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Research Paper

Connectedness in fuzzy closure space

U. D. Tapi , Bhagyashri A. Deole*

Department of Applied Mathematics and Computational Science Shri G. S. Institute of Technology and Science (M.P.), India
**Corresponding author E-mail: deolebhagyashri@gmail.com*

Abstract

A fuzzy Čech closure space (X, k) is a fuzzy set X with fuzzy Čech closure operator k: $I^X \to I^X$ where I^X is a power set of fuzzy subsets of X, which satisfies k (∅) = ∅, $\lambda_1 \le \lambda_2 \Rightarrow k(\lambda_1) \le k(\lambda_2)$, $k(\lambda_1 \cup \lambda_2) = k(\lambda_1) \cup k(\lambda_2)$ for all λ_1, $\lambda_2 \in I^X$. A fuzzy topological space X is said to be fuzzy connected if it has no proper fuzzy clopen set.Many properties which hold in fuzzy topological space hold in fuzzy Čech closure space as well. A Čech closure space (X, u) is said to be connected if and only if any continuous map f from X to the discrete space {0, 1} is constant. In this paper we introduce connectedness in fuzzy Čech closure space.

***Keywords**: Connectedness in Fuzzy Čech Closure Space, Connectedness in Fuzzy Topological Space, Fuzzy Čech Closure Operator, Fuzzy Čech Closure Space, Fuzzy Topological Space.*

1. Introduction

In 1965 Zadeh [1] in his classical paper generalized characteristic function to fuzzy set. Chang [2] in 1968 introduced the topological structure of fuzzy sets. Pu and Liu [3] defined the concept of fuzzy connectedness using fuzzy closed set. Lowen [4] also defined an extension of a connectedness in a restricted family of fuzzy topologies. Fuzzy Čech closure operator and fuzzy Čech closure space were first studied by A.S. Mashhour and M.H. Ghanim [5]. In this paper we introduce connectedness in fuzzy Čech closure space and study some of their properties.

2. Preliminaries

Definition 2.1 [6]: An operator u: P(X) →P(X) defined on the power set P(X) of a set X satisfying the axioms:

1) $u\phi=\phi$,
2) $A \subseteq uA$, for every $A \subseteq X$,
3) $u(A \cup B) = uA \cup uB$, for all $A, B \subseteq X$.

is called a Čech closure operator and the pair (X, u) is a Čech closure space.

Definition 2.2 [7]: Let X is a non-empty fuzzy set. A function k: $I^X \to I^X$ is called fuzzy Čech closure operator on X if it satisfies the following conditions

1) k (∅) =∅.
2) $\lambda \le k(\lambda)$, for all $\lambda \in I^X$.
3) $k(\lambda_1 \cup \lambda_2) = k(\lambda_1) \cup k(\lambda_2)$ for all $\lambda_1, \lambda_2 \in I^X$.

The pair (X, k) is called fuzzy Čech closure space.

Definition 2.3 [8]: A fuzzy topological space (X, k) is said to be connected if X cannot be represented as the union of two non-empty, disjoint fuzzy open subsets of X.

Definition 2.4 [9]: A Čech closure space (X, u) is said to be connected if and only if any continuous map f from X to the discrete space {0, 1} is constant. A subset A in a Čech closure space (X, u) is said to be connected if A with the subspace topology is a connected space.

Definition 2.5 [10]: Given fuzzy topological spaces (X, δ) and (Y, γ), a function f: X→Y is F- continuous if the inverse image under f of any fuzzy open set in Y is a fuzzy open set in X; i.e., if $f^{-1}(v) \in\delta$ whenever $v\in\gamma$.

3. Connectedness in fuzzy closure space

Definition 3.1: Let X is a nonempty fuzzy set .A function k: $I^X \to I^X$ is called fuzzy Čech closure operator on X. A fuzzy Čech closure space (X, k) is said to be connected if and only if any F-continuous map f from X to the fuzzy discrete space {0, 1} is constant.

