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Research paper

Abel Hill Inverse Problem for Two Non-Monotonic Cases

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Abstract

Abel's Hill as an inverse problem has been solved for piecewise monotonic potentials. This brief paper considers two cases which extend this solution. These cases correspond to the extreme possibilities for the potential's slope, zero and infinite. The effect on the measured return times are discussed and it is found that it is straightforward to identify potentials with these characteristics.

Keywords: *Abel's Hill; Inverse Problems; Monotonic Functions.*

1. Introduction

Abel's Hill is a well-studied inverse problem. [1-2] The problem is to determine the shape of a hill if one knew the time it took for an object slid up the hill to return. Different initial energies correspond to different return times. Assuming energies can be varied from zero to infinity, the theoretical challenge is to find what shape hills can be determined. Abel and others have shown this is only possible for piecewise monotonic functions. [1], [3] This is not strictly true.

This paper gives two examples that show that solutions exist for monotonic functions that include finite regions that have zero and infinite slopes. Characteristics can be identified in the time data that allow for the identification of these extreme slope cases. This broadens the known solution since these two cases are not strictly monotonic and cannot be identified by the standard method of analysis.

2. Solving the Abel Hill Problem

Consider a particle sliding up a frictionless hill. Varying the initial energy of the particle generates a return time function, T(E), that only depends on E. The challenge is, for a full range of energies, to find what shape hills can be determined. Keller [3] gives a thorough solution which will not be detailed here.

It has been proven that Abel's Hill is solvable for monotonic potentials. The solution being:

$$s = \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-\frac{1}{2}}\, T(E)\, dE \tag{1}$$

Where m is the particle's mass and V is the potential energy of the hill. When V is allowed to vary from zero to infinity there is a complete solution to the problem. Determining s for all values of V, uniquely identifies the hill shape. As detailed by Razavy [2], the above expression for s is found by using the energy integral method of classical mechanics. This consists of starting with the energy as a function of time and solving for the time as a function of energy. Then by identifying the time for the object to reach the turning point and considering the variation of V over all values, equation 1 is justified.

Keller [3] further shows that a piecewise monotonic hill has a solution with a very similar form. Keller's solution consists of a sum of expressions very similar to equation (1). (Keller's solution uses the same approach as that used to derive equation (1)) This is significant for this paper because considered below are cases where the potential is a piecewise monotonic hill outside of the sections that are a plateau or cliff. So it is only necessary to consider the solutions to these novel sections, and ignore the piecewise monotonic hill.

3. Extreme Slope Case

Abel and Keller's solutions are correct but do not include two additional possibilities. These are the cases where the slope of the potential is zero or infinite over a finite range. (Imagining a hill, these would be cases corresponding to a plateau or cliff potential.) Unsurprisingly, these two cases give very different changes in return times so they will be considered separately.

First, consider the case where the potential energy has a zero slope for a finite region and is monotonically increasing elsewhere. This would look something like:

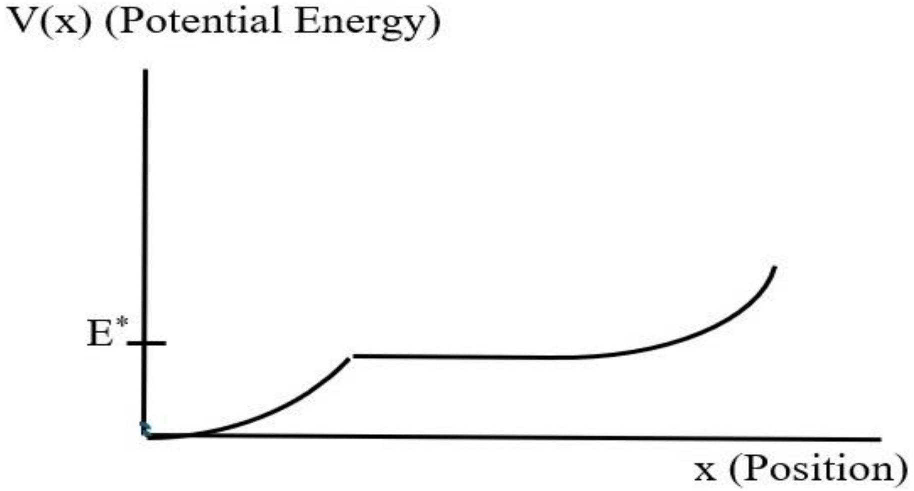

Fig. 1: A Potential Energy with A Constant Value at the Energy E^* And Monotonically Increasing Elsewhere.

