilon,M}^\gamma,\zeta_{\varepsilon,M}^\gamma,\rho_{\varepsilon,M}^\gamma,p_{\varepsilon,M}^\gamma\}$ satisfies the following systems :*

$$\begin{cases}\dfrac{\partial y_{\varepsilon,M}^\gamma}{\partial t}-\Delta y_{\varepsilon,M}^\gamma-f_M(y_{\varepsilon,M}^\gamma)=u_\varepsilon^\gamma & \text{in } Q,\\ y_{\varepsilon,M}^\gamma(0)=y^0 & \text{in } \Omega,\\ y_{\varepsilon,M}^\gamma=0 & \text{on } \Sigma.\end{cases}\tag{47}$$

$$\begin{cases}-\dfrac{\partial \zeta_{\varepsilon,M}^\gamma}{\partial t}-\Delta \zeta_{\varepsilon,M}^\gamma-\zeta_{\varepsilon,M}^\gamma f_M'(y_{\varepsilon,M}^\gamma)=-(y_{\varepsilon,M}^\gamma-y_d) & \text{in } Q,\\ \zeta_{\varepsilon,M}^\gamma(T,u_\varepsilon^\gamma)=0 & \text{in } \Omega,\\ \zeta_{\varepsilon,M}^\gamma=0 & \text{on } \Sigma.\end{cases}\tag{48}$$

$$\begin{cases}\dfrac{\partial \rho_{\varepsilon,M}^\gamma}{\partial t}-\Delta \rho_{\varepsilon,M}^\gamma-\rho_{\varepsilon,M}^\gamma f_M'(y_{\varepsilon,M}^\gamma)=0 & \text{in } Q,\\ \rho_{\varepsilon,M}^\gamma(0)=0 & \text{in } \Omega,\\ \rho_{\varepsilon,M}^\gamma=-\dfrac{\varepsilon^2}{2\gamma}\dfrac{\partial S(u_\varepsilon^\gamma)}{\partial v} & \text{on } \Sigma.\end{cases}\tag{49}$$

$$\begin{cases}-\dfrac{\partial p_{\varepsilon,M}^\gamma}{\partial t}-\Delta p_{\varepsilon,M}^\gamma-p_{\varepsilon,M}^\gamma f_M'(y_{\varepsilon,M}^\gamma)=y_{\varepsilon,M}^\gamma-y_d-\rho_{\varepsilon,M}^\gamma & \text{in } Q,\\ p_{\varepsilon,M}^\gamma(T,u_\varepsilon^\gamma)=0 & \text{in } \Omega,\\ p_{\varepsilon,M}^\gamma=0 & \text{on } \Sigma.\end{cases}\tag{50}$$

*In addition, we have that for all $\tilde{u}\in L^2(Q)$,*

$$p_{\varepsilon,M}^\gamma+(\alpha+1)u_\varepsilon^\gamma=\tilde{u}\quad \text{in } L^2(Q).\tag{51}$$

*Proof.* We have already shown that $y_{\varepsilon,M}^\gamma:=y_{\varepsilon,M}(u_\varepsilon^\gamma,0)$ satisfies (47). From the problem (33), we have $\zeta_{\varepsilon,M}^\gamma:=\zeta_{\varepsilon,M}(u_\varepsilon^\gamma)$ solution of (48). As $S(v)=\zeta_{\varepsilon,M}(v)-\zeta_{\varepsilon,M}(0)$, we obtain $S(u_\varepsilon^\gamma)=\zeta_{\varepsilon,M}(u_\varepsilon^\gamma)-\zeta_{\varepsilon,M}(0)$.
By the Proposition 3.1, let us denote by $\rho_{\varepsilon,M}^\gamma=\rho_{\varepsilon,M}(u_\varepsilon^\gamma,0)$ the solution of the following problem:

$$\begin{cases}\dfrac{\partial \rho_{\varepsilon,M}^\gamma}{\partial t}-\Delta \rho_{\varepsilon,M}^\gamma-\rho_{\varepsilon,M}^\gamma f_M'(y_{\varepsilon,M}^\gamma)=0 & \text{in } Q,\\ \rho_{\varepsilon,M}^\gamma(0)=0 & \text{in } \Omega,\\ \rho_{\varepsilon,M}^\gamma=-\dfrac{\varepsilon^2}{2\gamma}\dfrac{\partial S(u_\varepsilon^\gamma)}{\partial v} & \text{on } \Sigma.\end{cases}$$