Example 3.2: Let X= {a, b, c} be a non-empty fuzzy set. Define fuzzy Čech closure operator
k: $I^X \to I^X$ such that

$$k(x) = \begin{cases} 0_X; & A=0_X. \\ 1_{\{b, c\}}; & \text{if } 0<A \leq 1_{\{b, c\}} \\ 1_{\{b, c\}}; & \text{if } 0<A \leq 1_{\{b\}} \\ 1_{\{b, c\}}; & \text{if } 0<A \leq 1_{\{c\}} \\ 1_X; & \text{otherwise} \end{cases}$$

FOS(X) = {{a}, {b}, {c}, {a, b}, {a, c}, ∅, X}.
Then (X, k) is called fuzzy Čech closure space. We define an F-continuous function f: X→ {0, 1} such that
$f^{-1}\{1\}$ = {a, b} = {a, c} = {a} = {b} = {c} = X, $f^{-1}\{0\}$ = ∅.
Here function f is constant. Hence (X, k) is a fuzzy connected Čech closure space.

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k: $I^X \to I^X$ such that

$$k(x) = \begin{cases} 0_X; & A=0_X. \\ 1_{\{a, b\}}; & \text{if } 0<A \leq 1_{\{a\}} \\ 1_{\{b, c\}}; & \text{if } 0<A \leq 1_{\{b\}} \\ 1_{\{c, a\}}; & \text{if } 0<A \leq 1_{\{c\}} \\ 1_X; & \text{otherwise.} \end{cases}$$

FOS(X) = {{a}, {b}, {c}, {a, b}, {b, c}, {a, c}, ∅, X}.
Then (X, k) is called fuzzy Čech closure space. We define an F-continuous function f: X→ {0, 1} such that
$f^{-1}\{1\}$ = {a} = {b} = {c} = {a, b} = {b, c} = {c, a} = X, $f^{-1}\{0\}$ = ∅. Here function f is constant. Hence (X, k) is a fuzzy connected Čech closure space.

Theorem 3.4: *A fuzzy Čech closure space (X, k) is said to be disconnected if and only if there is a nonempty proper fuzzy subset of X, which is both fuzzy open and fuzzy closed.*

Proof: Necessary: Let fuzzy Čech closure space (X, k) is disconnected i.e. there exists an F-continuous function f: X→ {0, 1} is not constant. Consider a proper fuzzy subset λ of X such that λ = 1-δ. Since λ is fuzzy closed subset of X therefore δ is fuzzy open subset of X. But δ is also a fuzzy closed subset of fuzzy Čech closure space (X, k) therefore λ is fuzzy open subset of X. Hence λ is a clopen subset of X.
Sufficient: Let δ=X− λ, since λ is a nonempty proper fuzzy subset of X, so that fuzzy set δ is also nonempty. Consider an F-continuous function f: X→ {0, 1} such that f (λ) = 0 or 1, f (δ) = 1 or 0 that is an F-continuous function f is not constant. Hence (X, k) is fuzzy disconnected Čech closure space.

Theorem 3.5: *A continuous image of a fuzzy connected Čech closure space is fuzzy connected Čech closure space.*

Proof: Let fuzzy Čech closure space (X, k) is connected and consider an F-continuous function f: X →f(X) is surjective. If f(x) is not fuzzy connected Čech closure space, then there would be an F-continuous surjection g: f(x) → {0, 1} so that the composite function gof: X→ {0, 1} would also be an F-continuous surjection. It is contradiction to the connectedness of fuzzy Čech closure space (X, k). Hence f(x) is a fuzzy connected Čech closure space.

Theorem 3.6: *The union of any family of fuzzy connected subsets of fuzzy connected Čech closure space with a common point is connected.*

Proof: Let $\{X_\alpha\}$ be a family of fuzzy connected subsets of fuzzy connected Čech closure space (X, k) and $p\in X_\alpha$ for all α. Let f: $\cup X_\alpha \to \{0, 1\}$ be any F-continuous map and $f_\alpha : X_\alpha \to \{0, 1\}$be the restriction of f to X_α. Since f and f_α are

F-continuous functions. Each X_α is fuzzy connected Čech closure space so f_α is constant. Now let $p \in X_\alpha$, $f_\alpha(x_\alpha)=f(p)$, $\forall\alpha \Rightarrow p \in \cup X_\alpha$, $f(x_\alpha)=f(p)$ i.e. f is constant. Hence UX_α is fuzzy connected Čech closure space.

