

Right Circulant Matrices With Geometric Progression

Aldous Cesar F. Bueno

Department of Mathematics and Physics, Central Luzon State University
 Science City of Muñoz 3120, Nueva Ecija, Philippines
 E-mail: aldouz_cezar@yahoo.com

Abstract

In this paper, the right circulant matrix $RCIRC_n(\vec{g}) \in M_{n \times n}(\mathbb{R})$ with circulant vector $\vec{g} = (a, ar, ar^2, \dots, ar^{n-1})$, where $a \neq 0$ and $r \neq 0, 1$, was investigated and its inverse $RCIRC_n^{-1}(g)$ was obtained. The eigenvalues, determinant, Euclidean norm, and spectral norm of both $RCIRC_n(\vec{g})$ and $RCIRC_n^{-1}(\vec{g})$ were determined. Some examples were provided to illustrate the obtained results.

Keywords: Determinant, Eigenvalue, Euclidean norm, Right circulant matrix with geometric progression, Spectral norm

1 Introduction

A matrix $C \in M_{n \times n}(\mathbb{R})$ is said to be a right circulant matrix if it is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

The matrix C has the following structure:

1. Each row is a right cyclic shift of the row above it. Thus, C is determined by the first row

$$(c_0, c_1, c_2, \dots, c_{n-1})$$

2. $c_k = c_{i,j}$ whenever $i - j = k \pmod{n}$

Definition 1.1

1. Let $\vec{c} = (c_0, c_1, c_2, \dots, c_{n-1})$ then the right circulant matrix $C \in M_{n \times n}(\mathbb{R})$ is denoted by $\mathbf{RCIRC}_n(\vec{c})$
2. The vector $\vec{c} = (c_0, c_1, c_2, \dots, c_{n-1})$ is called the **circulant vector**
3. $\mathbf{RCIRC}_n(\mathbb{R}) = \{\mathbf{RCIRC}_n(\vec{c}) | \vec{c} \in \mathbb{R}^n\}$

Properties of $\mathbf{RCIRC}_n(\vec{c})$

1. The eigenvalues of $\mathbf{RCIRC}_n(\vec{c})$ are just the **Discrete Fourier Transform** of the circulant vector $\vec{c}$. That is

$$\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk}$$

where $\omega = e^{2\pi i/n}$ and $m=0, 1, \dots, n-1$

2. The eigenvectors of $\mathbf{RCIRC}_n(\vec{c})$ are the columns of the Fourier matrix F. That is

$$v_m = \frac{1}{\sqrt{n}} (1, \omega^m, \omega^{2m}, \dots, \omega^{(n-1)m})$$

3. The Fourier matrix F is a simultaneous, unitary, diagonalizing matrix for $\mathbf{RCIRC}_n(\vec{c})$. That is, for any $\mathbf{RCIRC}_n(\vec{c})$

$$\mathbf{RCIRC}_n(\vec{c}) = F D F^{-1},$$

where $D = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ and $F F^* = I$.

4. The determinant of $\mathbf{RCIRC}_n(\vec{c})$ denoted by $|\mathbf{RCIRC}_n(\vec{c})|$ is given by

$$|\mathbf{RCIRC}_n(\vec{c})| = \prod_{m=0}^{n-1} \lambda_m = \prod_{m=0}^{n-1} \left(\sum_{k=0}^{n-1} c_k \omega^{-mk} \right)$$

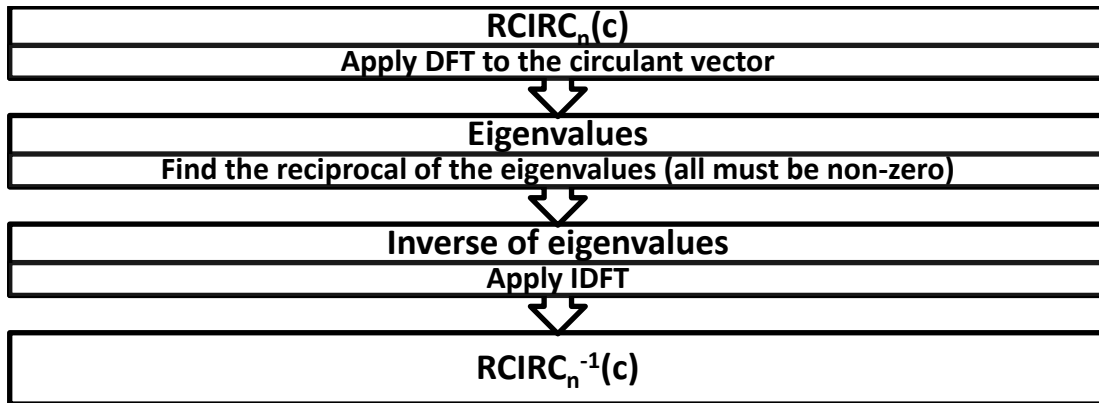
where $\omega = e^{2\pi i/n}$ and $m=0, 1, \dots, n-1$