Writing the Euler-Lagrange condition on $\mathscr{J}^{\gamma}_{\varepsilon a}$, we get for all $w \in L^2(Q)$:

$$\lim_{\theta\to 0}\frac{\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon}+\theta w)-\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon})}{\theta}=\left\langle \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w),\, y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)-y_d\right\rangle_{L^2(Q)}+\langle \alpha u^{\gamma}_{\varepsilon}+u^{\gamma}_{\varepsilon}-\tilde{u},\, w\rangle_{L^2(Q)}+$$
$$\frac{\varepsilon^2}{2\gamma}\left\langle \frac{\partial}{\partial v}\left(\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon,M})(w)\right),\, \frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon})\right\rangle_{L^2(\Sigma)}=0. \tag{52}$$

Indeed, for all $\theta \in ]0;1[$ and for all $w \in L^2(Q)$, we have:

$$\frac{\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon}+\theta w)-\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon})}{\theta}=\frac{\theta}{2}\left\|\frac{y_{\varepsilon,M}(u^{\gamma}_{\varepsilon}+\theta w,0)-y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)}{\theta}\right\|^2_{L^2(Q)}+\frac{\theta}{2}(\alpha+1)\|w\|^2_{L^2(Q)}$$
$$+\left\langle \frac{y_{\varepsilon,M}(u^{\gamma}_{\varepsilon}+\theta w,0)-y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)}{\theta},\, y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)-y_d\right\rangle_{L^2(Q)}+\alpha\langle u^{\gamma}_{\varepsilon},w\rangle_{L^2(Q)}+\langle u^{\gamma}_{\varepsilon}-\tilde{u},\, w\rangle_{L^2(Q)}$$
$$+\frac{\varepsilon^2\theta}{4\gamma}\left\|\frac{\frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon}+\theta w)-\frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon})}{\theta}\right\|^2_{L^2(\Sigma)}+\frac{\varepsilon^2}{2\gamma}\left\langle \frac{\frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon}+\theta w)-\frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon})}{\theta},\, \frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon})\right\rangle_{L^2(\Sigma)}.$$

By taking the limit, we obtain:

$$\lim_{\theta\to 0}\frac{\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon}+\theta w)-\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon})}{\theta}=\left\langle \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w),\, y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)-y_d\right\rangle_{L^2(Q)}+\langle \alpha u^{\gamma}_{\varepsilon}+u^{\gamma}_{\varepsilon}-\tilde{u},\, w\rangle_{L^2(Q)}$$
$$+\frac{\varepsilon^2}{2\gamma}\left\langle \frac{\partial}{\partial v}\left(\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})\right)(w),\, \frac{\partial S}{\partial v}(u^{\gamma}_{\varepsilon})\right\rangle_{L^2(\Sigma)}.$$

So, we have:

$$\lim_{\theta\to 0}\frac{y_{\varepsilon,M}(u^{\gamma}_{\varepsilon}+\theta w,0)-y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)}{\theta}=\frac{\partial y^{\gamma}_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)$$

and

$$\lim_{\theta\to 0}\frac{S(u^{\gamma}_{\varepsilon}+\theta w)-S(u^{\gamma}_{\varepsilon})}{\theta}=\lim_{\theta\to 0}\frac{\zeta_{\varepsilon,M}(u^{\gamma}_{\varepsilon}+\theta w)-\zeta_{\varepsilon,M}(u^{\gamma}_{\varepsilon})}{\theta}=\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w).$$

Furthermore, the terms $\left\|\frac{\partial y^{\gamma}_{\varepsilon,M}}{\partial v}\right\|^2_{L^2(Q)}$ and $\left\|\frac{\partial \zeta^{\gamma}_{\varepsilon,M}}{\partial v}\right\|^2_{L^2(\Sigma)}$ are bounded and tend to zero as $\theta$ tends to zero.