4. Connected subsets in a fuzzy closure space

Definition 4.1:-If $A \subset X$, (X, k) is a fuzzy Čech closure space, then A is said to be a fuzzy connected subset of X if A is fuzzy connected space as a fuzzy subspace of X. If $A \subset Y \subset X$, then A is a fuzzy connected subset of the fuzzy Čech closure space X if and only if it is a fuzzy connected subset of the fuzzy subspace Y of (X, k).

Example 4.2: Let X= {a, b, c} be a non-empty fuzzy set. Define fuzzy Čech closure operator
$k: I^X \to I^X$ such that

$$k(x) = \begin{cases} 0_X; & A=0_X. \\ 1_{\{b,c\}}; & \text{if } 0<A \le 1_{\{b,c\}} \\ 1_{\{b,c\}}; & \text{if } 0<A \le 1_{\{b\}} \\ 1_{\{b,c\}}; & \text{if } 0<A \le 1_{\{c\}} \\ 1_X; & \text{otherwise.} \end{cases}$$

FOS(X) = {{a}, {b}, {c}, {a, b}, {a, c}, X, ∅}.
Then (X, k) is called fuzzy Čech closure space. We define an F-continuous function f: X→ {0, 1} such that
$f^{-1}\{1\}$ = {a, b} = {a, c} = {a} = {b} = {c} = X, $f^{-1}\{0\}$ = ∅.
Here function f(x) is constant. Hence (X, k) is a fuzzy connected Čech closure space.
Consider a subset Y= {a, b} of X. Define a fuzzy Čech closure operator $k_Y: I^Y \to I^Y$ such that

$$k_Y(x) = \begin{cases} 0_X; & A=0_X. \\ 1_{\{a,b\}}; & \text{if } 0<A \le 1_{\{a\}} \\ 1_{\{b\}}; & \text{if } 0<A \le 1_{\{b\}} \\ 1_X; & \text{otherwise} \end{cases}$$

FOS(X) = {{a}, X, ∅}.Here (Y, k_Y) is a fuzzy Čech closure space. We define an F-continuous function f: Y→ {0, 1} such that $f^{-1}\{1\}$ = {a} = {X}, $f^{-1}\{0\}$ = ∅.
Hence (Y, k_Y) is a fuzzy connected Čech closure subspace of fuzzy connected Čech closure space (X, k).

Theorem 4.3: *If (X, k) is a fuzzy Čech closure space and A is a fuzzy connected subset of X and λ and δ are non-empty fuzzy open sets in X satisfying λ + δ=1, then either λ/A = 1 or δ/A =1.*

Proof: If A is a fuzzy connected subset of X than there exists a continuous function f: A→ {0, 1} is constant. Suppose there exists x_0 ,$y_0 \in A$ such that $\lambda(x_0)\neq 1$ and $\delta(y_0)\neq 1$.Then λ+δ=1 implies that λ/A+ δ/A =1,where λ/A ≠ 0 , δ/A ≠ 0 which implies that f(λ)≠f(δ) in A. So A is not a fuzzy connected Čech closure subset of X. Hence either λ/A = 1 or δ/A =1.