Inversion of $RCIRC_n(\vec{c})$

From the circulant vector we can obtain the eigenvalues of $RCIRC_n(\vec{c})$ through Discrete Fourier Transform (DFT). Furthermore, through the Inverse Discrete Fourier Transform (IDFT) of the eigenvalues of $RCIRC_n(\vec{c})$ the circulant vector can be obtained and hence $RCIRC_n(\vec{c})$ itself.

Note that if $\lambda \neq 0$ is an eigenvalue of an invertible matrix A , then λ^{-1} is also an eigenvalue of the matrix A^{-1} . With the help of this concept we can derive the matrix $RCIRC_n^{-1}(\vec{c})$, the inverse of $RCIRC_n(\vec{c})$.

The flowchart of finding $RCIRC_n^{-1}(c)$ is as given below.



Given $RCIRC_n(\vec{c})$ with eigenvalues $\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk} \neq 0$ where $m=0,1, \dots, n-1$, then $RCIRC_n^{-1}(\vec{c})$ is given by

$$RCIRC_n^{-1}(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{pmatrix}$$

where $C_k = \frac{1}{n} \sum_{m=0}^{n-1} \lambda_m^{-1} \omega^{mk}$ and $(C_0, C_1, \dots, C_{n-1})$ is the circulant vector.

As an example, consider the matrix

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 8 & 1 & 2 & 4 \\ 4 & 8 & 1 & 2 \\ 2 & 4 & 8 & 1 \end{pmatrix}$$

Its eigenvalues are $\lambda_0 = 15, \lambda_1 = -3 + 6i, \lambda_2 = -3 - 6i, \lambda_3 = -5$ with inverses $\frac{1}{15}, \frac{-1}{15} \pm \frac{2}{15}i, \frac{-1}{2}$ respectively. Performing the IDFT to each we will obtain the following: $C_0 = -\frac{1}{15}, C_1 = \frac{2}{15}, C_2 = C_3 = 0$.

Thus the inverse is given by

$$\begin{pmatrix} -\frac{1}{15} & \frac{2}{15} & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{2}{15} & 0 \\ 0 & 0 & -\frac{1}{15} & \frac{2}{15} \\ \frac{2}{15} & 0 & 0 & -\frac{1}{15} \end{pmatrix}.$$

Furthermore

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 8 & 1 & 2 & 4 \\ 4 & 8 & 1 & 2 \\ 2 & 4 & 8 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{15} & \frac{2}{15} & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{2}{15} & 0 \\ 0 & 0 & -\frac{1}{15} & \frac{2}{15} \\ \frac{2}{15} & 0 & 0 & -\frac{1}{15} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2 The Right Circulant Matrix $RCIRC_n(\vec{g})$

Let $\vec{g} = (a \ ar \ \dots \ ar^{n-1})$ be the circulant vector of a right circulant matrix where $a \neq 0, r \neq 0, 1$. Then the right circulant matrix with geometric progression denoted by $RCIRC_n(\vec{g})$ is the matrix of the form

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & \dots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \dots & ar^{n-3} & ar^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \dots & a & ar \\ ar & ar^2 & ar^3 & \dots & ar^{n-1} & a \end{pmatrix}$$

Now, for the rest of the paper the following notations will be used:

$\vec{g} = (a \ ar \ \dots \ ar^{n-1})$: circulant vector with geometric sequence whose first term is $a \neq 0$ and whose common ratio is $r \neq 0, 1$

$RCIRC_n(\vec{g})$: right circulant matrix with geometric sequence with dimension $n \times n$

$|RCIRC_n(\vec{g})|$: determinant of $RCIRC_n(\vec{g})$

$\|RCIRC_n(\vec{g})\|_E$: Euclidean norm of $RCIRC_n(\vec{g})$

$\|RCIRC_n(\vec{g})\|_2$: Spectral norm of $RCIRC_n(\vec{g})$
 $RCIRC_n^{-1}(\vec{g})$: inverse of $RCIRC_n(\vec{g})$