Thus, multiplying the equation of state for (49) by $\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)$, we have:

$$\left\langle \frac{\partial \rho^{\gamma}_{\varepsilon,M}}{\partial t}-\Delta\rho^{\gamma}_{\varepsilon,M}-\rho^{\gamma}_{\varepsilon,M}f'_M(y^{\gamma}_{\varepsilon,M}),\, \frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right\rangle_{L^2(Q)}=0$$

Then,

$$\left\langle \frac{\partial \rho^{\gamma}_{\varepsilon,M}}{\partial t},\, \frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right\rangle+\left\langle -\Delta\rho^{\gamma}_{\varepsilon,M},\, \frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right\rangle+\left\langle -\rho^{\gamma}_{\varepsilon,M}f'_M(y^{\gamma}_{\varepsilon,M}),\, \frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right\rangle=0.$$

And integrating by parts, we obtain that:

$$\left\langle \frac{\partial \rho^{\gamma}_{\varepsilon,M}}{\partial t},\, \frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right\rangle_{L^2(Q)}=\left\langle -\frac{\partial}{\partial t}\left(\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right),\, \rho^{\gamma}_{\varepsilon,M}\right\rangle_{L^2(Q)}.$$

By using Green's formula, we obtain:

$$\left\langle -\Delta\rho^{\gamma}_{\varepsilon,M},\, \frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right\rangle_{L^2(Q)}=\left\langle -\Delta\left(\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right),\, \rho^{\gamma}_{\varepsilon,M}\right\rangle_{L^2(Q)}-\frac{\varepsilon^2}{2\gamma}\left\langle \frac{\partial}{\partial v}\left(\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right),\, \frac{\partial S(u^{\gamma}_{\varepsilon})}{\partial v}\right\rangle_{L^2(\Sigma)}.$$

Therefore,

$$\frac{\varepsilon^2}{2\gamma}\left\langle \frac{\partial}{\partial v}\left(\frac{\partial \zeta_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon})(w)\right),\, \frac{\partial S(u^{\gamma}_{\varepsilon})}{\partial v}\right\rangle_{L^2(\Sigma)}=\left\langle \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w),\, -\rho^{\gamma}_{\varepsilon,M}\right\rangle_{L^2(Q)}.$$

The Euler-Lagrange condition becomes: $\forall w \in L^2(Q)$,

$$\left\langle \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0),\, y_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)(w)-y_d-\rho^{\gamma}_{\varepsilon,M}\right\rangle_{L^2(Q)}+\langle \alpha u^{\gamma}_{\varepsilon}+u^{\gamma}_{\varepsilon}-\tilde{u},\, w\rangle_{L^2(Q)}=0. \tag{53}$$

Then we define by $p^{\gamma}_{\varepsilon,M} := p_{\varepsilon,M}(u^{\gamma}_{\varepsilon},0)$ the solution to the adjoint problem of $\rho^{\gamma}_{\varepsilon,M}$ by:

$$\begin{cases} -\dfrac{\partial p^{\gamma}_{\varepsilon,M}}{\partial t} - \Delta p^{\gamma}_{\varepsilon,M} - p^{\gamma}_{\varepsilon,M} f'_M(y^{\gamma}_{\varepsilon,M}) = y^{\gamma}_{\varepsilon,M} - y_d - \rho^{\gamma}_{\varepsilon,M} & \text{in } Q, \\ p^{\gamma}_{\varepsilon,M}(T,u^{\gamma}_{\varepsilon}) = 0 & \text{in } \Omega, \\ p^{\gamma}_{\varepsilon,M} = 0 & \text{on } \Sigma. \end{cases}$$