Theorem 4.4: *Let $\{A_\alpha\}_{\alpha\in\wedge}$ be a family of fuzzy connected subsets of fuzzy Čech closure space (X, k) such that for each α and β in ∧ and α≠β, $\mu_{A\alpha}$ and $\mu_{A\beta}$ are not separated from each other. Then $\cup_{\alpha\in\wedge} A_\alpha$ is a fuzzy connected subset of fuzzy Čech closure space (X, k).*

Proof: Suppose $Y=\cup_{\alpha\in\wedge} A_\alpha$ is not a fuzzy connected subset of X that is F-continuous function f: Y→ {0, 1} is not constant. Let there exists non-zero fuzzy open sets a and b in Y s. t. f (a) ≠f (b) and a+b=1.Fix $\alpha_0 \in \wedge$.Then $A\alpha_0$ is a fuzzy connected subset of Y as it is so in fuzzy Čech closure space
(X, k).Therefore by theorem 4.3, either $\boldsymbol{\mu}_{A\alpha0}/A\alpha_0 = a/A\alpha_0$ or $\boldsymbol{\mu}_{A\alpha0}/A\alpha_0 = b/A\alpha_0$ .Without loss of generality assume that
$\boldsymbol{\mu}_{A\alpha0}/A\alpha_0 = a/A\alpha_0$ (1)
Define λ and δ as λ(x) =a(x) if x∈Y and λ(x) =0 if x∈X-Y and δ(x) =b(x) if x∈Y and δ(x) =0if x∈X-Y.
By theorem 4.3,
$k_Y(a)=k(\lambda)/Y$ and $k_Y(b)=k(\delta)/Y$ (2)
So (1) implies that $\boldsymbol{\mu}_{A\alpha0} \le \lambda$.Therefore
$k_Y(\boldsymbol{\mu}_{A\alpha0}) \le k(\lambda)$ (3)
Let $\alpha \in \wedge - \{\alpha_0\}$.Since A_α is a fuzzy connected closure subset of Y either
$\boldsymbol{\mu}_{A\alpha}/A_\alpha = a/A_\alpha$ or $\boldsymbol{\mu}_{A\alpha}/A_\alpha = b/A_\alpha$, we show that $\boldsymbol{\mu}_{A\alpha}/A_\alpha \neq b/A_\alpha$.
Suppose that $\boldsymbol{\mu}_{A\alpha}/A_\alpha = b/A_\alpha$. Therefore $\boldsymbol{\mu}_{A\alpha} \le \delta$.Hence
$k(\boldsymbol{\mu}_{A\alpha}) \le k(\delta)$ (4)
This gives a contradiction as $\boldsymbol{\mu}_{A\alpha0}$ and $\boldsymbol{\mu}_{A\alpha}$ are not separated from each other.
So $\boldsymbol{\mu}_{A\alpha}/A_\alpha \neq b/A_\alpha$ .Hence $\boldsymbol{\mu}_{A\alpha}/A_\alpha = a/A_\alpha$ for each $\alpha \in \wedge$.Which implies that $\boldsymbol{\mu}_Y = a$.

But a + b = 1. So b(x) = 0 for every x∈ Y.But b≠0.So the supposition that Y is not a fuzzy connected subset of X is false i.e. F-continuous function f: Y→ {0, 1} is constant. Hence $\bigcup_{\alpha\epsilon\wedge} A_\alpha$ is a fuzzy connected subset of fuzzy Čech closure space (X, k).

Theorem 4.5: *If $\{A_\alpha\}_{\alpha\epsilon\wedge}$ is a family of fuzzy connected subsets of a fuzzy connected Čech closure space (X, k) and $\bigcap_{\alpha\epsilon\wedge} A_\alpha \neq \emptyset$,then $\bigcup_{\alpha\epsilon\wedge} A_\alpha$ is a fuzzy connected subset of fuzzy connected Čech closure space X.*

Proof: Let A_α in the family is a fuzzy connected subset of fuzzy Čech closure space (X, k) i.e. A F-continuous function f :$\{A_\alpha\}_{\alpha\epsilon\wedge} \rightarrow \{0,1\}$ is constant. For any α, β ∈∧, α≠ β , we have $A_\alpha \cap A_\beta \neq \emptyset$.Hence $k(\boldsymbol{\mu}_{A\alpha}) + \boldsymbol{\mu}_{A\beta} > 1$ and $\boldsymbol{\mu}_{A\alpha} + k(\boldsymbol{\mu}_{A\beta}) > 1$.Thus characteristics functions of each pair of members of the family are not separated from each other. So $\bigcup_{\alpha\epsilon\wedge} A_\alpha$ is a fuzzy connected subset of fuzzy Čech closure space X.