3 Main Results

Theorem 3.1

$$|RCIRC_n(\vec{g})| = a^n(1 - r^n)^{n-1}$$

Proof:

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & \dots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \dots & ar^{n-3} & ar^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \dots & a & ar \\ ar & ar^2 & ar^3 & \dots & ar^{n-1} & a \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & r & r^2 & \dots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \dots & r^{n-3} & r^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \dots & 1 & r \\ r & r^2 & r^3 & \dots & r^{n-1} & 1 \end{pmatrix}$$

$$\text{Now let } K = \begin{pmatrix} 1 & r & r^2 & \dots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \dots & r^{n-3} & r^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \dots & 1 & r \\ r & r^2 & r^3 & \dots & r^{n-1} & 1 \end{pmatrix}$$

Note that $|cA| = c^n|A|$ so $|RCIRC_n(\vec{g})| = a^n|K|$

$$K = \begin{pmatrix} 1 & r & r^2 & \dots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \dots & r^{n-3} & r^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^2 & r^3 & r^4 & \dots & 1 & r \\ r & r^2 & r^3 & \dots & r^{n-1} & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & r & r^2 & \dots & r^{n-2} & r^{n-1} \\ 0 & -(r^n - 1) & -r(r^n - 1) & \dots & -r^{n-3}(r^n - 1) & -r^{n-2}(r^n - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(r^n - 1) & -r \\ 0 & 0 & 0 & \dots & 0 & -(r^n - 1) \end{pmatrix}$$

by applying the row operation $-r^{n-k}R_1 + R_{k+1} \rightarrow R_{k+1}$ where $k=1,2,3,\dots,n-1$
 From the equivalent lower diagonal matrix we get $|K| = (1 - r^n)^{n-1}$.
 Thus $|RCIRC_n(\vec{g})| = a^n(1 - r^n)^{n-1}$

Theorem 3.2

The eigenvalues of $RCIRC_n(\vec{g})$ are $\lambda_0 = S_n$ and $\lambda_m = \frac{a(r^n - 1)}{re^{-2\pi im/n} - 1}$ where $m=1,2,\dots,n-1$

Proof:

Note that $\lambda_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}$

For $m=0$, we have

$$\lambda_0 = \sum_{k=0}^{n-1} c_k = \sum_{k=0}^{n-1} ar^k = \frac{a(r^n - 1)}{r - 1} = S_n$$

For $m \neq 0$, we have

$$\begin{aligned} \lambda_m &= \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n} \\ &= \sum_{k=0}^{n-1} ar^k e^{-\frac{2\pi imk}{n}} = a \sum_{k=0}^{n-1} r^k e^{-2\pi imk/n} = \frac{a(r^n e^{-2\pi im} - 1)}{re^{-2\pi im/n} - 1} \\ &= \frac{a(r^n - 1)}{re^{-2\pi im/n} - 1} \end{aligned}$$

Theorem 3.3

$$\|RCIRC_n(\vec{g})\|_E = |a| \sqrt{\frac{n(1 - r^{2n})}{1 - r^2}}$$

Proof:

$$\begin{aligned} \|RCIRC_n(\vec{g})\|_E &= \sqrt{\sum_{i,j=0}^{n-1} a_{ij}^2} = \sqrt{\sum_{k=0}^{n-1} n(ar^k)^2} = \sqrt{a^2 n \sum_{k=0}^{n-1} r^{2k}} \\ &= |a| \sqrt{\frac{n(1 - r^{2n})}{1 - r^2}} \end{aligned}$$

Theorem 3.4

$$\|RCIRC_n(\vec{g})\|_2 = \max \left\{ |S_n|, \frac{|a(r^n - 1)|}{\sqrt{r^2 - 2rcos \frac{2\pi m}{n} + 1}} \right\}$$

Proof:

For $m=0$, $|\lambda_0| = |S_n|$

For $m \neq 0$, we have

$$\begin{aligned} |\lambda_m| &= \left| \frac{a(r^n - 1)}{re^{\frac{2\pi im}{n}} - 1} \right| = \frac{|a(r^n - 1)|}{\left| r \left(\cos \frac{2\pi im}{n} + i \sin \frac{2\pi im}{n} \right) - 1 \right|} \\ &= \frac{|a(r^n - 1)|}{\sqrt{\left(r \cos \frac{2\pi im}{n} - 1 \right)^2 + r^2 \sin^2 \frac{2\pi im}{n}}} \\ &= \frac{|a(r^n - 1)|}{\sqrt{r^2 - 2r \cos \frac{2\pi m}{n} + 1}} \end{aligned}$$