Thus, multiplying the equation of state in (50) by $\dfrac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)$, we have:

$$\left\langle -\frac{\partial p^{\gamma}_{\varepsilon,M}}{\partial t} - \Delta p^{\gamma}_{\varepsilon,M} - p^{\gamma}_{\varepsilon,M} f'_M(y^{\gamma}_{\varepsilon,M}), \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) \right\rangle_{L^2(Q)} = \left\langle y^{\gamma}_{\varepsilon,M} - y_d - \rho^{\gamma}_{\varepsilon,M}, \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) \right\rangle_{L^2(Q)}.$$

And integrating by parts, we obtain that:

$$\left\langle -\frac{\partial p^{\gamma}_{\varepsilon,M}}{\partial t}, \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial t}\left( \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)\right), p^{\gamma}_{\varepsilon,M} \right\rangle_{L^2(Q)}.$$

By applying Green's formula, we obtain:

$$\left\langle -\Delta p^{\gamma}_{\varepsilon,M}, \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) \right\rangle_{L^2(Q)} = \left\langle -\Delta\left( \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)\right), p^{\gamma}_{\varepsilon,M} \right\rangle_{L^2(Q)}.$$

Consequently,

$$\begin{aligned} \left\langle y^{\gamma}_{\varepsilon,M} - y_d - \rho^{\gamma}_{\varepsilon,M}, \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) \right\rangle_{L^2(Q)} &= \left\langle \frac{\partial}{\partial t}\left( \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)\right), p^{\gamma}_{\varepsilon,M} \right\rangle_{L^2(Q)} + \left\langle -\Delta\left( \frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)\right), p^{\gamma}_{\varepsilon,M} \right\rangle_{L^2(Q)} \\ &+ \left\langle -\frac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) f'_M(y^{\gamma}_{\varepsilon,M}), p^{\gamma}_{\varepsilon,M} \right\rangle_{L^2(Q)} = \langle p^{\gamma}_{\varepsilon,M}, w\rangle_{L^2(Q)}. \end{aligned}$$

Indeed, $\dfrac{\partial}{\partial t}\left( \dfrac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)\right) - \Delta\left( \dfrac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w)\right) - \dfrac{\partial y_{\varepsilon,M}}{\partial v}(u^{\gamma}_{\varepsilon},0)(w) f'_M(y^{\gamma}_{\varepsilon,M}) = w$ in $Q$.

The condition (53) becomes: $\langle p^{\gamma}_{\varepsilon,M} + \alpha u^{\gamma}_{\varepsilon} + u^{\gamma}_{\varepsilon} - \tilde{u}, w\rangle_{L^2(Q)} = 0,\ \forall w \in L^2(Q)$.

From which we can deduce (51). □

### 3.2. Singular optimality system

In this section, we will characterize the low-regret control and the no-regret control of the problem (1).

#### 3.2.1. Characterization of low-regret control

The lemmas below set out the initial assumptions needed for our analysis.

**Lemma 3.4.** *For all $\varepsilon > 0$ and for all fixed $(\gamma, M) > 0$, there exist various constants $C > 0$ such that:*

$$\|u^{\gamma}_{\varepsilon}\|_{L^2(Q)} \le C, \qquad \left\|y^{\gamma}_{\varepsilon,M}\right\|_{L^2(Q)} \le C, \qquad \frac{1}{2}\frac{\varepsilon}{\sqrt{\gamma}}\left\|\frac{\partial S(u^{\gamma}_{\varepsilon})}{\partial \nu}\right\|_{L^2(\Sigma)} \le C, \tag{54}$$

$$\begin{aligned} &\left\|\frac{\partial \zeta^{\gamma}_{\varepsilon,M}}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \left\|\zeta^{\gamma}_{\varepsilon,M}\right\|_{L^2(0,T;H^1(\Omega))} \le C, \\ &\left\|\frac{\partial \rho^{\gamma}_{\varepsilon,M}}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \left\|\rho^{\gamma}_{\varepsilon,M}\right\|_{L^2(0,T;H^1(\Omega))} \le C, \\ &\left\|\frac{\partial p^{\gamma}_{\varepsilon,M}}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \left\|p^{\gamma}_{\varepsilon,M}\right\|_{L^2(0,T;H^1(\Omega))} \le C. \end{aligned} \tag{55}$$

