

Some results of a class of univalent functions with negative coefficients

Mustafa Abbas Fadhel ^{1*}, Rafid Habib Buti ²

¹ Department of General Sciences, College of Basic Education, University of Al-Muthanna, Iraq

² Department of Mathematics and Computer Applications, College of Sciences, University of Al-Muthanna, Iraq

*Corresponding author E-mail: mustaffa8@hotmail.com

Copyright © 2014 Mustafa Abbas Fadhel, Rafid Habib Buti. This is an open access article distributed under the [Creative Commons Attribution License](https://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we study a new subclass of univalent analytic functions with positive coefficients in the unit disk; we obtain main result, distortion theorem and some properties of this subclass.

Keywords: Univalent Functions, Distortion Theorem, Linear Combination.

1. Introduction

Let R denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

Which are analytic and univalent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$?

Let R^* be a subclass of a class H consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \quad (2)$$

A function $f \in R^*$ is said to be starlike function of order γ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \quad (0 \leq \gamma < 1; z \in U). \quad (3)$$

Definition 1: A function $f \in H^*$ is said to be in the class $RM(B, \tau, \theta)$ if f satisfies the condition:

$$\left| \frac{\left[\left(\frac{zf'(z)}{f(z)} - 1 \right) (\tau + \theta) + B \left(\frac{zf'(z)}{f(z)} - 1 \right) \right]}{(1 - \theta) + \theta \left(\frac{zf'(z)}{f(z)} - 1 \right)} \right| < 1, \quad (4)$$

Where, $0 \leq B \leq 1$, $0 \leq \theta < 1$, $0 \leq \tau \leq 1$.

In the following theorem, we obtain a sufficient condition for the function f to be in the class $RM(B, \tau, \theta)$.

Theorem 1: A function f defined by (2) be in the class $RM(B, \tau, \theta)$ if

$$\sum_{n=2}^{\infty} [(n-2)\theta + 1 + (n-1)(\tau + \theta + B)] a_n \leq 1 - \varepsilon. \quad (5)$$

The result is sharp.

Proof: For $|z|=1$, we have $\left| \left[z f'(z) - f(z)(\tau + \theta) + B(z f'(z) - f(z)) \right] \right|$

$$\begin{aligned} &= \left| f(z)(1-\theta) + \theta(z f'(z) - f(z)) \right| \\ &= \left| \sum_{n=2}^{\infty} (n-1)(\tau + \theta + B) a_n z^n \right| \\ &= \left| z(1-\theta) + \sum_{n=2}^{\infty} [(n-2)\theta + 1] a_n z^n \right| \\ &= \sum_{n=2}^{\infty} [(n-2)\theta + 1 + (n-1)(\tau + \theta + B)] a_n - (1-\theta) \leq 0. \end{aligned}$$

This by maximum modulus theorem $f \in RM(B, \tau, \theta)$

The result is sharp for the function f given by the form

$$f_n(z) = z + \frac{1-\theta}{[(n-2)\theta + 1 + (n-1)(\tau + \theta + B)]} z^n \quad (6)$$

There are many authors who have studied the various interesting properties of the classes, H. Silverman [4], H. J. A. Hussein and R. H. Buti [3], K. K. Dixit and Y. K. Mishra [2], N. E. Cho, S. H. Lee and S. Owa [1].

In the next, we obtain distortion theorem for the class $RM(B, \tau, \theta)$.

Theorem 2: Let $f(z)$ defined by (2) be in the class $RM(B, \tau, \theta)$. Then

$$|f(z)| \leq |z| + \frac{1-\theta}{[1+\tau+\theta+B]} |z|^2 \quad (7)$$

and

$$|f(z)| \geq |z| - \frac{1-\theta}{[1+\tau+\theta+B]} |z|^2. \quad (8)$$

The inequalities in (7) and (8) are attained for the function

$$f(z) = z + \frac{1-\theta}{[1+\tau+\theta+B]} z^2 \quad (9)$$

Proof: By using Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\theta}{[1+\tau+\theta+B]} \quad (10)$$

So by using (2) and (10), we have

$$\begin{aligned} |f(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{1-\theta}{[1+\tau+\theta+B]} |z|^2, \end{aligned}$$

Which gives (7), we also have

$$\begin{aligned} |f(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{1-\theta}{[1+\tau+\theta+B]} |z|^2 \end{aligned}$$

Which gives (8)?

Now, we shall prove that class $RM(B, \tau, \theta)$ is closed under convex linear combinations.

Let the function f_k ($k=1, 2, \dots, m$) be defined by

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{n,k} z^n, \quad (a_{n,k} \geq 0, n \geq 2). \quad (11)$$

Theorem 3: Let the function $f_k(z)$ defined by (11) be in the class $RM(B, \tau, \theta)$, ($0 \leq \theta < 1$). Then the following function g defined by

$$g(z) = z + \frac{1}{m} \sum_{n=2}^{\infty} \left[\sum_{k=2}^m a_{n,k} \right] z^n, \quad (k=1, 2, \dots, m) \quad (12)$$

Is in the class $RM(B, \tau, \theta)$, where $\theta = \min_{2 \leq k \leq m} \{\theta_k\}$. (13)

Proof: Since $f_k \in RM(B, \tau, \theta)$ for each ($k=1, 2, \dots, m$), we note that

$$\sum_{n=2}^{\infty} [(n-2)\theta_k + 1 + (n-1)(\tau + \theta_k + B)] a_{n,k} \leq 1 - \varepsilon_k.$$

Therefore

$$\begin{aligned} & \sum_{n=2}^{\infty} [(n-2)\theta_k + 1 + (n-1)(\tau + \theta_k + B)] \left[\frac{1}{m} \sum_{k=2}^m a_{n,k} \right] \\ &= \frac{1}{m} \sum_{k=2}^m \left[\sum_{n=2}^{\infty} [(n-2)\theta_k + 1 + (n-1)(\tau + \theta_k + B)] a_{n,k} \right] \\ &\leq \frac{1}{m} \sum_{k=2}^m (1 - \theta_k) \leq (1 - \theta) . \end{aligned}$$

Thus, we get

$$\sum_{n=2}^{\infty} [(n-2)\theta_k + 1 + (n-1)(\tau + \theta_k + B)] \left[\frac{1}{m} \sum_{k=2}^m a_{n,k} \right] \leq 1 - \theta .$$

Hence, by Theorem 1, we have $g \in RM(B, \tau, \theta)$.

In the next we show that the integral operator in the class $RM(B, \tau, \theta)$.

References

- [1] N. E. Cho, S. H. Lee and S. Owa, *A class of meromorphic univalent functions with positive coefficients*, Koebe J. Math., 4 (1987), 43-50
- [2] K. K. Dixit and Y. K. Mishra, *Meromorphic starlike functions with positive missing coefficients*, Acta Ciencia Indica, Vol. XXV, M. No. 2(1999), 211-219.
- [3] H. J. A. Hussein and R. H. Buti, *Some Geometric Properties of a Class of Univalent Functions with Negative Coefficients Defined by Hadamard Product with Fractional Calculus I*, I.M.Forum, Vol. 6, 2011, no. 64, 3179 – 3188.
- [4] H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl. 172(1993), 574-581. <http://dx.doi.org/10.1006/jmaa.1993.1044>.