This plateau at the energy E* leads to a discontinuity in the function T(E) as shown in Figure 2:

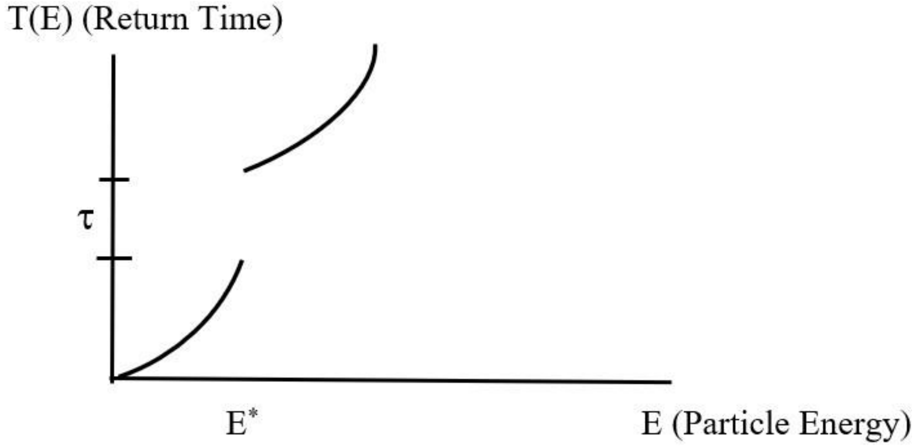

Fig. 2: The Return Time T(E) As A Function of E with A Time Discontinuity T Due to Potential Energy Given in Figure 1.

The time parameter t corresponds to the time that the particle is moving along the plateau. Along the plateau the object will be moving at a constant speed, v. The energy, E^*, of the particle on the plateau is given by ½ mv^2. Since the object moves the length L in the time t, the speed is L/ t. Therefore, the length of the plateau, L, can easily be written in terms of t and is given by:

$$L = \sqrt{\frac{2E^*}{m}}\,\tau \tag{2}$$

The second case for consideration here, is when the slope of the potential energy curve is infinite for a finite range of energies.
An example of this is given in Figure 3:

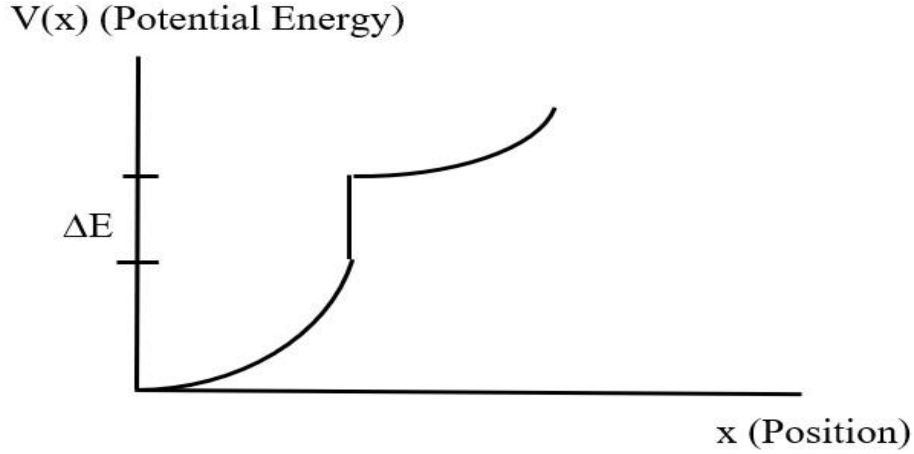

Fig. 3: A Potential Energy with A Finite Region of Infinite Slope.

Assuming the case of a gravitational potential energy mgh, the added time is just the return time for a projectile fired directly upward with an initial energy of **ε**. This initial energy is related to the initial speed, v, by ½ mv^2. The return time for an object acting only under the acceleration due to gravity, g, is well known to be 2v/g. [4] It is straightforward to show the time added with increased energy along the

ΔE section is:

$$\mathbf{t} = \frac{2}{g}\sqrt{\frac{2\varepsilon}{m}} \tag{3}$$

Where is $\boldsymbol{\varepsilon}$ is the additional energy used to enter the ΔE region.
Note that this results in a section of the T(E) that goes as the square root of E. So, a section of the hill with an infinite slope would result in a T(E) section identified by this square root dependence, which is not the relationship that would be given by equation (1).

4. Conclusion

The standard method used for solving Abel's Hill is to use an Energy Integral. (Equation 1) However, this method would not work for the two cases discussed in this paper. The plateau (slope of zero) cannot be analysed by Abel's method because of the discontinuity in T(E) and the cliff (infinite slope) cannot be analysed because the infinite slope section is not accounted for in the integral.
Abel's Hill problem has been solved for piecewise monotonic functions. It is also argued that these are the only solvable cases. As shown in this paper, this is not strictly correct as cases of a plateau and cliff, while not analysable by equation (1) can be identified from the return time data.
Real world applications of Abel's Hill exist. In particular, studies that probe the structure of the Ionosphere. In this case the plateau would be of interest as it would identify regions of constant density, and the cliff would be of interest as it could identify abrupt changes in density. Though the former is not possible under a gravitational potential, the latter is a possibility.

References

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