*Proof.* As $u^{\gamma}_{\varepsilon}$ is a solution to (43), therefore $\mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon}) \le \mathscr{J}^{\gamma}_{\varepsilon a}(v),\ \forall v \in L^2(Q)$.
In particular, for $v = 0$, we obtain:

$$\begin{aligned} \mathscr{J}^{\gamma}_{\varepsilon a}(u^{\gamma}_{\varepsilon}) &= \frac{1}{2}\|y^{\gamma}_{\varepsilon,M} - y_d\|^2_{L^2(Q)} + \frac{\alpha}{2}\|u^{\gamma}_{\varepsilon}\|^2_{L^2(Q)} + \frac{1}{2}\|u^{\gamma}_{\varepsilon} - \tilde{u}\|^2_{L^2(Q)} + \frac{\varepsilon^2}{4\gamma}\left\|\frac{\partial S(u^{\gamma}_{\varepsilon})}{\partial \nu}\right\|^2_{L^2(\Sigma)} \\ &\le \frac{1}{2}\left(\|y_d\|^2_{L^2(Q)} + \|\tilde{u}\|^2_{L^2(Q)}\right) = C. \end{aligned}$$

So, we get (54).
The estimates in (55) can be easily derived by applying the method used in the proof of Lemma 2.1. □

**Lemma 3.5.** *For all fixed $\gamma > 0$, there exist various constants $C > 0$ independent of $M$ such that:*

$$\left\|\frac{\partial y_M^\gamma}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \|y_M^\gamma\|_{L^2(0,T;H_0^1(\Omega))} \leq C, \quad \left\|\frac{\partial \zeta_M^\gamma}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \|\zeta_M^\gamma\|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{56}$$

$$\left\|\frac{\partial \rho_M^\gamma}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \|\rho_M^\gamma\|_{L^2(0,T;H_0^1(\Omega))} \leq C, \quad \left\|\frac{\partial p_M^\gamma}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \|p_M^\gamma\|_{L^2(0,T;H_0^1(\Omega))} \leq C. \tag{57}$$

*Proof.* The following inequalities follow directly from the reasoning set out in subsection 2.2. □

By applying Proposition 3.6, and by passing to the limits when $\varepsilon \to 0$ and $M \to \infty$, we obtain the singular optimality system for the low-regret control of the original problem.

**Theorem 3.2.** *Let $u^\gamma \in L^2(Q)$ be the low-regret control associated to the state $y^\gamma := y(u^\gamma, 0)$. Then, there exist $\rho^\gamma$, $p^\gamma$ such that $\{y^\gamma, \zeta^\gamma, \rho^\gamma, p^\gamma\}$ satisfies the following.*

$$\begin{cases} \dfrac{\partial y^\gamma}{\partial t} - \Delta y^\gamma - e^{y^\gamma} = u^\gamma & \text{in } Q, \\ y^\gamma(0) = y^0 & \text{in } \Omega, \\ y^\gamma = 0 & \text{on } \Sigma. \end{cases} \tag{58}$$

$$\begin{cases} -\dfrac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma - \zeta^\gamma e^{y^\gamma} = -(y^\gamma - y_d) & \text{in } Q, \\ \zeta^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ \zeta^\gamma = 0 & \text{on } \Sigma. \end{cases} \tag{59}$$

$$\begin{cases} \dfrac{\partial \rho^\gamma}{\partial t} - \Delta \rho^\gamma - \rho^\gamma e^{y^\gamma} = 0 & \text{in } Q, \\ \rho^\gamma(0) = 0 & \text{in } \Omega, \\ \rho^\gamma = 0 & \text{on } \Sigma. \end{cases} \tag{60}$$

$$\begin{cases} -\dfrac{\partial p^\gamma}{\partial t} - \Delta p^\gamma - p^\gamma e^{y^\gamma} = y^\gamma - y_d - \rho^\gamma & \text{in } Q, \\ p^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ p^\gamma = 0 & \text{on } \Sigma. \end{cases} \tag{61}$$