Theorem 4.6: *If C is a fuzzy connected subset of a fuzzy connected Čech closure space X, V⊂ X-C and $\boldsymbol{\mu}_v$ /X-C is a fuzzy clopen in X-C, and then C⋃V is a fuzzy connected subset of fuzzy connected Čech closure space (X, k).*

Proof: Suppose Y=C⋃V is not a fuzzy connected Čech closure subset of fuzzy connected closure space (X, k).Then there exist fuzzy open sets λ and δ such that f (λ) ≠ f (δ), and

λ /Y + δ /Y =1 (1)

Since C is a fuzzy connected Čech closure subset of Y (as it is so in X),

By theorem 4.3, λ /C = μ_C /C or δ / C = μ_C /C .Without loss of generality assume that

λ /C = μ_C /C . So by equation (1),

δ / C = 0 (2)

Therefore δ / V ≠ 0 (as δ /Y ≠ 0) (3)

Let us define a fuzzy set δ_1 in X as

$\delta_1(x) = \delta(x)$, if x∈ V,

= 0, if x∈ X-V.

We shall now show that δ_1 is fuzzy closed as well as fuzzy open in X.

Now δ_1 /V = δ / V and by equation (1) δ / V is fuzzy closed in V.

Therefore δ_1 /V is fuzzy closed in V. Also $\boldsymbol{\mu}_v$ is fuzzy closed in X-C. Therefore δ_1 /X-C is fuzzy closed in X-C. Now

δ_1 /X-C = δ /X-C⋂ $\boldsymbol{\mu}_v$ /X-C. (4)

Therefore δ_1 /X-C is fuzzy open in X-C.

Thus δ_1 /X-C is fuzzy clopen in X-C

Further δ_1 /Y = δ/Y (because of (2)).As δ /Y is fuzzy clopen in Y (because of (1)), therefore (5)

δ_1 /Y is fuzzy clopen in Y.

Now by (4) and (5) and δ_1 is fuzzy clopen in (X-C) ∪ Y = X. As δ_1 is a proper fuzzy set, we get a contradiction to the fact that X is fuzzy connected Čech closure space. Hence C⋃V is a fuzzy connected subset of fuzzy connected Čech closure space (X, k).

Theorem 4.7: *If A and B are fuzzy subsets of a fuzzy Čech closure space (X, k) and $\boldsymbol{\mu}_A \leq \boldsymbol{\mu}_B \leq k(\boldsymbol{\mu}_A)$ and A is fuzzy connected closure subset of X , then B is also a fuzzy connected Čech closure subset of fuzzy Čech closure space (X, k).*

Proof: If we suppose that B is not a fuzzy connected subset then there exist fuzzy open sets λ and δ in X such that

λ /B ≠ 0, δ /B ≠ 0, and f(λ)≠f(δ) (1)

We first show that λ /A≠ 0. If λ /A = 0, then $\lambda + \boldsymbol{\mu}_A \leq 1$, which implies that $\lambda + k(\boldsymbol{\mu}_A) \leq 1$; so $\lambda + \boldsymbol{\mu}_B \leq 1$ (because $\boldsymbol{\mu}_B \leq k(\boldsymbol{\mu}_A)$).This is turn implies that λ /B = 0, which is a contradiction ,as λ /B ≠ 0.Therefore λ /A ≠ 0 . Similarly we can show that δ /A ≠ 0.Now (1) and $\boldsymbol{\mu}_A \leq \boldsymbol{\mu}_B$ imply λ /A + δ /A = 1.So A is not fuzzy connected, which is a contradiction.

5. Conclusion

In this paper the idea of connectedness in fuzzy Čech closure space was introduced and relationship between the connectedness and fuzzy Čech closure space were explained.

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