Corollary 3.5

$$|\mathbf{RCIRC}_n^{-1}(\vec{g})| = \frac{1}{a^n(1 - r^n)^{n-1}}$$

Corollary 3.6

The eigenvalues of $\mathbf{RCIRC}_n^{-1}(\vec{g})$ are $\lambda_0^{-1} = \frac{1}{S_n}$ and $\lambda_m^{-1} = \frac{re^{-2\pi im/n-1}}{a(r^n-1)}$ where $m=1,2,\dots,n-1$

Corollary 3.7

$$\|\mathbf{RCIRC}_n^{-1}(\vec{g})\|_E \geq \frac{1}{|a| \sqrt{\frac{n(1-r^{2n})}{1-r^2}}}$$

Theorem 3.8

For $n \geq 3$,

$$\begin{aligned} \mathbf{RCIRC}_n^{-1}(\vec{g}) &= \begin{pmatrix} \mathcal{C}_0 & \mathcal{C}_1 & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{C}_0 & \mathcal{C}_1 & \cdots & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & \mathcal{C}_0 & \mathcal{C}_1 \\ \mathcal{C}_1 & 0 & 0 & \cdots & 0 & \mathcal{C}_0 \end{pmatrix} \\ &= \frac{1}{a(r^n - 1)} \begin{pmatrix} -1 & r & 0 & \cdots & 0 & 0 \\ 0 & -1 & r & \cdots & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & -1 & r \\ r & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \end{aligned}$$

where $\mathcal{C}_0 = \frac{-1}{a(r^n-1)}$ and $\mathcal{C}_1 = \frac{r}{a(r^n-1)}$

Proof:

Note that the first row entries of a circulant matrix which determines the matrix is just the Inverse Discrete Fourier Transform (IDFT) of its eigenvalues. From Corollary 3.6, the eigenvalues of $\mathbf{RCIRC}_n^{-1}(\vec{g})$ are $\lambda_0^{-1} = \frac{1}{s_n}$ and $\lambda_m^{-1} = \frac{re^{-2\pi im/n-1}}{a(r^n-1)}$ where $m=1,2,\dots,n-1$. By performing IDFT to them, we will get the entries of $\mathbf{RCIRC}_n^{-1}(\vec{g})$.

$$\mathcal{C}_k = \frac{1}{n} \sum_{m=0}^{n-1} \frac{r\theta^{-m} - 1}{a(r^n - 1)} \theta^{km} \text{ where } \theta = e^{\frac{2\pi i}{n}}$$

For $k=0$

$$\begin{aligned} \mathcal{C}_0 &= \frac{1}{n} \sum_{m=0}^{n-1} \frac{r\theta^{-m} - 1}{a(r^n - 1)} = \frac{1}{na(r^n - 1)} \sum_{m=0}^{n-1} (r\theta^{-m} - 1) \\ &= \frac{1}{na(r^n - 1)} \left[\frac{r(1 - \theta^n)}{1 - \theta} - n \right] = \frac{-1}{a(r^n - 1)} \end{aligned}$$

For $k=1$

$$\begin{aligned} \mathcal{C}_1 &= \frac{1}{n} \sum_{m=0}^{n-1} \frac{r\theta^{-m} - 1}{a(r^n - 1)} \theta^m \\ &= \frac{1}{na(r^n - 1)} \sum_{m=0}^{n-1} (r - \theta^m) = \frac{1}{na(r^n - 1)} \left(rn - \frac{1 - \theta^n}{1 - \theta} \right) \\ &= \frac{r}{a(r^n - 1)} \end{aligned}$$

For $k=2, 3, \dots, n-1$

$$\begin{aligned} \mathcal{C}_k &= \frac{1}{n} \sum_{m=0}^{n-1} \frac{r\theta^{-m} - 1}{a(r^n - 1)} \theta^{km} \\ &= \frac{1}{na(r^n - 1)} \sum_{m=0}^{n-1} (r\theta^{m(k-1)} - \theta^{km}) \\ &= \frac{1}{na(r^n - 1)} \left[\frac{r(1 - \theta^{(k-1)n})}{1 - \theta^{k-1}} - \frac{1 - \theta^{kn}}{1 - \theta^k} \right] = 0 \end{aligned}$$

which is as desired.