*In addition, we have that for all $\tilde{u} \in L^2(Q)$,*

$$p^\gamma + (\alpha+1)u^\gamma = \tilde{u} \quad \text{in } L^2(Q). \tag{62}$$

*Proof.* By using the Proposition 3.6, the fact that $f_M$ is Lipschitz and bounded by $e^M$, and the Lemma 3.4, we can extract subsequences of the sequences $(u_\varepsilon^\gamma)_\varepsilon$, $(y_{\varepsilon,M}^\gamma)_\varepsilon$, $\left(\dfrac{\varepsilon}{2\sqrt{\gamma}}\dfrac{\partial S(u_\varepsilon^\gamma)}{\partial \nu}\right)_\varepsilon$, $(\rho_{\varepsilon,M}^\gamma)_\varepsilon$, $(p_{\varepsilon,M}^\gamma)_\varepsilon$ et $(\zeta_{\varepsilon,M}^\gamma)_\varepsilon$, which we shall denote in the same way, such that:

$$\begin{array}{rcll} u_\varepsilon^\gamma & \rightharpoonup & u^\gamma & \text{weakly in} \quad L^2(Q), \\ y_{\varepsilon,M}^\gamma & \rightharpoonup & y_M^\gamma & \text{weakly in} \quad L^2(Q), \\ \dfrac{\varepsilon}{2\sqrt{\gamma}}\dfrac{\partial S(u_\varepsilon^\gamma)}{\partial \nu} & \rightharpoonup & \lambda_M^\gamma(0) & \text{weakly in} \quad L^2(\Sigma), \\ \rho_{\varepsilon,M}^\gamma & \rightharpoonup & \rho_M^\gamma & \text{weakly in} \quad L^2(0,T;H^1(\Omega)), \\ p_{\varepsilon,M}^\gamma & \rightharpoonup & p_M^\gamma & \text{weakly in} \quad L^2(0,T;H^1(\Omega)), \\ \zeta_{\varepsilon,M}^\gamma & \rightharpoonup & \zeta_M^\gamma & \text{weakly in} \quad L^2(0,T;H^1(\Omega)). \end{array} \tag{63}$$

As $\dfrac{\varepsilon}{2\sqrt{\gamma}}\dfrac{\partial S(u_\varepsilon^\gamma)}{\partial \nu} \rightharpoonup \lambda_M^\gamma(0)$ weakly in $L^2(\Sigma)$ then $\dfrac{\varepsilon^2}{4\gamma}\dfrac{\partial S(u_\varepsilon^\gamma)}{\partial \nu} \longrightarrow 0$ when $\varepsilon \longrightarrow 0$

For any $\gamma > 0$ and a fixed $M > 0$, the adjoint state $p_{\varepsilon,M}^\gamma$ is also bounded in $\varepsilon$, and the result (62).

By the Lemma 3.5, we can extract from the sequences $(y_M^\gamma)_M$, $(\zeta_M^\gamma)_M$, $(\rho_M^\gamma)_M$ and $(p_M^\gamma)_M$ subsequences denoted in the same way such that:

$$\begin{array}{rcll} y_M^\gamma & \rightharpoonup & y^\gamma & \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)), \\ \rho_M^\gamma & \rightharpoonup & \rho^\gamma & \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)), \\ \zeta_M^\gamma & \rightharpoonup & \zeta^\gamma & \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)), \\ p_M^\gamma & \rightharpoonup & p^\gamma & \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)), \end{array}$$

and by the property of $f_M$, we have:

$$f_M(y_M^\gamma) \longrightarrow e^{y^\gamma} \text{ in } L^2(Q).$$

□

### 3.2.2. Characterization of no-regret control

The optimality system for the no-regret control of problem (1)-(38) is a consequence of the limit in Theorem 3.2 as $\gamma$ tends to zero.

**Theorem 3.3.** *Let $\tilde{u} \in L^2(Q)$ be the no-regret control associated to the state $y := y(\tilde{u}, 0)$. Then, there exist $\rho$, $p$ such that $\{y, \zeta, \rho, p\}$ satisfies the following.*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y - e^y = \tilde{u} & \text{in } Q, \\ y(0) = y^0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma. \end{cases} \tag{64}$$

$$\begin{cases} -\dfrac{\partial \zeta}{\partial t} - \Delta \zeta - \zeta\, e^y = -(y - y_d) & \text{in } Q, \\ \zeta(T, \tilde{u}) = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \Sigma. \end{cases} \tag{65}$$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} - \Delta \rho - \rho\, e^y = 0 & \text{in } Q, \\ \rho(0) = 0 & \text{in } \Omega, \\ \rho = 0 & \text{on } \Sigma. \end{cases} \tag{66}$$

$$\begin{cases} -\dfrac{\partial p}{\partial t} - \Delta p - p\, e^y = y - y_d - \rho & \text{in } Q, \\ p(T, \tilde{u}) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Sigma. \end{cases} \tag{67}$$

*In addition, we have that for all $\tilde{u} \in L^2(Q)$,*

$$p + \alpha \tilde{u} = 0 \quad \text{in } L^2(Q). \tag{68}$$

*Proof.* By the Theorem 3.2 and by taking the limit as $\gamma$ tends to zero, we have:

$$\begin{cases} \zeta^\gamma \rightharpoonup \zeta = 0, \\ y^\gamma \rightharpoonup y = 0, \\ \rho^\gamma \rightharpoonup \rho = 0, \\ p^\gamma \rightharpoonup p = 0. \end{cases} \quad \text{on } \Sigma \qquad \begin{cases} \zeta^\gamma \rightharpoonup \zeta = 0, \\ y^\gamma(0) \rightharpoonup y(0) = y^0, \\ \rho^\gamma \rightharpoonup \rho = 0, \\ p^\gamma(T, u^\gamma) \rightharpoonup p(T, \tilde{u}) = 0. \end{cases} \quad \text{in } \Omega$$

From equation (62), we deduce that:

$$u^\gamma \rightharpoonup \tilde{u} \quad \text{weakly in} \quad L^2(Q),$$

Consequently, there exists a no-regret control $\tilde{u}$, characterized by $\{y, \zeta, \rho, p\}$, which is a solution to problem (1)-(38). □

## 4. Conclusion

In this paper, we studied the control of a semilinear parabolic problem involving a blow-up nonlinearity and missing boundary data. The introduced regularisation allowed us to reformulate the original problem (1) as a family of well-posed problems (3), the convergence of which to the solution of the original problem as $\varepsilon \to 0$ and $M \to \infty$ we established. By combining low-regret and no-regret control methods and studying the convergence of low-regret controls as the perturbation parameter $\gamma$ tends to zero, we characterised the no-regret control of problem (1) by means of a singular optimality system. The present work opens up broader research perspectives. The tools developed here have natural applications in problems governed by exponentially reactive dynamics, such as environmental modelling, heat transfer in combustion processes and the control of exothermic industrial reactors. However, it does have certain limitations that should be made explicit. The analysis was carried out without numerical validation, which constitutes the most immediate limitation of this work. Indeed, the theoretical results obtained require the construction of robust discretisation schemes. The question of extending the no-regret framework to other classes of parabolic equations with blow-up nonlinearities beyond the exponential case treated here remains entirely open. As a next step, we plan to develop sufficiently robust discretisation schemes to demonstrate the convergence of the regularised system (3) to the original model (1) and to approximate the singular optimality system that characterises the no-regret control. Overall, this work marks a significant initial step towards a deeper understanding of no-regret control for ill-posed nonlinear systems and paves the way for promising theoretical, numerical and applied developments.

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