Remarks: For $n=1, 2$

(a) has inverse $(1/a)$

$\begin{pmatrix} a & ar \\ ar & a \end{pmatrix}$ has inverse $\frac{-1}{a^2(r^2-1)} \begin{pmatrix} -1 & r \\ r & -1 \end{pmatrix}$

Theorem 3.9

$$\|RCIRC_n^{-1}(\vec{g})\|_E = \frac{\sqrt{n(r^2 + 1)}}{|a(r^n - 1)|}$$

Proof:

$$\begin{aligned} \|RCIRC_n^{-1}(\vec{g})\|_E &= \sqrt{\sum_{k=0}^{n-1} n\mathcal{C}_k^2} = \sqrt{n\left(\frac{-1}{a(r^n - 1)}\right)^2 + n\left(\frac{r}{a(r^n - 1)}\right)^2} \\ &= \sqrt{\frac{n(r^2 + 1)}{[a(r^n - 1)]^2}} = \frac{\sqrt{n(r^2 + 1)}}{|a(r^n - 1)|} \end{aligned}$$

Corollary 3.10

$$\|RCIRC_n^{-1}(\vec{g})\|_2 = \max \left\{ \frac{1}{|S_n|}, \frac{\sqrt{r^2 - 2rcos\frac{2\pi m}{n} + 1}}{|a(r^n - 1)|} \right\}$$

4 Examples

Consider the right circulant matrix with the circulant vector $\vec{g} = (4, 12, 36, 108)$

$$\text{Hence } RCIRC_4(\vec{g}) = \begin{pmatrix} 4 & 12 & 36 & 108 \\ 108 & 4 & 12 & 36 \\ 36 & 108 & 4 & 12 \\ 12 & 36 & 108 & 4 \end{pmatrix} = 4 \begin{pmatrix} 1 & 3 & 9 & 27 \\ 27 & 1 & 3 & 9 \\ 9 & 27 & 1 & 3 \\ 3 & 9 & 27 & 1 \end{pmatrix}$$

$$|RCIRC_4(\vec{g})| = 4^4(1 - 4^4)^3 = -4244832000$$

Eigenvalues of $RCIRC_4(\vec{g})$:

$$\begin{aligned} \lambda_0 &= 160 \\ \lambda_1 &= -32 + 96i \\ \lambda_2 &= -32 - 96i \\ \lambda_3 &= -80 \end{aligned}$$

$$\|RCIRC_4(\vec{g})\|_E = |4| \sqrt{\frac{4(1-3^8)}{1-3^2}} = 4 \sqrt{\frac{4(-6560)}{-8}} = 8\sqrt{820} = 16\sqrt{205}$$

$$\|RCIRC_4(\vec{g})\|_2 = \max\{\lambda_m\} = \lambda_0 = 160$$

$$\begin{aligned} RCIRC_4^{-1}(\vec{g}) &= \frac{1}{4(3^4-1)} \begin{pmatrix} -1 & 3 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 \\ 3 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{320} & \frac{3}{320} & 0 & 0 \\ 0 & \frac{-1}{320} & \frac{3}{320} & 0 \\ 0 & 0 & \frac{-1}{320} & \frac{3}{320} \\ \frac{3}{320} & 0 & 0 & \frac{-1}{320} \end{pmatrix} \end{aligned}$$

$$|RCIRC_4^{-1}(\vec{g})| = \frac{-1}{4244832000}$$

Eigenvalues of $RCIRC_4^{-1}(\vec{g})$

$$\begin{aligned} \lambda_0^{-1} &= \frac{1}{160} \\ \lambda_1^{-1} &= \frac{1}{-32+96i} = \frac{-32}{10240} + \frac{96}{10240}i = \frac{-1}{320} + \frac{3}{320}i \\ \lambda_2^{-1} &= \frac{1}{-32-96i} = \frac{-32}{10240} - \frac{96}{10240}i = \frac{-1}{320} - \frac{3}{320}i \\ \lambda_3^{-1} &= -\frac{1}{80} \end{aligned}$$

$$\|RCIRC_4^{-1}(\vec{g})\|_E = \frac{\sqrt{4(3^2+1)}}{|4(3^4-1)|} = \frac{\sqrt{10}}{160}$$

$$\|RCIRC_4^{-1}(\vec{g})\|_2 = \max\{|\lambda_m^{-1}|\} = |\lambda_m^{-1}| = \frac{1}{80}$$

References

- [1] M. Bahsi and S. Solak, “On the Circulant Matrices with Arithmetic Sequence”, *International Journal of Contemporary Mathematical. Sciences*, Vol. 5, 2010, no. 25, 1213 – 1222.
- [2] R. M. Gray, “Toeplitz and Circulant Matrices: A Review,” Technical Report, Stanford University, USA.